

Quasi-variational inequalities for phase transitions

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Abstract. In this paper, we consider a quasi-variational problem for irreversible phase change with temperature. We propose a mathematical model of a class of irreversible phase change described by a system of PDEs including a quasi-variational inequality. One of our interests is an existence of solutions of this system. The existence of solutions is obtained as a limit of approximate solutions. Our approximate problems are formulated by using the Moreau-Yosida approximation. The convergence of approximate solutions is based on some uniform estimates and monotonicity techniques in the nonlinear operator theory.

1. Introduction

We consider the following system of PDEs:

$$\theta_t + w_t - \Delta\theta = h(t, x) \quad \text{in } Q := (0, T) \times \Omega, \quad (1.1)$$

$$w_t + \partial I_{\theta, N}(w_t) - \nu \Delta w \ni f(\theta, w) \quad \text{in } Q, \quad (1.2)$$

subject to the boundary conditions

$$\frac{\partial \theta}{\partial n} = \frac{\partial w}{\partial n} = 0 \quad \text{on } \Sigma := (0, T) \times \Gamma \quad (1.3)$$

and the initial conditions

$$\theta(0, \cdot) = \theta_0, \quad w(0, \cdot) = w_0 \quad \text{in } \Omega, \quad (1.4)$$

where Ω is a bounded domain in $\mathbf{R}^3$ with smooth boundary Γ , T is a finite positive number, ν is a positive constant, θ_t and w_t are the time derivatives of θ and w , Δ denotes the Laplace operator in space variable x and $\frac{\partial}{\partial n}$ denotes the outward normal derivative on Γ ; f is a given function on $\mathbf{R}^2$, h is a given function on Q ; θ_0 and w_0 are the initial data of θ and w , respectively; $I_{\theta, N}(\cdot)$ is the indicator function of the interval $[g(\theta), g(\theta) + N]$ with a non-negative bounded smooth function g on $\mathbf{R}$ and a sufficiently large positive number N ;

$$I_{\theta, N}(w_t) := \begin{cases} +\infty & \text{if } w_t < g(\theta) \text{ or } g(\theta) + N < w_t, \\ 0 & \text{if } g(\theta) \leq w_t \leq g(\theta) + N; \end{cases}$$

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$\partial I_{\theta,N}(w_t)$ is the subdifferential with respect to w_t , namely, it is a set-valued mapping defined by

$$\partial I_{\theta,N}(w_t) := \begin{cases} \emptyset & \text{if } w_t < g(\theta) \text{ or } g(\theta) + N < w_t, \\ (-\infty, 0] & \text{if } w_t = g(\theta), \\ \{0\} & \text{if } g(\theta) < w_t < g(\theta) + N, \\ [0, +\infty) & \text{if } w_t = g(\theta) + N. \end{cases}$$

For instance, in the context of solidification of multi-composite materials, the unknowns θ and w of the system $(P) := \{(1.1)-(1.4)\}$ are explored, respectively, as the temperature and the irreversible solidification parameter. (w is often called a phase change parameter or an order parameter.) Since the mapping $\partial I_{\theta,N}(\cdot)$ in (1.2) requires that w_t is within $(0 \leq) g(\theta) \leq w_t \leq g(\theta) + N$, our system possibly describes the irreversibility effect. As for a mathematical treatment of irreversible phase change, there are some related works [3,6,7,8,13,15,16] and so on, however, in any case the restriction of w_t does not depend on the unknown functions θ and w . In our setting, $\partial I_{\theta,N}(\cdot)$ depends on the unknown function θ , which is one of new aspects of our work.

In this paper, we give an existence result for the system (P) under some assumptions on the data f, g, h, θ_0, w_0 . Concerning the system (P) we have already discussed the case when Ω is a bounded domain in $\mathbf{R}^2$ (cf.[3]). In the paper of [3], we used the abstract quasi-variational evolution inequality established in [2] to get approximate solutions:

$$\partial \phi_{u(t)}(u'(t)) + \partial \psi(u(t)) \ni G(t, u(t)) \text{ in } X \text{ for a.e. } t \in (0, T),$$

where X is a real Hilbert space, ϕ_u is a proper lower semi-continuous convex function on X for each $u \in D(\psi) := \{z \in X; \psi(z) < +\infty\}$, ψ is a proper lower semi-continuous convex function on X , $\partial \phi_u$ and $\partial \psi$ are their subdifferentials in X , G is a single-valued operator from X into itself. Since we cannot apply such a procedure to (P) , we shall employ a fixed point argument to construct approximate solutions of (P) and obtain a solution of (P) by showing their convergence.

Throughout this paper, H denotes the real Hilbert space $L^2(\Omega)$ with the usual inner product $(\cdot, \cdot)$ and V denotes the Sobolev space $H^1(\Omega)$ and it is a Hilbert space equipped with the following inner product:

$$(z, v)_V := (z, v) + a(z, v), \quad a(z, v) := \int_{\Omega} \nabla z(x) \cdot \nabla v(x) dx, \quad \forall z, v \in V$$

and norm $|z|_V := \sqrt{(z, z)_V}$. We use the notation Δ_0 to indicate the operator Δ with homogeneous Neumann boundary condition; note here that $-\Delta_0$ is linear, closed, non-negative and self-adjoint in H ; in fact, we have

$$D(-\Delta_0) = \left\{ z \in H^2(\Omega); \frac{\partial z}{\partial n} = 0 \text{ in } H^{\frac{1}{2}}(\Gamma) \right\}$$

and

$$-\Delta_0 z = -\Delta z \text{ in } H, \quad \forall z \in D(-\Delta_0).$$

Notation $|\cdot|_\infty$ stands for various L^∞ -norms, for instance, $L^\infty(Q)$, $L^\infty(\Omega)$ and so on.

Next we recall some basic properties on convex functions and their subdifferentials in a real Hilbert space; precisely see [4,5,9,12]. Let W be a real Hilbert space with inner product $(\cdot, \cdot)_W$ and norm $|\cdot|_W$. Let φ be a proper lower semi-continuous and convex function on W . The subset $D(\varphi) = \{z \in W; \varphi(z) < +\infty\}$ of W is called the effective domain of φ . The subdifferential $\partial\varphi$ of φ is a set-valued operator from W into itself defined by

$$z^* \in \partial\varphi(z) \stackrel{\text{def}}{\iff} (z^*, v - z)_W \leq \varphi(v) - \varphi(z), \quad \forall v \in W \quad (1.5)$$

and its domain is defined by $D(\partial\varphi) := \{z \in W; \partial\varphi(z) \neq \emptyset\}$. For each $\varepsilon > 0$, we define $J_\varepsilon^\varphi := (I + \varepsilon\partial\varphi)^{-1}$ which is called the resolvent of $\partial\varphi$, where I is the identity operator in W . The Moreau-Yosida approximation φ_ε of φ and its subdifferential $\partial\varphi_\varepsilon$ are defined by

$$\varphi_\varepsilon(z) = \inf_{v \in W} \left\{ \frac{1}{2\varepsilon} |z - v|_W^2 + \varphi(v) \right\}, \quad \partial\varphi_\varepsilon(z) := \frac{z - J_\varepsilon^\varphi z}{\varepsilon}, \quad \forall z \in W. \quad (1.6)$$

Concerning the Moreau-Yosida approximation and the resolvent J_ε^φ , the following facts are often used in this paper:

$$\varphi_\varepsilon(z) = \varphi(J_\varepsilon^\varphi z) + \frac{1}{2\varepsilon} |z - J_\varepsilon^\varphi z|_W^2, \quad \forall \varepsilon > 0, \forall z \in W, \quad (1.7)$$

$$\varphi(J_\varepsilon^\varphi z) \leq \varphi_\varepsilon(z) \leq \varphi(z), \quad \lim_{\varepsilon \searrow 0} \varphi_\varepsilon(z) = \varphi(z), \quad \forall \varepsilon > 0, \forall z \in W, \quad (1.8)$$

$$|\partial\varphi_\varepsilon(z)|_W \leq |\partial\varphi(z)| := \inf\{|w|_W; w \in \partial\varphi(z)\}, \quad \forall \varepsilon > 0, \forall z \in D(\partial\varphi). \quad (1.9)$$

Especially, in the case that Ω is a bounded domain with smooth boundary, for every $z \in H$, define

$$\varphi(z) = \begin{cases} \frac{1}{2}a(z, z) & \text{if } z \in V, \\ +\infty & \text{otherwise.} \end{cases} \quad (1.10)$$

Then φ is a proper lower semi-continuous and convex function on H and $\partial\varphi = -\Delta_0$ in H (cf. [5]).

2. Main result

We make the following assumptions on the data:

- (1) f is a Lipschitz continuous function from $\mathbf{R}^2$ into $\mathbf{R}$ and g is a non-negative function of C^2 -class from $\mathbf{R}$ into itself such that the derivatives g' and g'' are bounded on $\mathbf{R}$.
- (2) $h \in L^\infty(Q)$, $\theta_0 \in V \cap L^\infty(\Omega)$ and $w_0 \in D(-\Delta_0)$.

Now we give the definition of a solution of (P).

Definition 2.1. A pair of functions $\{\theta, w\}$ is called a solution of (P) if it satisfies the following conditions (a1)-(a4):

$$(a1) \quad \theta, w \in W^{1,2}(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega)).$$

$$(a2) \quad \theta'(t) + w'(t) - \Delta_0 \theta(t) = h(t) \text{ in } H \text{ for a.e. } t \in (0, T).$$

$$(a3) \quad \text{There exists a function } \xi \in L^2(0, T; H) \text{ with } \xi \in \partial I_{\theta, N}(w') \text{ a.e. on } Q \text{ such that}$$

$$w'(t) + \xi(t) - \nu \Delta_0 w(t) = f(\theta(t), w(t)) \text{ in } H \text{ for a.e. } t \in (0, T).$$

$$(a4) \quad \theta(0) = \theta_0 \text{ and } w(0) = w_0 \text{ in } H.$$

We denote the time-derivatives of θ and w by θ' and w' , respectively.

Theorem 2.1. *Under the assumptions (1) and (2), problem (P) has at least one solution $\{\theta, w\}$ in the sense of Definition 2.1 such that $\theta \in L^\infty(Q)$ and $w \in W^{1,2}(0, T; V) \cap L^\infty(0, T; H^2(\Omega))$.*

The above existence result will be proved in the sections 3, 4 and 5.

3. Approximate problem

In this section, we consider the following approximate problem $(P_\varepsilon) := \{(3.1)-(3.3)\}$:

$$\theta_t + w_t - \Delta_0 \theta = h \quad \text{in } Q, \quad (3.1)$$

$$w_t + \partial I_{\theta, N}(w_t) + \nu \partial \varphi_\varepsilon(w) \ni f(\theta, J_\varepsilon^\varphi w) \quad \text{in } Q, \quad (3.2)$$

$$\theta(0, \cdot) = \theta_0, \quad w(0, \cdot) = w_0 \quad \text{in } \Omega, \quad (3.3)$$

where φ_ε is the Moreau-Yosida approximation of φ defined by (1.10), $\partial \varphi_\varepsilon$ is the subdifferential of φ_ε in H and $J_\varepsilon^\varphi = (I + \varepsilon \partial \varphi)^{-1}$.

Definition 3.1. For every fixed $\varepsilon > 0$, a pair of functions $\{\theta_\varepsilon, w_\varepsilon\}$ is called a solution of (P_ε) if it satisfies the following conditions (b1)-(b4):

$$(b1) \quad \theta_\varepsilon, J_\varepsilon^\varphi w_\varepsilon \in W^{1,2}(0, T; H) \cap L^\infty(0, T; V), \quad w_\varepsilon \in W^{1,2}(0, T; H).$$

$$(b2) \quad \theta'_\varepsilon(t) + w'_\varepsilon(t) - \Delta_0 \theta_\varepsilon(t) = h(t) \text{ in } H \text{ for a.e. } t \in (0, T).$$

$$(b3) \quad \text{There exists a function } \xi_\varepsilon \in L^2(0, T; H) \text{ with } \xi_\varepsilon \in \partial I_{\theta_\varepsilon, N}(w'_\varepsilon) \text{ a.e. on } Q \text{ such that}$$

$$w'_\varepsilon(t) + \xi_\varepsilon(t) + \nu \partial \varphi_\varepsilon(w_\varepsilon(t)) = f(\theta_\varepsilon(t), J_\varepsilon^\varphi w_\varepsilon(t)) \text{ in } H \text{ for a.e. } t \in (0, T).$$

$$(b4) \quad \theta_\varepsilon(0) = \theta_0 \text{ and } w_\varepsilon(0) = w_0 \text{ in } H.$$

Theorem 3.1. *Under the assumption (1), for any $\theta_0, w_0 \in V, h \in L^2(0, T; H)$ and for each $\varepsilon > 0$, there exists at least one solution $\{\theta_\varepsilon, w_\varepsilon\}$ of (P_ε) in the sense of Definition 3.1.*

First, we construct a local in time solution of (P_ε) by using the fixed point argument. To do so, prepare a set $X_T^\varepsilon(M_0)$ defined by

$$X_T^\varepsilon(M_0) := \left\{ (\bar{\theta}, \bar{w}) \left| \begin{array}{l} \bar{\theta} \in W^{1,2}(0, T; H) \cap L^\infty(0, T; V), \quad \bar{\theta}(0) = \theta_0, \\ J_\varepsilon^\varphi \bar{w} \in W^{1,2}(0, T; H) \cap L^\infty(0, T; V), \quad \bar{w}(0) = w_0, \\ |\bar{\theta}'|_{L^2(0, T; H)}^2 \leq 1 + M_0, \quad \sup_{t \in [0, T]} |\nabla \bar{\theta}(t)|_H^2 \leq 1 + M_0, \\ |\bar{w}'|_{L^2(0, T; H)}^2 \leq 1 + M_0, \quad \sup_{t \in [0, T]} |\nabla J_\varepsilon^\varphi \bar{w}(t)|_H^2 \leq 1 + M_0 \end{array} \right. \right\}, \quad (3.4)$$

where M_0 is a positive constant dependent on the norm of initial data and a fixed number ν , more precisely, $M_0 := |\theta_0|_V^2 + 2(1 + \nu)|w_0|_V^2$. We see that $X_T^\varepsilon(M_0)$ is the convex and compact subset of the product space $C([0, T]; H) \times C_w([0, T]; H)$. We fix $\varepsilon > 0$ and take an element $(\bar{\theta}, \bar{w})$ in $X_T^\varepsilon(M_0)$, and substitute $\bar{\theta}, \bar{w}$ for θ, w in the right side of (3.2), namely

$$w'(t) + \partial I_{\bar{\theta}(t), N}(w'(t)) + \nu \partial \varphi_\varepsilon(w(t)) \ni f(\bar{\theta}(t), J_\varepsilon^\varphi \bar{w}(t)) \quad \text{in } H \text{ for a.e. } t \in (0, T). \quad (3.5)$$

We denote by $(P_\varepsilon)_{(\bar{\theta}, \bar{w})}$ the system (3.1), (3.5) for each $(\bar{\theta}, \bar{w}) \in X_T^\varepsilon(M_0)$ and (3.3). For every $(\bar{\theta}, \bar{w}) \in X_T^\varepsilon(M_0)$, (3.5) can be written in the form

$$w'(t) = \left(I + \partial I_{\bar{\theta}(t), N} \right)^{-1} \left(f(\bar{\theta}(t), J_\varepsilon^\varphi \bar{w}(t)) - \nu \partial \varphi_\varepsilon(w(t)) \right),$$

where I is the identity in H . Noting that $(I + \partial I_{\bar{\theta}, N})^{-1}$ and $\partial \varphi_\varepsilon$ are Lipschitz continuous in H and the equation (3.1) is linear, we can find a unique solution $\{\theta, w\}$ of $(P_\varepsilon)_{(\bar{\theta}, \bar{w})}$. Now, taking a number T_0 with $0 < T_0 \leq T$, (determined later), we define a mapping S from $X_T^\varepsilon(M_0)$ into $W^{1,2}(0, T; H) \cap L^\infty(0, T; V)$ by the formula

$$S(\bar{\theta}, \bar{w})(t) = \begin{cases} (\theta(t), w(t)) & \text{if } 0 \leq t \leq T_0, \\ (\theta(T_0), w(T_0)) & \text{if } T_0 \leq t \leq T, \end{cases}$$

where $\{\theta, w\}$ is the solution of $(P_\varepsilon)_{(\bar{\theta}, \bar{w})}$. As to this mapping S , we see the following lemma:

Lemma 3.1. *There exists T_0 with $0 < T_0 \leq T$ such that*

$$S(X_T^\varepsilon(M_0)) \subset X_T^\varepsilon(M_0).$$

Proof. Let $(\bar{\theta}, \bar{w}) \in X_T^\varepsilon(M_0)$ and $\{\theta, w\}$ be the solution of $(P_\varepsilon)_{(\bar{\theta}, \bar{w})}$. From the assumption (1), without loss of generality, we may assume that g and g' are Lipschitz continuous on $\mathbb{R}$. Multiplying (3.5) by $w' - g(\bar{\theta})$ in H and noting that

- $(w'(t), w'(t) - g(\bar{\theta}(t))) \geq \frac{3}{4}|w'(t)|_H^2 - |g(\bar{\theta}(t))|_H^2,$
- $(\xi(t), w'(t) - g(\bar{\theta}(t))) \geq 0, \quad \forall \xi(t) \in \partial I_{\bar{\theta}(t), N}(w'(t)),$
- $(\partial \varphi_\varepsilon(w(t)), w'(t) - g(\bar{\theta}(t))) = \frac{d}{dt} \varphi_\varepsilon(w(t)) - (\partial \varphi_\varepsilon(w(t)), g(\bar{\theta}(t))),$

$$\begin{aligned}
& \bullet \left(f(\bar{\theta}(t), J_\varepsilon^\varphi \bar{w}(t)), w'(t) - g(\bar{\theta}(t)) \right) \\
&= \left(f(\bar{\theta}(t), J_\varepsilon^\varphi \bar{w}(t)), w'(t) \right) - \left(f(\bar{\theta}(t), J_\varepsilon^\varphi \bar{w}(t)), g(\bar{\theta}(t)) \right) \\
&\leq \frac{1}{4} |w'(t)|_H^2 + |f(\bar{\theta}(t), J_\varepsilon^\varphi \bar{w}(t))|_H^2 + \frac{1}{2} |f(\bar{\theta}(t), J_\varepsilon^\varphi \bar{w}(t))|_H^2 + \frac{1}{2} |g(\bar{\theta}(t))|_H^2 \\
&\leq \frac{1}{4} |w'(t)|_H^2 + K_1 \left(|\bar{\theta}(t)|_H^2 + |J_\varepsilon^\varphi \bar{w}(t)|_H^2 + 1 \right),
\end{aligned}$$

we have

$$\frac{1}{2} |w'(t)|_H^2 + \nu \frac{d}{dt} \varphi_\varepsilon(w(t)) \leq K_1 \left(|\bar{\theta}(t)|_H^2 + |J_\varepsilon^\varphi \bar{w}(t)|_H^2 + 1 \right) + \nu (\partial \varphi_\varepsilon(w(t)), g(\bar{\theta}(t)))$$

for a.e. $t \in (0, T)$, where K_1 is a positive constant depending only on Lipschitz constants of f and g , and norms $|f|_\infty$ and $|g|_\infty$. Combining the above inequality with the following inequalities:

$$\begin{aligned}
(\partial \varphi_\varepsilon(w(t)), g(\bar{\theta}(t))) &= (-\Delta_0 J_\varepsilon^\varphi w(t), g(\bar{\theta}(t))) \\
&= (\nabla J_\varepsilon^\varphi w(t), \nabla g(\bar{\theta}(t))) \\
&\leq \frac{1}{2} |\nabla J_\varepsilon^\varphi w(t)|_H^2 + \frac{1}{2} |\nabla g(\bar{\theta}(t))|_H^2 \\
&\leq \varphi_\varepsilon(w(t)) + \frac{|g'|_\infty^2 (1 + M_0)}{2},
\end{aligned}$$

$$\begin{aligned}
|\bar{\theta}(t)|_H^2 + |J_\varepsilon^\varphi \bar{w}(t)|_H^2 &\leq |\bar{\theta}(t)|_H^2 + |\bar{w}(t)|_H^2 \\
&= \left| \bar{\theta}(0) + \int_0^t \bar{\theta}'(s) ds \right|_H^2 + \left| \bar{w}(0) + \int_0^t \bar{w}'(s) ds \right|_H^2 \\
&\leq 2|\theta_0|_H^2 + 2 \int_0^t |\bar{\theta}'(s)|_H^2 ds + 2|w_0|_H^2 + 2 \int_0^t |\bar{w}'(s)|_H^2 ds \\
&\leq 2|\theta_0|_H^2 + 2t |\bar{\theta}'|_{L^2(0,T;H)}^2 + 2|w_0|_H^2 + 2t |\bar{w}'|_{L^2(0,T;H)}^2 \\
&\leq 2 \left(|\theta_0|_H^2 + |w_0|_H^2 \right) + 4T(1 + M_0),
\end{aligned}$$

we see that

$$\frac{1}{2} |w'(t)|_H^2 + \nu \frac{d}{dt} \varphi_\varepsilon(w(t)) \leq \nu \varphi_\varepsilon(w(t)) + K_2 \quad (3.6)$$

for a.e. $t \in (0, T)$, where K_2 is a positive constant depending only on the Lipschitz constants of f, g , norms $|f|_\infty, |g|_\infty, |\theta_0|_H, |w_0|_H$ and constants ν and M_0 . Applying the Gronwall's lemma to (3.6), we obtain that

$$\nu \varphi_\varepsilon(w(t)) \leq (\nu \varphi_\varepsilon(w_0) + K_2 T) e^T \leq (\nu |\nabla w_0|_H^2 + K_2 T) e^T =: K_3, \quad \forall t \in [0, T].$$

Hence by (3.6), the following holds:

$$\frac{1}{2} |w'(t)|_H^2 + \nu \frac{d}{dt} \varphi_\varepsilon(w(t)) \leq K_2 + K_3 \quad \text{for a.e. } t \in (0, T).$$

Integrating the above in t over $[0, T']$ with $(0 < T' \leq T)$, we obtain that

$$\frac{1}{2} \int_0^{T'} |w'(t)|_H^2 dt + \nu \varphi_\varepsilon(w(T')) \leq \nu \varphi_\varepsilon(w_0) + T'(K_2 + K_3), \quad \forall T' \in (0, T]. \quad (3.7)$$

Next multiplying θ' by (3.1) in H , we get that

$$\frac{1}{2} |\theta'(t)|_H^2 + \frac{1}{2} \frac{d}{dt} |\nabla \theta(t)|_H^2 \leq |w'(t)|_H^2 + |h(t)|_H^2 \quad \text{for a.e. } t \in (0, T).$$

Integrating the above in t over $[0, \tilde{T}]$ ($0 < \tilde{T} \leq T'$) and using (3.7), we see that

$$\begin{aligned} \frac{1}{2} \int_0^{\tilde{T}} |\theta'(t)|_H^2 dt + \frac{1}{2} |\nabla \theta(\tilde{T})|_H^2 &\leq \frac{1}{2} |\nabla \theta_0|_H^2 + \int_0^{T'} |w'(t)|_H^2 dt + \int_0^{T'} |h(t)|_H^2 dt \\ &\leq K_0 + 2T'(K_2 + K_3) + \int_0^{T'} |h(t)|_H^2 dt, \end{aligned}$$

where

$$K_0 := \frac{1}{2} |\nabla \theta_0|_H^2 + \nu |\nabla w_0|_H^2 \leq \frac{M_0}{2}.$$

Taking T_0 with $T_0 \leq T$ such that

$$2T_0 \left(1 + \frac{1}{\nu}\right) (K_2 + K_3) + \int_0^{T_0} |h(t)|_H^2 dt \leq \frac{1}{2},$$

we have the conclusion. $\diamond$

Lemma 3.2. *For any $\varepsilon > 0$ and any $\theta_0, w_0 \in V, h \in L^2(0, T; H)$, there exists a solution $\{\theta, w\}$ of (P_ε) on the time-interval $[0, T_0]$ such that $\theta, J_\varepsilon^\varphi w \in W^{1,2}(0, T_0; H) \cap L^\infty(0, T_0; V)$ and $w \in W^{1,2}(0, T_0; H)$, where T_0 is a (small) positive number determined in Lemma 3.1.*

Proof. In order to get the conclusion of this lemma, we shall use the Schauder's fixed point theorem for the mapping S . First we show S is continuous in the topology of $C([0, T]; H) \times C_w([0, T]; H)$. We take a sequence $\{(\bar{\theta}_n, \bar{w}_n)\} \subset X_T^\varepsilon(M_0)$ converging to some element $(\bar{\theta}, \bar{w}) \in X_T^\varepsilon(M_0)$ in the topology of $C([0, T]; H) \times C_w([0, T]; H)$. Let $\{\theta_i, w_i\}$ be the solution of $(P_\varepsilon)_{(\bar{\theta}_i, \bar{w}_i)}$ each for $i \in \mathbb{N}$. Then the couple $\{\theta_i, w_i\}$ of functions satisfies

$$\theta'_i(t) + w'_i(t) - \Delta_0 \theta_i(t) = h(t) \quad \text{in } H \quad \text{for a.e. } t \in (0, T_0), \quad (3.8)$$

$$w'_i(t) + \xi_i(t) + \nu \partial \varphi_\varepsilon(w_i(t)) = f(\bar{\theta}_i(t), J_\varepsilon^\varphi \bar{w}_i(t)) \quad \text{in } H \quad \text{for a.e. } t \in (0, T_0), \quad (3.9)$$

$$\theta_i(0) = \theta_0 \quad \text{and} \quad w_i(0) = w_0 \quad \text{in } H, \quad (3.10)$$

where $\xi_i(t) \in \partial I_{\bar{\theta}_i(t), N}(w'_i(t))$ in H for a.e. $t \in (0, T_0)$. By (3.8) and (3.9), two solutions $\{\theta_i, w_i\}$, $i = m, n$, satisfy that

$$\theta'_m(t) - \theta'_n(t) + w'_m(t) - w'_n(t) - \Delta_0(\theta_m(t) - \theta_n(t)) = 0 \quad \text{in } H \quad (3.11)$$

and

$$\begin{aligned} w'_m(t) - w'_n(t) + \xi_m(t) - \xi_n(t) + \nu \partial \varphi_\varepsilon(w_m(t)) - \nu \partial \varphi_\varepsilon(w_n(t)) \\ = f(\bar{\theta}_m(t), J_\varepsilon^\varphi \bar{w}_m(t)) - f(\bar{\theta}_n(t), J_\varepsilon^\varphi \bar{w}_n(t)) \quad \text{in } H \end{aligned} \quad (3.12)$$

for a.e. $t \in (0, T_0)$. For the sake of simplicity, we denote $w_m - w_n$, $\theta_m - \theta_n$, $\xi_m - \xi_n$, $\bar{\theta}_m - \bar{\theta}_n$ and $\bar{w}_m - \bar{w}_n$ by $\hat{w}$, $\hat{\theta}$, $\hat{\xi}$, $\hat{\bar{\theta}}$ and $\hat{\bar{w}}$, respectively. Multiplying (3.12) by $w'_m - g(\bar{\theta}_m) - (w'_n - g(\bar{\theta}_n))$ in H and noting that

- $$\begin{aligned}
& \bullet \left(w'_m(t) - w'_n(t), w'_m(t) - g(\bar{\theta}_m(t)) - (w'_n(t) - g(\bar{\theta}_n(t))) \right) \\
&= |w'_m(t) - w'_n(t)|_H^2 - (w'_m(t) - w'_n(t), g(\bar{\theta}_m(t)) - g(\bar{\theta}_n(t))) \\
&\geq \frac{3}{4} |w'_m(t) - w'_n(t)|_H^2 - |g(\bar{\theta}_m(t)) - g(\bar{\theta}_n(t))|_H^2 \\
&\geq \frac{3}{4} |\hat{w}'(t)|_H^2 - L_g^2 |\hat{\theta}(t)|_H^2,
\end{aligned}$$
- $$\bullet \left(\xi_m(t) - \xi_n(t), w'_m(t) - g(\bar{\theta}_m(t)) - (w'_n(t) - g(\bar{\theta}_n(t))) \right) \geq 0,$$
- $$\begin{aligned}
& \bullet \left(\partial\varphi_\varepsilon(w_m(t)) - \partial\varphi_\varepsilon(w_n(t)), w'_m(t) - g(\bar{\theta}_m(t)) - (w'_n(t) - g(\bar{\theta}_n(t))) \right) \\
&= (\partial\varphi_\varepsilon(w_m(t) - w_n(t)), w'_m(t) - w'_n(t)) - (\partial\varphi_\varepsilon(w_m(t) - w_n(t)), g(\bar{\theta}_m(t)) - g(\bar{\theta}_n(t))) \\
&\geq \frac{d}{dt} \varphi_\varepsilon(w_m(t) - w_n(t)) - \frac{1}{2} |\partial\varphi_\varepsilon(w_m(t) - w_n(t))|_H^2 - \frac{1}{2} |g(\bar{\theta}_m(t)) - g(\bar{\theta}_n(t))|_H^2 \\
&\geq \frac{d}{dt} \varphi_\varepsilon(\hat{w}(t)) - \frac{1}{2\varepsilon^2} |\hat{w}(t)|_H^2 - \frac{L_g^2}{2} |\hat{\theta}(t)|_H^2,
\end{aligned}$$
- $$\begin{aligned}
& \bullet \left(f(\bar{\theta}_m(t), J_\varepsilon^\varphi \bar{w}_m(t)) - f(\bar{\theta}_n(t), J_\varepsilon^\varphi \bar{w}_n(t)), w'_m(t) - g(\bar{\theta}_m(t)) - (w'_n(t) - g(\bar{\theta}_n(t))) \right) \\
&= \left(f(\bar{\theta}_m(t), J_\varepsilon^\varphi \bar{w}_m(t)) - f(\bar{\theta}_n(t), J_\varepsilon^\varphi \bar{w}_n(t)), w'_m(t) - w'_n(t) \right) \\
&\quad - \left(f(\bar{\theta}_m(t), J_\varepsilon^\varphi \bar{w}_m(t)) - f(\bar{\theta}_n(t), J_\varepsilon^\varphi \bar{w}_n(t)), g(\bar{\theta}_m(t)) - g(\bar{\theta}_n(t)) \right) \\
&\leq \frac{1}{4} |w'_m(t) - w'_n(t)|_H^2 + |f(\bar{\theta}_m(t), J_\varepsilon^\varphi \bar{w}_m(t)) - f(\bar{\theta}_n(t), J_\varepsilon^\varphi \bar{w}_n(t))|_H^2 \\
&\quad + \frac{1}{2} |f(\bar{\theta}_m(t), J_\varepsilon^\varphi \bar{w}_m(t)) - f(\bar{\theta}_n(t), J_\varepsilon^\varphi \bar{w}_n(t))|_H^2 + \frac{1}{2} |g(\bar{\theta}_m(t)) - g(\bar{\theta}_n(t))|_H^2 \\
&\leq \frac{1}{4} |\hat{w}'(t)|_H^2 + 3L_f^2 \left(|\hat{\theta}(t)|_H^2 + |J_\varepsilon^\varphi \hat{w}(t)|_H^2 \right) + \frac{L_g^2}{2} |\hat{\theta}(t)|_H^2,
\end{aligned}$$

we have that

$$\frac{1}{2} |\hat{w}'(t)|_H^2 + \nu \frac{d}{dt} \varphi_\varepsilon(\hat{w}(t)) \leq K_4 \left(|\hat{\theta}(t)|_H^2 + |J_\varepsilon^\varphi \hat{w}(t)|_H^2 + |\hat{w}(t)|_H^2 \right) \quad (3.13)$$

for a.e. $t \in (0, T_0)$, where K_4 is a positive constant dependent on $\varepsilon > 0$ and L_f and L_g are the Lipschitz constants of f and g , respectively. By the simple calculation, we have

$$\frac{d}{dt} |\hat{w}(t)|_H^2 = 2(\hat{w}'(t), \hat{w}(t)) \leq |\hat{w}'(t)|_H^2 + |\hat{w}(t)|_H^2. \quad (3.14)$$

It follows from (3.13) and (3.14) that

$$\frac{d}{dt} \left\{ \frac{1}{2} |\hat{w}(t)|_H^2 + \nu \varphi_\varepsilon(\hat{w}(t)) \right\} \leq K_5 \left(|\hat{\theta}(t)|_H^2 + |J_\varepsilon^\varphi \hat{w}(t)|_H^2 \right) + K_6 \left(\frac{1}{2} |\hat{w}(t)|_H^2 + \nu \varphi_\varepsilon(\hat{w}(t)) \right)$$

for a.e. $t \in (0, T_0)$, where K_5 and K_6 are positive constants. Applying the Gronwall's lemma to the above inequality, we have

$$\frac{1}{2}|\hat{w}(t)|_H^2 + \nu\varphi_\varepsilon(\hat{w}(t)) \leq e^{K_6 T_0} \left\{ K_5 \int_0^{T_0} \left(|\hat{\theta}(t)|_H^2 + |J_\varepsilon^\varphi \hat{w}(t)|_H^2 \right) dt \right\}, \quad \forall t \in [0, T_0].$$

This implies that

$$|\hat{w}(t)|_H^2 \leq 2e^{K_6 T_0} K_5 \int_0^{T_0} \left(|\hat{\theta}(t)|_H^2 + |J_\varepsilon^\varphi \hat{w}(t)|_H^2 \right) dt, \quad \forall t \in [0, T_0].$$

Then (3.13) gives

$$\frac{1}{2}|\hat{w}'(t)|_H^2 + \nu \frac{d}{dt} \varphi_\varepsilon(\hat{w}(t)) \leq K_7 \left\{ |\hat{\theta}(t)|_H^2 + |J_\varepsilon^\varphi \hat{w}(t)|_H^2 + \int_0^{T_0} \left(|\hat{\theta}(t)|_H^2 + |J_\varepsilon^\varphi \hat{w}(t)|_H^2 \right) dt \right\}$$

for a.e. $t \in (0, T_0)$, where K_7 is a positive constant. Integrating the above in t over $[0, T_0]$, then

$$\frac{1}{2} \int_0^{T_0} |\hat{w}'(t)|_H^2 dt + \nu \varphi_\varepsilon(\hat{w}(T_0)) \leq K_7(1 + T_0) \int_0^{T_0} \left(|\hat{\theta}(t)|_H^2 + |J_\varepsilon^\varphi \hat{w}(t)|_H^2 \right) dt.$$

Taking $n, m \rightarrow +\infty$, we see that

$$K_7(1 + T_0) \int_0^{T_0} \left(|\hat{\theta}(t)|_H^2 + |J_\varepsilon^\varphi \hat{w}(t)|_H^2 \right) dt \rightarrow 0.$$

This shows that $\{w'_n\}$ is a Cauchy sequence in $L^2(0, T_0; H)$. Hence there exists a function $w \in W^{1,2}(0, T_0; H)$ such that

$$w_n \rightarrow w \text{ in } C([0, T_0]; H) \text{ and } w'_n \rightarrow w' \text{ in } L^2(0, T_0; H) \text{ as } n \rightarrow +\infty. \quad (3.15)$$

For every fixed $\varepsilon > 0$, from (3.15) and the following inequality

$$\begin{aligned} \int_0^{T_0} |\partial \varphi_\varepsilon(w_n(t))|_H^2 dt &= \int_0^{T_0} |\partial \varphi_\varepsilon(w_n(t)) - \partial \varphi_\varepsilon(0)|_H^2 dt \\ &\leq \frac{1}{\varepsilon^2} \int_0^{T_0} |w_n(t)|_H^2 dt \end{aligned}$$

we see that $\{\partial \varphi_\varepsilon(w_n)\}_{n=1}^\infty$ is bounded in $L^2(0, T_0; H)$. Putting

$$\xi_n(t) := -w'_n(t) - \nu \partial \varphi_\varepsilon(w_n(t)) + f(\bar{\theta}_n(t), J_\varepsilon^\varphi \bar{w}_n(t)) \quad \text{in } H \text{ for a.e. } t \in (0, T_0),$$

we see that $\xi_n(t) \in \partial I_{\bar{\theta}_n(t), N}(w'_n(t))$ in H for a.e. $t \in (0, T_0)$ and $\{\xi_n\}$ is bounded in $L^2(0, T_0; H)$. We may assume that for a subsequence $\{n_k\}$, $\{\xi_{n_k}\}$ converges weakly to ξ in $L^2(0, T_0; H)$ as $k \rightarrow +\infty$ and $\xi = -w' - \nu \partial \varphi_\varepsilon(w) + f(\bar{\theta}, J_\varepsilon^\varphi \bar{w})$, because $J_\varepsilon^\varphi \bar{w}_n \rightarrow J_\varepsilon^\varphi \bar{w}$ in $C([0, T_0]; H)$ as $n \rightarrow +\infty$. For simplicity, we use again n instead of n_k . Moreover, we can easily show that

$$\limsup_{n \rightarrow +\infty} \int_0^{T_0} (\xi_n(t), w'_n(t)) dt \leq \int_0^{T_0} (\xi(t), w'(t)) dt, \quad (3.16)$$

because we see that

$$\begin{aligned} \int_0^{T_0} (\xi_n(t), w'_n(t)) dt &= \int_0^{T_0} \left(-w'_n(t) - \nu \partial \varphi_\varepsilon(w_n(t)) + f(\bar{\theta}_n(t), J_\varepsilon^\varphi \bar{w}_n(t)), w'_n(t) \right) dt \\ &= - \int_0^{T_0} |w'_n(t)|_H^2 dt - \nu \varphi_\varepsilon(w_n(T_0)) + \nu \varphi_\varepsilon(w_0) \\ &\quad + \int_0^{T_0} \left(f(\bar{\theta}_n(t), J_\varepsilon^\varphi \bar{w}_n(t)), w'_n(t) \right) dt, \end{aligned}$$

then

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \int_0^{T_0} (\xi_n(t), w'_n(t)) dt &\leq - \liminf_{n \rightarrow +\infty} |w'_n|_{L^2(0, T_0; H)}^2 - \nu \liminf_{n \rightarrow +\infty} \varphi_\varepsilon(w_n(T_0)) + \nu \varphi_\varepsilon(w_0) \\ &\quad + \lim_{n \rightarrow +\infty} \int_0^{T_0} \left(f(\bar{\theta}_n(t), J_\varepsilon^\varphi \bar{w}_n(t)), w'_n(t) \right) dt \\ &\leq - |w'|_{L^2(0, T_0; H)}^2 - \nu \varphi_\varepsilon(w(T_0)) + \nu \varphi_\varepsilon(w_0) \\ &\quad + \int_0^{T_0} \left(f(\bar{\theta}(t), J_\varepsilon^\varphi \bar{w}(t)), w'(t) \right) dt \\ &= \int_0^{T_0} \left(-w'(t) - \nu \partial \varphi_\varepsilon(w(t)) + f(\bar{\theta}(t), J_\varepsilon^\varphi \bar{w}(t)), w'(t) \right) dt \\ &= \int_0^{T_0} (\xi(t), w'(t)) dt. \end{aligned}$$

Since $I_{\bar{\theta}_n, N}(\cdot) \rightarrow I_{\bar{\theta}, N}(\cdot)$ on H in the sense of Mosco (cf.[4,12,17]) as $n \rightarrow +\infty$, by the usual monotonicity technique with the Mosco convergence and (3.16), we have the inclusion $\xi(t) \in \partial I_{\bar{\theta}(t), N}(w'(t))$ in H for a.e. $t \in (0, T_0)$. Finally, we have the following:

$$w'(t) + \xi(t) + \nu \partial \varphi_\varepsilon(w(t)) = f(\bar{\theta}(t), J_\varepsilon^\varphi \bar{w}(t)), \quad \xi(t) \in \partial I_{\bar{\theta}(t), N}(w'(t)) \text{ in } H \quad (3.17)$$

for a.e. $t \in (0, T_0)$. Multiplying (3.11) by $\hat{\theta}'$ in H with the following calculations

- $(\theta'_m(t) - \theta'_n(t), \theta'_m(t) - \theta'_n(t)) = (\hat{\theta}'(t), \hat{\theta}'(t)) = |\hat{\theta}'(t)|_H^2,$
- $(w'_m(t) - w'_n(t), \theta'_m(t) - \theta'_n(t)) = (\hat{w}'(t), \hat{\theta}'(t)) \geq -\frac{1}{2} |\hat{\theta}'(t)|_H^2 - \frac{1}{2} |\hat{w}'(t)|_H^2,$
- $(-\Delta_0(\theta_m(t) - \theta_n(t)), \theta'_m(t) - \theta'_n(t)) = (-\Delta_0 \hat{\theta}(t), \hat{\theta}'(t)) = \frac{1}{2} \frac{d}{dt} |\nabla \hat{\theta}(t)|_H^2,$

we have that

$$|\hat{\theta}'(t)|_H^2 + \frac{d}{dt} |\nabla \hat{\theta}(t)|_H^2 \leq |\hat{w}'(t)|_H^2 \quad \text{for a.e. } t \in (0, T_0). \quad (3.18)$$

Then on account of (3.15), the above inequality implies that $\{\theta_n\}$ is a Cauchy sequence in $W^{1,2}(0, T_0; H) \cap L^\infty(0, T_0; V)$. Therefore we may assume that there exists a function $\theta \in W^{1,2}(0, T_0; H) \cap L^\infty(0, T_0; V)$ such that

$$\theta_n \rightarrow \theta \text{ in } C([0, T_0]; H) \text{ and } \theta'_n \rightarrow \theta' \text{ in } L^2(0, T_0; H) \text{ as } n \rightarrow +\infty. \quad (3.19)$$

It follows from (3.15) and (3.19) that

$$-\Delta_0 \theta_n \rightarrow -\Delta_0 \theta \text{ weakly in } L^2(0, T_0; H) \text{ as } n \rightarrow +\infty.$$

Hence the limit functions θ and w enjoy

$$\theta'(t) + w'(t) - \Delta_0 \theta(t) = h(t) \text{ in } H \text{ for a.e. } t \in (0, T_0). \quad (3.20)$$

Therefore from (3.17) and (3.20), the pair of limit functions $\{\theta, w\}$ is the solution to $(P_\varepsilon)_{(\bar{\theta}, \bar{w})}$ on $(0, T_0)$ with the regularities $\theta, J_\varepsilon^\varphi w \in W^{1,2}(0, T_0; H) \cap L^\infty(0, T_0; V)$ and $w \in W^{1,2}(0, T_0; H)$. Here, extend θ and w on $[0, T_0]$ onto the time interval $[0, T]$ by $\theta(T_0)$ and $w(T_0)$. Then $S(\bar{\theta}, \bar{w}) = (\theta, w)$ and S is continuous in the topology of $C([0, T]; H) \times C_w([0, T]; H)$. Hence we can apply the Schauder's fixed point theorem with respect to the mapping S in $X_T^\varepsilon(M_0)$ to find a fixed point (θ, w) of S which is a solution to (P_ε) on $[0, T_0]$. $\diamond$

Lemma 3.3. *For every fixed $\varepsilon > 0$, the solution $\{\theta, w\}$ of (P_ε) is unique on any time interval $[0, T']$ ($0 < T' \leq T$).*

Proof. Let $\{\theta_m, w_m\}$ and $\{\theta_n, w_n\}$ be the solutions to (P_ε) on $[0, T']$ ($0 < T' \leq T$) with the same initial data, namely, they satisfy the following equations:

$$\theta'_i(t) + w'_i(t) - \Delta_0 \theta_i(t) = h(t) \text{ in } H \text{ for a.e. } t \in (0, T'), \quad (3.21)$$

$$w'_i(t) + \xi_i(t) + \nu \partial \varphi_\varepsilon(w_i(t)) = f(\theta_i(t), J_\varepsilon^\varphi w_i(t)) \text{ in } H \text{ for a.e. } t \in (0, T'), \quad (3.22)$$

$$\theta_i(0) = \theta_0 \text{ and } w_i(0) = w_0 \text{ in } H, \quad (3.23)$$

where $\xi_i(t) \in \partial I_{\theta_i(t), N}(w'_i(t))$ in H for a.e. $t \in (0, T')$, $i = m, n$. By the above equations, two solutions $\{\theta_i, w_i\}$, $i = m, n$, satisfy that

$$\theta'_m(t) - \theta'_n(t) + w'_m(t) - w'_n(t) - \Delta_0(\theta_m(t) - \theta_n(t)) = 0 \text{ in } H \quad (3.24)$$

and

$$\begin{aligned} & w'_m(t) - w'_n(t) + \xi_m(t) - \xi_n(t) + \nu \partial \varphi_\varepsilon(w_m(t)) - \nu \partial \varphi_\varepsilon(w_n(t)) \\ & = f(\theta_m(t), J_\varepsilon^\varphi w_m(t)) - f(\theta_n(t), J_\varepsilon^\varphi w_n(t)) \text{ in } H \end{aligned} \quad (3.25)$$

for a.e. $t \in (0, T')$. Then by the same calculations to get (3.13) and (3.18) as in Lemma 3.2: $(3.24) \times \theta'$ and $(3.25) \times \{w'_m - g(\theta_m) - (w'_n - g(\theta_n))\}$,

- $(w'_m(t) - w'_n(t), w'_m(t) - g(\theta_m(t)) - (w'_n(t) - g(\theta_n(t))))$

$$= |w'_m(t) - w'_n(t)|_H^2 - (w'_m(t) - w'_n(t), g(\theta_m(t)) - g(\theta_n(t)))$$

$$\geq \frac{3}{4} |w'_m(t) - w'_n(t)|_H^2 - |g(\theta_m(t)) - g(\theta_n(t))|_H^2$$

$$\geq \frac{3}{4} |\hat{w}'(t)|_H^2 - L_g^2 |\hat{\theta}(t)|_H^2,$$
- $(\xi_m(t) - \xi_n(t), w'_m(t) - g(\theta_m(t)) - (w'_n(t) - g(\theta_n(t)))) \geq 0,$
- $\frac{1}{2} |\nabla J_\varepsilon^\varphi(\hat{w}(t))|_H^2 = \varphi(J_\varepsilon^\varphi \hat{w}(t)) \leq \varphi_\varepsilon(\hat{w}(t)),$

- $$\begin{aligned}
\frac{1}{2}|\nabla g(\theta_m(t)) - \nabla g(\theta_n(t))|_H^2 &= \frac{1}{2}|g'(\theta_m(t))\nabla\theta_m(t) - g'(\theta_n(t))\nabla\theta_n(t)|_H^2 \\
&\leq |g'(\theta_m(t))\nabla\theta_m(t) - g'(\theta_m(t))\nabla\theta_n(t)|_H^2 + |g'(\theta_m(t))\nabla\theta_n(t) - g'(\theta_n(t))\nabla\theta_n(t)|_H^2 \\
&\leq |g'(\theta_m(t))|_H^2|\nabla\theta_m(t) - \nabla\theta_n(t)|_H^2 + |\nabla\theta_n(t)|_H^2|g'(\theta_m(t)) - g'(\theta_n(t))|_H^2 \\
&\leq |g'(\theta_m(t))|_H^2|\nabla\hat{\theta}(t)|_H^2 + L_{g'}^2|\nabla\theta_n(t)|_H^2|\hat{\theta}(t)|_H^2 \\
&\leq K_8(|\nabla\hat{\theta}(t)|_H^2 + |\hat{\theta}(t)|_H^2),
\end{aligned}$$
- $$\begin{aligned}
&(\partial\varphi_\varepsilon(w_m(t)) - \partial\varphi_\varepsilon(w_n(t)), w'_m(t) - g(\theta_m(t)) - (w'_n(t) - g(\theta_n(t)))) \\
&= (\partial\varphi_\varepsilon(w_m(t) - w_n(t)), w'_m(t) - w'_n(t)) - (\partial\varphi_\varepsilon(w_m(t) - w_n(t)), g(\theta_m(t)) - g(\theta_n(t))) \\
&\geq \frac{d}{dt}\varphi_\varepsilon(w_m(t) - w_n(t)) - \frac{1}{2}|\nabla J_\varepsilon^\varphi(w_m(t) - w_n(t))|_H^2 - \frac{1}{2}|\nabla g(\theta_m(t)) - \nabla g(\theta_n(t))|_H^2 \\
&\geq \frac{d}{dt}\varphi_\varepsilon(\hat{w}(t)) - \varphi_\varepsilon(\hat{w}(t)) - K_8(|\nabla\hat{\theta}(t)|_H^2 + |\hat{\theta}(t)|_H^2),
\end{aligned}$$
- $$\begin{aligned}
&(f(\theta_m(t), J_\varepsilon^\varphi w_m(t)) - f(\theta_n(t), J_\varepsilon^\varphi w_n(t)), w'_m(t) - g(\theta_m(t)) - (w'_n(t) - g(\theta_n(t)))) \\
&= (f(\theta_m(t), J_\varepsilon^\varphi w_m(t)) - f(\theta_n(t), J_\varepsilon^\varphi w_n(t)), w'_m(t) - w'_n(t)) \\
&\quad - (f(\theta_m(t), J_\varepsilon^\varphi w_m(t)) - f(\theta_n(t), J_\varepsilon^\varphi w_n(t)), g(\theta_m(t)) - g(\theta_n(t))) \\
&\leq \frac{1}{4}|w'_m(t) - w'_n(t)|_H^2 + |f(\theta_m(t), J_\varepsilon^\varphi w_m(t)) - f(\theta_n(t), J_\varepsilon^\varphi w_n(t))|_H^2 \\
&\quad + \frac{1}{2}|f(\theta_m(t), J_\varepsilon^\varphi w_m(t)) - f(\theta_n(t), J_\varepsilon^\varphi w_n(t))|_H^2 + \frac{1}{2}|g(\theta_m(t)) - g(\theta_n(t))|_H^2 \\
&\leq \frac{1}{4}|\hat{w}'(t)|_H^2 + 3L_f^2(|\hat{\theta}(t)|_H^2 + |J_\varepsilon^\varphi \hat{w}(t)|_H^2) + \frac{L_g^2}{2}|\hat{\theta}(t)|_H^2 \\
&\leq \frac{1}{4}|\hat{w}'(t)|_H^2 + K_9(|\hat{w}(t)|_H^2 + |\hat{\theta}(t)|_H^2),
\end{aligned}$$

we deduce that

$$\frac{1}{2}|\hat{w}'(t)|_H^2 + \nu \frac{d}{dt}\varphi_\varepsilon(\hat{w}(t)) \leq \nu\varphi_\varepsilon(\hat{w}(t)) + K_{10}(|\hat{\theta}(t)|_H^2 + |\nabla\hat{\theta}(t)|_H^2 + |\hat{w}(t)|_H^2) \quad (3.26)$$

and

$$|\hat{\theta}'(t)|_H^2 + \frac{d}{dt}|\nabla\hat{\theta}(t)|_H^2 \leq |\hat{w}'(t)|_H^2 \quad (3.27)$$

for a.e. $t \in (0, T')$, where $\hat{\theta} = \theta_m - \theta_n$, $\hat{w} = w_m - w_n$ and K_8 , K_9 and K_{10} are positive constants independent of $\varepsilon > 0$ and $L_{g'}$ is a Lipschitz constant of g' . Computing (3.26) + (3.27) $\times \frac{1}{4}$, we have that

$$\frac{1}{4}|\hat{w}'(t)|_H^2 + \frac{1}{4}|\hat{\theta}'(t)|_H^2 + \frac{d}{dt}\left\{\nu\varphi_\varepsilon(\hat{w}(t)) + \frac{1}{4}|\nabla\hat{\theta}(t)|_H^2\right\} \leq \nu\varphi_\varepsilon(\hat{w}(t)) + K_{10}(|\hat{\theta}(t)|_H^2 + |\hat{w}(t)|_H^2)$$

for a.e. $t \in (0, T')$. Making use of (3.14) for both w and θ , we have the following

$$\frac{d}{dt} \left\{ \frac{1}{4} |\hat{\theta}(t)|_V^2 + \frac{1}{4} |\hat{w}(t)|_H^2 + \nu \varphi_\varepsilon(\hat{w}(t)) \right\} \leq K_{11} \left(\frac{1}{4} |\hat{\theta}(t)|_V^2 + \frac{1}{4} |\hat{w}(t)|_H^2 + \nu \varphi_\varepsilon(\hat{w}(t)) \right)$$

for a.e. $t \in (0, T')$, where K_{11} is a positive constant. Applying the Gronwall's lemma to the above inequality, the uniqueness follows at once. $\diamond$

Lemma 3.4. *For every fixed $\varepsilon > 0$, the solution $\{\theta, w\}$ of (P_ε) can be extended in time to the interval $[0, T]$.*

Proof. Let T^* be the supremum of all $T_0 \in [0, T]$ such that (P_ε) has a (unique) solution $\{\theta, w\}$ on $[0, T_0]$. By Lemma 3.3, $\{\theta, w\}$ is uniquely determined on the interval $[0, T^*)$. Let T_0 be any number such that $0 < T_0 < T^*$. The solution $\{\theta, w\}$ satisfies that:

$$\theta'(t) + w'(t) - \Delta_0 \theta(t) = h(t) \quad \text{in } H, \quad (3.28)$$

$$w'(t) + \xi(t) + \nu \partial \varphi_\varepsilon(w(t)) = f(\theta(t), J_\varepsilon^\varphi w(t)) \quad \text{in } H, \quad (3.29)$$

$$\theta(0) = \theta_0 \quad \text{and} \quad w(0) = w_0 \quad \text{in } H, \quad (3.30)$$

where $\xi(t) \in \partial I_{\theta(t), N}(w'(t))$ in H . Multiplying (3.29) by $w' - g(\theta)$ and (3.28) by θ' in H with the following calculations:

- $(w'(t), w'(t) - g(\theta(t))) \geq \frac{3}{4} |w'(t)|_H^2 - |g(\theta(t))|_H^2,$
- $(\xi(t), w'(t) - g(\theta(t))) \geq 0, \quad \forall \xi(t) \in \partial I_{\theta(t), N}(w'(t)),$
- $(\partial \varphi_\varepsilon(w(t)), w'(t) - g(\theta(t))) = \frac{d}{dt} \varphi_\varepsilon(w(t)) - (\partial \varphi_\varepsilon(w(t)), g(\theta(t))),$
- $(\partial \varphi_\varepsilon(w(t)), g(\theta(t))) \leq \varphi_\varepsilon(w(t)) + \frac{|g'|_\infty^2}{2} |\nabla \theta(t)|_H^2,$
- $(f(\theta(t), J_\varepsilon^\varphi w(t)), w'(t) - g(\theta(t)))$
 $= (f(\theta(t), J_\varepsilon^\varphi w(t)), w'(t)) - (f(\theta(t), J_\varepsilon^\varphi w(t)), g(\theta(t)))$
 $\leq \frac{1}{4} |w'(t)|_H^2 + |f(\theta(t), J_\varepsilon^\varphi w(t))|_H^2 + \frac{1}{2} |f(\theta(t), J_\varepsilon^\varphi w(t))|_H^2 + \frac{1}{2} |g(\theta(t))|_H^2$
 $\leq \frac{1}{4} |w'(t)|_H^2 + K_{12} (|\theta(t)|_H^2 + |J_\varepsilon^\varphi w(t)|_H^2 + 1),$

we have that

$$\frac{1}{2} |w'(t)|_H^2 + \nu \frac{d}{dt} \varphi_\varepsilon(w(t)) \leq K_{13} (|\theta(t)|_V^2 + |w(t)|_H^2 + \nu \varphi_\varepsilon(w(t)) + 1) \quad (3.31)$$

and

$$\frac{1}{2} |\theta'(t)|_H^2 + \frac{1}{2} \frac{d}{dt} |\nabla \theta(t)|_H^2 \leq |w'(t)|_H^2 + |h(t)|_H^2 \quad (3.32)$$

for a.e. $t \in (0, T_0)$, respectively, where K_{12} and K_{13} are positive constants. Computing (3.31) + (3.32) $\times \frac{1}{4}$, we have that

$$\begin{aligned} & \frac{1}{4}|w'(t)|_H^2 + \frac{1}{8}|\theta'(t)|_H^2 + \frac{d}{dt} \left\{ \frac{1}{8}|\nabla\theta(t)|_H^2 + \nu\varphi_\varepsilon(w(t)) \right\} \\ & \leq K_{14} \left(|\theta(t)|_V^2 + |w(t)|_H^2 + \nu\varphi_\varepsilon(w(t)) + |h(t)|_H^2 + 1 \right) \end{aligned} \quad (3.33)$$

for a.e. $t \in (0, T_0)$, where K_{14} is a positive constant. Using (3.14) for both θ and w with the suitable arrangement, we have the following:

$$\frac{d}{dt}E(t) \leq K_{15} \left(E(t) + |h(t)|_H^2 + 1 \right) \quad \text{for a.e. } t \in (0, T_0), \quad (3.34)$$

where

$$E(t) := \frac{1}{8}|\theta(t)|_V^2 + \frac{1}{4}|w(t)|_H^2 + \nu\varphi_\varepsilon(w(t)) \quad \text{for a.e. } t \in (0, T_0)$$

and K_{15} is a positive constant. Applying the Gronwall's lemma to (3.34), we obtain that

$$E(t) \leq \left(E(0) + K_{15} \int_0^{T_0} |h(t)|_H^2 dt + K_{15}T_0 \right) e^{K_{15}T_0}, \quad \forall t \in [0, T_0]. \quad (3.35)$$

Integrating (3.33) in t over $[0, T_0]$, then by (3.35) we have that

$$\begin{aligned} & \frac{1}{4} \int_0^{T_0} |w'(t)|_H^2 dt + \frac{1}{8} \int_0^{T_0} |\theta'(t)|_H^2 dt + \frac{1}{8} |\nabla\theta(T_0)|_H^2 + \nu\varphi_\varepsilon(w(T_0)) \\ & \leq K_{16} \left(\varphi_\varepsilon(w_0) + |\theta_0|_V^2 + \int_0^T |h(t)|_H^2 dt + 1 \right), \end{aligned} \quad (3.36)$$

where K_{16} is a positive constant. Noting that (3.36) is valid for any $T_0 \in [0, T^*)$ because the value of right hand side of (3.36) is independent of T_0 , and $|(J_\varepsilon^\varphi w)'|_{L^2(0, T_0; H)} \leq |w'|_{L^2(0, T_0; H)}$, we obtain that

$$\theta, J_\varepsilon^\varphi w \in W^{1,2}(0, T^*; H) \cap L^\infty(0, T^*; V) \quad \text{and} \quad w \in W^{1,2}(0, T^*; H).$$

Therefore the following limits exist:

$$\lim_{t \nearrow T^*} \theta(t) =: \theta^* \quad \text{and} \quad \lim_{t \nearrow T^*} w(t) =: w^* \quad \text{in } H.$$

Hence by the local existence result again we see that $\{\theta, w\}$ can be extended to the time beyond T^* . It contradicts the hypothesis of T^* . Finally, we obtain that $T = T^*$. $\diamond$

Proof of Theorem 3.1: It follows immediately from Lemmas 3.2-3.4. $\diamond$

4. Convergence of approximate solutions

In this section we discuss the convergence of approximate solutions. Let $\{\theta_\varepsilon, w_\varepsilon\}$ be the solution of (P_ε) obtained in Theorem 3.1, namely, it satisfies that

$$\theta'_\varepsilon(t) + w'_\varepsilon(t) - \Delta_0 \theta_\varepsilon(t) = h(t) \text{ in } H \text{ a.e. } t \in (0, T), \quad (4.1)$$

$$w'_\varepsilon(t) + \partial I_{\theta_\varepsilon(t), N}(w'_\varepsilon(t)) + \nu \partial \varphi_\varepsilon(w_\varepsilon(t)) \ni f(\theta_\varepsilon(t), J_\varepsilon^\varphi w_\varepsilon(t)) \text{ in } H \text{ a.e. } t \in (0, T), \quad (4.2)$$

$$\theta_\varepsilon(0) = \theta_0 \text{ in } H \text{ and } w_\varepsilon(0) = w_0 \text{ in } H. \quad (4.3)$$

We need some uniform estimates of approximate solutions $\{\theta_\varepsilon, w_\varepsilon\}$ to discuss the convergence.

Lemma 4.1. *Any approximate solution $\{\theta_\varepsilon, w_\varepsilon\}$ satisfies*

$$|\theta_\varepsilon|_\infty, |w'_\varepsilon|_\infty \leq M_1 + M_1 T,$$

where $M_1 = |\theta_0|_\infty + |h|_\infty + |g|_\infty + N$.

Proof. Define a function p on $[0, T]$ by

$$p(t) := M_1 + M_1 t.$$

Noting that $|w'_\varepsilon|_\infty \leq |g|_\infty + N$ holds for any $\varepsilon > 0$ by the definition of a solution of (P_ε) , we observe that

$$(\theta_\varepsilon - p)' - \Delta_0(\theta_\varepsilon - p) = h - w'_\varepsilon - M_1 \leq 0 \text{ in } Q. \quad (4.4)$$

Multiplying (4.4) by $[\theta_\varepsilon - p]^+$ in H , we have that

$$\frac{1}{2} \frac{d}{dt} |[\theta_\varepsilon(t) - p(t)]^+|_H^2 + |\nabla[\theta_\varepsilon(t) - p(t)]^+|_H^2 \leq 0 \text{ for a.e. } t \in (0, T).$$

Integrating the above inequality in t , we see that

$$|[\theta_\varepsilon(t) - p(t)]^+|_H^2 \leq |[\theta_0 - p(0)]^+|_H^2 = 0, \quad \forall t \in [0, T].$$

This implies that $\theta_\varepsilon \leq p \leq M_1 + M_1 T$. On the other hand,

$$(-\theta_\varepsilon - p)' - \Delta_0(-\theta_\varepsilon - p) = -h + w'_\varepsilon - M_1 \leq 0 \text{ in } Q. \quad (4.5)$$

Multiplying (4.5) by $[-\theta_\varepsilon - p]^+$ in H , we have that

$$\frac{1}{2} \frac{d}{dt} |[-\theta_\varepsilon(t) - p(t)]^+|_H^2 + |\nabla[-\theta_\varepsilon(t) - p(t)]^+|_H^2 \leq 0 \text{ for a.e. } t \in (0, T).$$

Integrating the above in t , we see that

$$|[-\theta_\varepsilon(t) - p(t)]^+|_H^2 \leq |[-\theta_0 - p(0)]^+|_H^2 = 0, \quad \forall t \in [0, T],$$

which gives $\theta_\varepsilon(t) \geq -p(t) \geq -M_1 - M_1 T$. Hence we complete the proof. $\diamond$

Lemma 4.2. *There exists a positive constant R_1 independent of $\varepsilon > 0$ such that*

$$|w'_\varepsilon|_{L^2(0,T;H)}^2 + |\theta'_\varepsilon|_{L^2(0,T;H)}^2 + |\Delta_0 \theta_\varepsilon|_{L^2(0,T;H)}^2 \leq R_1$$

and

$$\sup_{t \in [0,T]} |\nabla \theta_\varepsilon(t)|_H^2 + \sup_{t \in [0,T]} |\nabla J_\varepsilon^\varphi w_\varepsilon(t)|_H^2 \leq R_1.$$

Proof. Multiplying $w'_\varepsilon - g(\theta_\varepsilon)$ by (4.2) in H and noting that

$$(\xi_\varepsilon(t), w'_\varepsilon(t) - g(\theta_\varepsilon)) \geq 0, \quad \forall \xi_\varepsilon(t) \in \partial I_{\theta_\varepsilon(t), N}(w'_\varepsilon(t)) \text{ in } H \text{ for a.e. } t \in (0, T),$$

we have with the similar calculation to obtain (3.31)

$$\frac{1}{2} |w'_\varepsilon(t)|_H^2 + \nu \frac{d}{dt} \varphi_\varepsilon(w_\varepsilon(t)) \leq N_1 (|\theta_\varepsilon(t)|_H^2 + |w_\varepsilon(t)|_H^2 + 1) + \nu (\partial \varphi_\varepsilon(w_\varepsilon(t)), g(\theta_\varepsilon(t))) \quad (4.6)$$

for a.e. $t \in (0, T)$, where N_1 is a positive constant independent of $\varepsilon > 0$. Noting that $\partial \varphi_\varepsilon(w_\varepsilon(t)) = -\Delta_0 J_\varepsilon^\varphi w_\varepsilon(t)$ in H and the boundedness of g , we have

$$\begin{aligned} (\partial \varphi_\varepsilon(w_\varepsilon(t)), g(\theta_\varepsilon(t))) &= (\nabla J_\varepsilon^\varphi w_\varepsilon(t), \nabla g(\theta_\varepsilon(t))) \\ &\leq \frac{1}{2} |\nabla J_\varepsilon^\varphi w_\varepsilon(t)|_H^2 + \frac{1}{2} |\nabla g(\theta_\varepsilon(t))|_H^2 \\ &\leq \varphi_\varepsilon(w_\varepsilon(t)) + \frac{|g'|_\infty^2}{2} |\nabla \theta_\varepsilon(t)|_H^2 \end{aligned}$$

for a.e. $t \in (0, T)$. From the above inequality and (4.6) we observe that

$$\frac{1}{2} |w'_\varepsilon(t)|_H^2 + \nu \frac{d}{dt} \varphi_\varepsilon(w_\varepsilon(t)) \leq N_2 (|\theta_\varepsilon(t)|_H^2 + |\nabla \theta_\varepsilon(t)|_H^2 + |w_\varepsilon(t)|_H^2 + 1) + \nu \varphi_\varepsilon(w_\varepsilon(t)) \quad (4.7)$$

for a.e. $t \in (0, T)$, where N_2 is a positive constant independent of $\varepsilon > 0$. Next, multiplying (4.1) by θ'_ε and $-\Delta_0 \theta_\varepsilon$ in H , we have

$$\frac{1}{2} |\theta'_\varepsilon(t)|_H^2 + \frac{1}{2} \frac{d}{dt} |\nabla \theta_\varepsilon(t)|_H^2 \leq |w'_\varepsilon(t)|_H^2 + |h(t)|_H^2 \quad (4.8)$$

and

$$\frac{1}{2} \frac{d}{dt} |\nabla \theta_\varepsilon(t)|_H^2 + \frac{1}{2} |\Delta_0 \theta_\varepsilon(t)|_H^2 \leq |w'_\varepsilon(t)|_H^2 + |h(t)|_H^2 \quad (4.9)$$

for a.e. $t \in (0, T)$, respectively. Computing (4.7) + (4.8) $\times \frac{1}{8}$ + (4.9) $\times \frac{1}{8}$, we infer that

$$\begin{aligned} &\frac{1}{16} |\theta'_\varepsilon(t)|_H^2 + \frac{1}{4} |w'_\varepsilon(t)|_H^2 + \frac{1}{16} |\Delta_0 \theta_\varepsilon(t)|_H^2 + \frac{d}{dt} \left\{ \frac{1}{8} |\nabla \theta_\varepsilon(t)|_H^2 + \nu \varphi_\varepsilon(w_\varepsilon(t)) \right\} \\ &\leq N_3 (|\theta_\varepsilon(t)|_H^2 + |\nabla \theta_\varepsilon(t)|_H^2 + |w_\varepsilon(t)|_H^2 + 1) + \nu \varphi_\varepsilon(w_\varepsilon(t)) \end{aligned} \quad (4.10)$$

for a.e. $t \in (0, T)$, where N_3 is a positive constant independent of $\varepsilon > 0$. By (3.14) with some suitable arrangements in (4.10), we deduce that

$$\frac{d}{dt} E(t) \leq N_4 (E(t) + 1) \quad \text{for a.e. } t \in (0, T), \quad (4.11)$$

where N_4 is a positive constant independent of $\varepsilon > 0$ and

$$E(t) := \frac{1}{16}|\theta_\varepsilon(t)|_H^2 + \frac{1}{8}|\nabla\theta_\varepsilon(t)|_H^2 + \frac{1}{4}|w_\varepsilon(t)|_H^2 + \nu\varphi_\varepsilon(w_\varepsilon(t)), \quad \forall t \in [0, T]. \quad (4.12)$$

Applying the Gronwall's lemma to (4.11), we have the following:

$$E(t) \leq (E(0) + N_4 T)e^{N_4 T}, \quad \forall t \in [0, T]. \quad (4.13)$$

Combining (4.12) with (4.13), we can find a positive constant N_5 independent of $\varepsilon > 0$ such that

$$\sup_{t \in [0, T]} |\nabla\theta_\varepsilon(t)|_H^2 + \sup_{t \in [0, T]} |\nabla J_\varepsilon^\varphi w_\varepsilon(t)|_H^2 \leq N_5. \quad (4.14)$$

By (4.10) and (4.13), we can find a positive constant N_6 independent of $\varepsilon > 0$ such that

$$|w'_\varepsilon|_{L^2(0, T; H)}^2 + |\theta'_\varepsilon|_{L^2(0, T; H)}^2 + |\Delta_0\theta_\varepsilon|_{L^2(0, T; H)}^2 \leq N_6. \quad (4.15)$$

By (4.14) and (4.15), put $R_1 := N_5 + N_6$ to get the conclusion. $\diamond$

Lemma 4.3. *There exists a positive constant R_2 independent of $\varepsilon > 0$ such that*

$$|\Delta_0 g(\theta_\varepsilon(t))|_H^2 \leq R_2 \left(|\Delta_0\theta_\varepsilon(t)|_H^2 + 1 \right) \quad \text{for a.e. } t \in (0, T).$$

Proof. From the fact that

$$\Delta_0 g(\theta_\varepsilon(t)) = g'(\theta_\varepsilon(t))\Delta_0\theta_\varepsilon(t) + g''(\theta_\varepsilon(t))|\nabla\theta_\varepsilon(t)|^2,$$

it follows that

$$|\Delta_0 g(\theta_\varepsilon(t))|_H^2 \leq N_7 \left(|\nabla\theta_\varepsilon(t)|_{L^4(\Omega)}^4 + |\Delta_0\theta_\varepsilon(t)|_H^2 \right) \quad (4.16)$$

for a.e. $t \in (0, T)$, where $N_7 := 2 \max\{|g'|_\infty, |g''|_\infty\}$. By the Gagliardo-Nirenberg interpolation inequality (cf.[18]):

$$|\nabla z|_{L^4(\Omega)} \leq C|z|_{H^2(\Omega)}^{\frac{1}{2}} |z|_\infty^{\frac{1}{2}}, \quad \forall z \in L^\infty(\Omega) \cap H^2(\Omega)$$

and Lemma 4.1 and 4.2, the following inequalities hold:

$$\begin{aligned} |\nabla\theta_\varepsilon(t)|_{L^4(\Omega)}^4 &\leq C^4 |\theta_\varepsilon(t)|_{H^2(\Omega)}^2 |\theta_\varepsilon(t)|_\infty^2 \\ &\leq C^4 \left(|\theta_\varepsilon(t)|_H^2 + |\nabla\theta_\varepsilon(t)|_H^2 + |\Delta_0\theta_\varepsilon(t)|_H^2 \right) |\theta_\varepsilon(t)|_\infty^2 \\ &\leq N_8 \left(|\Delta_0\theta_\varepsilon(t)|_H^2 + 1 \right) \quad \text{for a.e. } t \in (0, T), \end{aligned}$$

where C and N_8 are positive constants independent of $\varepsilon > 0$. In virtue of (4.16) we can find the desired constant R_2 . $\diamond$

Remark 4.1. In the case that $\Omega \subset \mathbf{R}^2$, we see that the above constant R_2 is independent of both parameters ε and N . By the Gagliardo-Nirenberg interpolation inequality for 2-dimensional case

$$|z|_{L^4(\Omega)} \leq \tilde{C}|z|_{H^2}^{\frac{1}{2}} |z|_V^{\frac{1}{2}}, \quad \forall z \in V,$$

and Lemma 4.2 we can get the following inequalities without the L^∞ -estimate of θ_ε obtained in Lemma 4.1:

$$\begin{aligned}
|\Delta_0 g(\theta_\varepsilon(t))|_H^2 &\leq 2|g'(\theta_\varepsilon(t))\Delta_0 \theta_\varepsilon(t)|_H^2 + 2|g''(\theta_\varepsilon(t))|\nabla \theta_\varepsilon(t)|^2|_H^2 \\
&\leq N_7 \left(|\Delta_0 \theta_\varepsilon(t)|_H^2 + |\nabla \theta_\varepsilon(t)|_{L^4(\Omega)}^4 \right) \\
&\leq N_7 \left(|\Delta_0 \theta_\varepsilon(t)|_H^2 + \tilde{C}^4 |\nabla \theta_\varepsilon(t)|_H^2 |\nabla \theta_\varepsilon(t)|_V^2 \right) \\
&\leq \tilde{C}^4 N_7 R_1^4 + N_7 (1 + \tilde{C}^4 R_1^2) |\Delta_0 \theta_\varepsilon(t)|_H^2 \\
&\leq N_9 \left(|\Delta_0 \theta_\varepsilon(t)|_H^2 + 1 \right) \quad \text{for a.e. } t \in (0, T),
\end{aligned}$$

where $\tilde{C}$ and N_9 are positive constants independent of ε and N .

Lemma 4.4. *There exists a positive constant R_3 independent of $\varepsilon > 0$ such that*

$$|\nabla J_\varepsilon^\varphi w'_\varepsilon|_{L^2(0,T;H)}^2 + \sup_{t \in [0,T]} |\Delta_0 J_\varepsilon^\varphi w_\varepsilon(t)|_H^2 \leq R_3.$$

Proof. Multiplying (4.2) by $\partial \varphi_\varepsilon(w'_\varepsilon - g(\theta_\varepsilon))$ in H and note that

- $(w'_\varepsilon(t), \partial \varphi_\varepsilon(w'_\varepsilon(t) - g(\theta_\varepsilon(t)))) = (w'_\varepsilon(t), \partial \varphi_\varepsilon(w'_\varepsilon(t))) - (w'_\varepsilon(t), \partial \varphi_\varepsilon(g(\theta_\varepsilon(t))))$,
- $(w'_\varepsilon(t), \partial \varphi_\varepsilon(w'_\varepsilon(t))) = (w'_\varepsilon(t) - J_\varepsilon^\varphi w'_\varepsilon(t), \partial \varphi_\varepsilon(w'_\varepsilon(t))) + (J_\varepsilon^\varphi w'_\varepsilon(t), \partial \varphi_\varepsilon(w'_\varepsilon(t)))$
 $= \varepsilon |\partial \varphi_\varepsilon(w'_\varepsilon(t))|_H^2 + |\nabla J_\varepsilon^\varphi w'_\varepsilon(t)|_H^2$
 $\geq |\nabla J_\varepsilon^\varphi w'_\varepsilon(t)|_H^2$,
- $(w'_\varepsilon(t), \partial \varphi_\varepsilon(g(\theta_\varepsilon(t)))) = (\nabla J_\varepsilon^\varphi w'_\varepsilon(t), \nabla g(\theta_\varepsilon(t)))$
 $\leq \frac{1}{4} |\nabla J_\varepsilon^\varphi w'_\varepsilon(t)|_H^2 + |\nabla g(\theta_\varepsilon(t))|_H^2$
 $\leq \frac{1}{4} |\nabla J_\varepsilon^\varphi w'_\varepsilon(t)|_H^2 + |g'|_\infty^2 |\nabla \theta_\varepsilon(t)|_H^2$
 $\leq \frac{1}{4} |\nabla J_\varepsilon^\varphi w'_\varepsilon(t)|_H^2 + N_{10}$,
- $(\xi_\varepsilon(t), \partial \varphi_\varepsilon(w'_\varepsilon(t) - g(\theta_\varepsilon(t)))) \geq 0, \quad \forall \xi_\varepsilon(t) \in \partial I_{\theta_\varepsilon(t), N}(w'_\varepsilon(t))$,
- $(\partial \varphi_\varepsilon(w_\varepsilon(t)), \partial \varphi_\varepsilon(w'_\varepsilon(t) - g(\theta_\varepsilon(t))))$
 $= (\partial \varphi_\varepsilon(w_\varepsilon(t)), \partial \varphi_\varepsilon(w'_\varepsilon(t))) - (\partial \varphi_\varepsilon(w_\varepsilon(t)), \partial \varphi_\varepsilon(g(\theta_\varepsilon(t))))$
 $= \frac{1}{2} \frac{d}{dt} |\partial \varphi_\varepsilon(w_\varepsilon(t))|_H^2 - (\partial \varphi_\varepsilon(w_\varepsilon(t)), \partial \varphi_\varepsilon(g(\theta_\varepsilon(t))))$,
- $(\partial \varphi_\varepsilon(w_\varepsilon(t)), \partial \varphi_\varepsilon(g(\theta_\varepsilon(t))))$
 $\leq \frac{1}{2} |\partial \varphi_\varepsilon(w_\varepsilon(t))|_H^2 + \frac{1}{2} |\partial \varphi_\varepsilon(g(\theta_\varepsilon(t)))|_H^2$

$$\begin{aligned}
&\leq \frac{1}{2}|\partial\varphi_\varepsilon(w_\varepsilon(t))|_H^2 + \frac{1}{2}|\partial\varphi(g(\theta_\varepsilon(t)))|_H^2 \\
&= \frac{1}{2}|\partial\varphi_\varepsilon(w_\varepsilon(t))|_H^2 + \frac{1}{2}|\Delta_0 g(\theta_\varepsilon(t))|_H^2 \\
&\leq \frac{1}{2}|\partial\varphi_\varepsilon(w_\varepsilon(t))|_H^2 + \frac{R_2}{2}(|\Delta_0\theta_\varepsilon(t)|_H^2 + 1), \\
&\bullet (f(\theta_\varepsilon(t), J_\varepsilon^\varphi w_\varepsilon(t)), \partial\varphi_\varepsilon(w'_\varepsilon(t) - g(\theta_\varepsilon(t)))) \\
&\quad = (f(\theta_\varepsilon(t), J_\varepsilon^\varphi w_\varepsilon(t)), \partial\varphi_\varepsilon(w'_\varepsilon(t))) - (f(\theta_\varepsilon(t), J_\varepsilon^\varphi w_\varepsilon(t)), \partial\varphi_\varepsilon(g(\theta_\varepsilon(t)))), \\
&\bullet (f(\theta_\varepsilon(t), J_\varepsilon^\varphi w_\varepsilon(t)), \partial\varphi_\varepsilon(w'_\varepsilon(t))) \\
&\quad \leq \frac{1}{4}|\nabla J_\varepsilon^\varphi w'_\varepsilon(t)|_H^2 + |\nabla f(\theta_\varepsilon(t), J_\varepsilon^\varphi w_\varepsilon(t))|_H^2 \\
&\quad \leq \frac{1}{4}|\nabla J_\varepsilon^\varphi w'_\varepsilon(t)|_H^2 + N_{11}(|\nabla\theta_\varepsilon(t)|_H^2 + |\nabla J_\varepsilon^\varphi(w_\varepsilon(t))|_H^2) \\
&\quad \leq \frac{1}{4}|\nabla J_\varepsilon^\varphi w'_\varepsilon(t)|_H^2 + N_{11}(|\nabla\theta_\varepsilon(t)|_H^2 + \varphi_\varepsilon(w_\varepsilon(t))) \\
&\quad \leq \frac{1}{4}|\nabla J_\varepsilon^\varphi w'_\varepsilon(t)|_H^2 + N_{12}, \\
&\bullet (f(\theta_\varepsilon(t), J_\varepsilon^\varphi w_\varepsilon(t)), \partial\varphi_\varepsilon(g(\theta_\varepsilon(t)))) \\
&\quad \leq \frac{1}{2}|f(\theta_\varepsilon(t), J_\varepsilon^\varphi w_\varepsilon(t))|_H^2 + \frac{1}{2}|\partial\varphi_\varepsilon(g(\theta_\varepsilon(t)))|_H^2 \\
&\quad \leq N_{13}(|\theta_\varepsilon(t)|_H^2 + |J_\varepsilon^\varphi w_\varepsilon(t)|_H^2 + |\Delta_0\theta_\varepsilon(t)|_H^2 + 1) \\
&\quad \leq N_{14}(|\Delta_0\theta_\varepsilon(t)|_H^2 + 1),
\end{aligned}$$

we have with the help of Lemmas 4.2 and 4.3 that

$$\frac{1}{2}|\nabla J_\varepsilon^\varphi w'_\varepsilon(t)|_H^2 + \frac{\nu}{2} \frac{d}{dt} |\partial\varphi_\varepsilon(w_\varepsilon(t))|_H^2 \leq N_{15}(|\Delta_0\theta_\varepsilon(t)|_H^2 + |\partial\varphi_\varepsilon(w_\varepsilon(t))|_H^2 + 1) \quad (4.17)$$

for a.e. $t \in (0, T)$, where N_i ($i = 10, 11, 12, 13, 14, 15$) are positive constants independent of $\varepsilon > 0$. Moreover, multiply (4.1) by $-\Delta_0\theta_\varepsilon$ to have

$$\frac{1}{2} \frac{d}{dt} |\nabla\theta_\varepsilon(t)|_H^2 + \frac{1}{2} |\Delta_0\theta_\varepsilon(t)|_H^2 \leq |w'_\varepsilon(t)|_H^2 + |h(t)|_H^2 \quad (4.18)$$

for a.e. $t \in (0, T)$. Integrating (4.18) in t over $[0, T]$ and using Lemma 4.2, we have that

$$\int_0^T |\Delta_0\theta_\varepsilon(t)|_H^2 dt \leq |\theta_0|_V^2 + 2R_1^2 + 2 \int_0^T |h(t)|_H^2 dt =: N_{16}. \quad (4.19)$$

Applying the Gronwall's lemma to (4.17) and using (4.19), we obtain that

$$\sup_{t \in [0, T]} |\partial\varphi_\varepsilon(w_\varepsilon(t))|_H^2 \leq \left(\frac{2(N_{15}N_{16} + N_{15}T)}{\nu} + |w_0|_{H^2(\Omega)}^2 \right) \exp \left\{ \frac{2N_{15}T}{\nu} \right\} =: N_{17}. \quad (4.20)$$

Again by (4.17), we have with the above estimate (4.20) that

$$\int_0^T |\nabla J_\varepsilon^\varphi w'_\varepsilon(t)|_H^2 dt \leq \nu |w_0|_{H^2(\Omega)}^2 + 2N_{15}(N_{16} + N_{17}T + T) =: N_{18}.$$

Putting $R_3 := N_{17} + N_{18}$, we complete the proof. $\diamond$

5. Proof of Theorem 2.1

We are now in a position to give a proof of Theorem 2.1.

Proof of Theorem 2.1 Let $\{\theta_\varepsilon, w_\varepsilon\}$ be the solution of (P_ε) . By the uniform estimates in Lemmas 4.1-4.4 we can choose a sequence $\{\varepsilon_n\}$ tending to 0 as $n \rightarrow +\infty$ with functions $\theta \in W^{1,2}(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega))$ and $w \in W^{1,2}(0, T; V) \cap L^\infty(0, T; H^2(\Omega))$ such that

$$\theta_n := \theta_{\varepsilon_n} \rightarrow \theta \text{ in } C([0, T]; H) \text{ as } n \rightarrow +\infty \quad (5.1)$$

and

$$J_{\varepsilon_n}^\varphi w_n := J_{\varepsilon_n}^\varphi w_{\varepsilon_n} \rightarrow w \text{ in } C([0, T]; V) \text{ as } n \rightarrow +\infty. \quad (5.2)$$

It follows from Lemma 4.4 and (1.6) that

$$|w_n(t) - J_{\varepsilon_n}^\varphi w_n(t)|_H \leq \varepsilon_n |\partial \varphi_{\varepsilon_n}(w_n(t))|_H \leq \varepsilon_n R_3, \quad \forall t \in [0, T].$$

This implies that

$$w_n \rightarrow w \text{ in } C([0, T]; H) \text{ as } n \rightarrow +\infty. \quad (5.3)$$

Now, let $n \rightarrow +\infty$. Then it follows that

$$\partial \varphi_{\varepsilon_n}(w_n) \rightarrow \partial \varphi(w) = -\Delta_0 w \text{ weakly in } L^2(0, T; H) \text{ as } n \rightarrow +\infty.$$

Here, we observe that the function

$$\xi_n(t) := -w'_n(t) - \nu \partial \varphi_{\varepsilon_n}(w_n(t)) + f(\theta_n(t), J_{\varepsilon_n}^\varphi w_n(t)) \text{ in } H \text{ for a.e. } t \in (0, T).$$

satisfies that $\xi_n(t) \in \partial I_{\theta_n(t), N}(w'_n(t))$ in H for a.e. $t \in (0, T)$ and $\{\xi_n\}$ is bounded in $L^2(0, T; H)$. Therefore there exist a subsequence $\{n_k\}$ of n and a function $\xi \in L^2(0, T; H)$ such that

$$\xi_{n_k} \rightarrow \xi \text{ weakly in } L^2(0, T; H) \text{ as } k \rightarrow +\infty.$$

Clearly $\xi = -w' + \nu \Delta_0 w + f(\theta, w)$ in $L^2(0, T; H)$. Now let us show the inclusion $\xi(t) \in \partial I_{\theta(t), N}(w'(t))$ in H for a.e. $t \in (0, T)$. In order to do so, we employ the usual monotonicity technique. Since $I_{\theta_n, N}(\cdot) \rightarrow I_{\theta, N}(\cdot)$ on H in the sense of Mosco, we have only to show

$$\limsup_{k \rightarrow +\infty} \int_0^T (\xi_{n_k}(t), w'_{n_k}(t)) dt \leq \int_0^T (\xi(t), w'(t)) dt. \quad (5.4)$$

This can be proved as follows:

$$\begin{aligned}
& \limsup_{k \rightarrow +\infty} \int_0^T (\xi_{n_k}(t), w'_{n_k}(t)) dt \\
& \leq -\liminf_{k \rightarrow +\infty} |w'_{n_k}|_{L^2(0,T;H)}^2 + \nu \lim_{k \rightarrow +\infty} \varphi_{\varepsilon_{n_k}}(w_0) - \nu \liminf_{k \rightarrow +\infty} \varphi_{\varepsilon_{n_k}}(w_{n_k}(T)) \\
& \quad + \lim_{k \rightarrow +\infty} \int_0^T (f(\theta_{n_k}(t), J_{\varepsilon_{n_k}}^\varphi w_{n_k}(t)), w'_{n_k}(t)) dt \\
& \leq -|w'|_{L^2(0,T;H)}^2 + \frac{\nu}{2} |\nabla w_0|_H^2 - \frac{\nu}{2} |\nabla w(T)|_H^2 + \int_0^T (f(\theta(t), w(t)), w'(t)) dt \\
& = \int_0^T (\xi(t), w'(t)) dt.
\end{aligned}$$

Therefore we obtain $\xi(t) \in \partial I_{\theta(t),N}(w'(t))$ in H for a.e. $t \in (0, T)$ and

$$w'(t) + \xi(t) - \nu \Delta_0 w(t) = f(\theta(t), w(t)) \quad \text{in } H \quad \text{for a.e. } t \in (0, T).$$

By using regularity $w \in L^\infty(0, T; H^2(\Omega))$ and Lemma 4.1, we obtain $w \in W^{1,\infty}(Q)$. Since the equation (4.1) is linear, it is easy to discuss the convergence for (4.1) and obtain that

$$\theta'(t) + w'(t) - \Delta_0 \theta(t) = h(t) \quad \text{in } H \quad \text{for a.e. } t \in (0, T).$$

This completes the proof of Theorem 2.1. $\diamond$

Remark 5.1. In the case of $\Omega \subset \mathbf{R}^2$, from Remark 4.1 it is possible to take $N \rightarrow +\infty$ in $I_{\theta(t),N}(\cdot)$ and hence the solutions obtained in the above satisfy the following system $(P)'$:

$$(P)' \quad \begin{cases} \theta_t + w_t - \Delta_0 \theta = h(t, x) & \text{in } Q, \\ w_t + \partial I_\theta(w_t) - \nu \Delta_0 w \ni f(\theta, w) & \text{in } Q, \\ \theta(0, \cdot) = \theta_0, \quad w(0, \cdot) = w_0 & \text{in } \Omega, \end{cases}$$

where

$$I_\theta(w_t) := \begin{cases} +\infty & \text{if } w_t < g(\theta), \\ 0 & \text{if } w_t \geq g(\theta); \end{cases}$$

and $\partial I_\theta(w_t)$ is its subdifferential:

$$\partial I_\theta(w_t) := \begin{cases} \emptyset & \text{if } w_t < g(\theta), \\ (-\infty, 0] & \text{if } w_t = g(\theta), \\ \{0\} & \text{if } w_t \geq g(\theta). \end{cases}$$

$(P)'$ is the same system that we have already discussed in [3]. We note here that if N is a sufficiently large positive number, $I_{\theta,N}(\cdot)$ is close to $I_\theta(\cdot)$. Therefore we can consider that the problem (P) is one of approximate problems of $(P)'$, because the indicator function $I_{\theta,N}(\cdot)$ can be regarded as an approximation of $I_\theta(\cdot)$.

Remark 5.2. In the case of $\Omega \subset \mathbf{R}^3$, we are very interested in the situation when the fixed large number N goes to $+\infty$. In that case we can not obtain the uniform estimate in Lemma 4.1. This unables us to discuss the convergence of approximate solutions.

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