

Marginal worth vectors for TU games

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1. Introduction

The Shapley value ([Shapley 1953]) is a solution for cooperative TU games. Marginal worth in cooperative games is a kind of difference of the characteristic function. The Shapley value could be expressed as a convex combination of marginal worth vectors. In this article taking marginal worths of several players into consideration we give relations between the Shapley values for subgames. We characterize the Shapley value as a sum of differences of several players, which satisfies the efficiency. See [Grabisch/Marichal/Roubens 2000] with respect to extensions of this kind of discussions under Boolean function etc. See, for example, [Lange/Grabisch 2011] with respect to topics around the Shapley value.

A TU game is a pair (N, v) , where $N = \{1, \dots, n\}$ is a finite set of players and v is a real-valued function defined on the subsets of N with $v(\emptyset) = 0$. v is called the *characteristic function* of the TU game. A subset of N is called a *coalition*. For a TU game (N, v) and a coalition $S \subseteq N$, a *subgame* (S, v) is a TU game where S is the player set and v is a restriction to the subsets of S . For any set Z , $|Z|$ denotes the cardinality of Z . For a coalition S , $\mathbf{R}^S$ is the $|S|$ -dimensional product space $\mathbf{R}^{|S|}$ with coordinates indexed by players in S . An element x of $\mathbf{R}^N$ is a *payoff* vector. The i -th component of $x \in \mathbf{R}^S$ is denoted by x_i . For $S \subseteq N$ and $x \in \mathbf{R}^N$, we define $x(S) = \sum_{i \in S} x_i$ (if $S \neq \emptyset$) and $= 0$ (if $S = \emptyset$). For a TU game (N, v) let

$$X(N, v) \equiv \{x \in \mathbf{R}^N : x(N) = v(N)\}. \quad (1)$$

An element of $X(N, v)$ is called a *pre-imputation*. The *Shapley value* ([Shapley 1953]) for a TU game (N, v) is a payoff vector $\varphi(N, v) \in \mathbf{R}^N$ where

$$\varphi_i(N, v) = \sum_{S: i \in S} \frac{(|S| - 1)!(n - |S|)!}{n!} [v(S) - v(S \setminus \{i\})]. \quad (2)$$

It is well-known that the Shapley value is a pre-imputation, that is, $\varphi(N, v) \in X(N, v)$.

2. Marginal worth vector and the Shapley value

Let $a : 2^N \rightarrow \mathbf{R}$ be a function. For each $R \subseteq N$, let

$$c_R(a) = \sum_{T \subseteq R} (-1)^{|R \setminus T|} a(T), \quad (3)$$

and for $R \subseteq N$, define a characteristic function v_R by

$$v_R(S) = \begin{cases} 1, & \text{if } R \subseteq S \subseteq N; \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

The next lemma was used in [Shapley 1953] to define and characterize the Shapley value.

Lemma 2.1. ([Shapley 1953]) For any TU game (N, v) and all $S \subseteq N$,

$$v(S) = \sum_{\emptyset \neq R \subseteq N} c_R(v) v_R(S). \quad (5)$$

Let $a : 2^N \rightarrow \mathbf{R}$ be a function. For $i \in N$ and $S \subseteq N$, define

$$\Delta_i a(S) \equiv a(S) - a(S \setminus \{i\}). \quad (6)$$

When $a = v$ for some TU game (N, v) , $\Delta_i v(S)$ is called the *marginal worth* of $i \in N$ at S . The next lemmas are obtained straightforwardly from Lemma 2.1, and given (at least implicitly) in [Shapley 1953].

Lemma 2.2. For a TU game (N, v) and $S \subseteq N$,

$$\Delta_i v(S) = \sum_{i \in R \subseteq S} c_R(v). \quad (7)$$

Proof:

$$\begin{aligned} \Delta_i v(S) &= \sum_{\emptyset \neq R \subseteq N} c_R(v) [v_R(S) - v_R(S \setminus \{i\})] \\ &= \sum_{i \in R \subseteq S} c_R(v). \end{aligned}$$

□

Lemma 2.3. For a TU game (N, v) and for each $i \in N$ and $S \subseteq N$ such that $i \in S$,

$$\varphi_i(N, v) = \Delta_i v(S) + \sum_{i \in R, R \not\subseteq S} \frac{c_R(v)}{|R|} + \sum_{i \in R \subseteq S} \left(\frac{1}{|R|} - 1 \right) c_R(v). \quad (8)$$

Proof: By Lemma 2.2 and from [Shapley 1953]

$$\varphi_i(N, v) = \sum_{i \in R \subseteq N} \frac{1}{|R|} c_R(v). \quad \square$$

Let $a : 2^N \rightarrow \mathbf{R}$ be a function. For $|S| \geq 2$, let $S = \{i_1, \dots, i_{|S|}\}$. Define inductively with (6), the marginal worth of several players, by

$$\Delta_{i_1, \dots, i_{|S|}} a(S) = \Delta_{i_1, \dots, i_{|S|-1}} a(S) - \Delta_{i_1, \dots, i_{|S|-1}} a(S \setminus \{i_{|S|}\}).$$

Lemma 2.4. ([Grabisch/Marichal/Roubens 2000]) Let $a : 2^N \rightarrow \mathbf{R}$ be a function. For $S \subseteq T \subseteq N$,

$$\Delta_{i_1, \dots, i_{|S|}} a(T) = \sum_{S \subseteq R \subseteq T} c_R(a), \quad (9)$$

where $S = \{i_1, \dots, i_{|S|}\}$.

Proof: Let $s = |S|$. By induction on s . When $s = 1$, that is, $S = \{i\}$. We define $b : 2^N \rightarrow \mathbf{R}$ by $b(R) = a(R) - a(\emptyset)$ for all $R \in 2^N$. Since $b(\emptyset) = 0$, (N, b) could be regarded as a TU game. It is easy to see $c_R(a) = c_R(b)$ for all $R \in 2^N$. Applying Lemma 2.2, we have

$$\sum_{i \in R \subseteq T} c_R(a) = \sum_{i \in R \subseteq T} c_R(b) = \Delta_i b(T) = \Delta_i a(T).$$

So it holds for $s = 1$. Suppose $s \geq 2$.

$$\begin{aligned} \Delta_{i_1, i_2, \dots, i_s} a(T) &= \Delta_{i_1, \dots, i_{s-1}} a(T) - \Delta_{i_1, \dots, i_{s-1}} a(T \setminus \{i_s\}) \\ &= \sum_{\{i_1, \dots, i_{s-1}\} \subseteq R \subseteq T} c_R(a) - \sum_{\{i_1, \dots, i_{s-1}\} \subseteq R \subseteq T \setminus \{i_s\}} c_R(a) \\ &= \sum_{\{i_1, \dots, i_{s-1}, i_s\} \subseteq R \subseteq T} c_R(a). \quad \square \end{aligned}$$

From Lemma 2.4 we could express $\Delta_{i_1, \dots, i_{|S|}} a(T)$ as $\Delta_S a(T)$ without confusion. When $a = v$ for some TU game (N, v) , $\Delta_S v(T)$ is called the *marginal worth* of $S \subseteq N$ at T . The next lemma is from [Grabisch/Marichal/Roubens 2000].

Lemma 2.5. For a TU game (N, v) and $i \in N$,

$$\varphi_i(N, v) = v(\{i\}) + \sum_{R: i \in R \subseteq N, |R| \geq 2} \frac{\Delta_R v(R)}{|R|}. \quad (10)$$

Proof: This is by Lemmas 2.3 and 2.4 and by $v(\emptyset) = 0$. $\square$

Example 1. Let $n = 3$. By Lemmas 2.4 and 2.5,

$$\begin{aligned} \varphi(N, v) &= x^{123} + \left(\frac{\Delta_{1,2} v(12)}{2} + \frac{\Delta_{1,3} v(13)}{2} + \frac{\Delta_{1,2,3} v(123)}{3} \right) (1, 0, 0) \\ &\quad + \left(\frac{\Delta_{1,2} v(12)}{2} + \frac{\Delta_{2,3} v(23)}{2} + \frac{\Delta_{1,2,3} v(123)}{3} \right) (0, 1, 0) \\ &\quad + \left(\frac{\Delta_{1,3} v(13)}{2} + \frac{\Delta_{2,3} v(23)}{2} + 2 \frac{\Delta_{1,2,3} v(123)}{3} \right) (0, 0, 1) \\ &= x^{123} + \frac{1}{2} \Delta_{1,2} v(12) (1, 1, 0) + \frac{1}{2} \Delta_{1,3} v(13) (1, 0, 1) + \frac{1}{2} \Delta_{2,3} v(23) (0, 1, 1) \\ &\quad + \frac{1}{3} \Delta_{1,2,3} v(123) (1, 1, 1) \end{aligned} \quad (11)$$

where $x^{123} = (v(1), v(2), v(3))$.

For $S \subseteq N$, let $\varphi(S, v)$ be the Shapley value for the subgame (S, v) . The next theorem gives a relation between the Shapley values for subgames.

Theorem 2.6. For every $i \in N$,

$$\begin{aligned} \varphi_i(N, v) &= \sum_{i \in S \subseteq N} (-1)^{n+1-|S|} \varphi_i(S, v) + \frac{\Delta_N v(N)}{n} \\ \text{or} \\ \sum_{i \in S \subseteq N} (-1)^{n+1-|S|} \varphi_i(S, v) + \frac{\Delta_N v(N)}{n} &= 0. \end{aligned} \quad (12)$$

Proof: By induction on n . Assume that it holds for all (R, v) such that $|R| \leq n-1$. Then

$$\sum_{i \in S \subseteq R} (-1)^{|R|+1-|S|} \varphi_i(S, v) + \frac{\Delta_R v(R)}{|R|} = 0.$$

From this and Lemma 2.5,

$$\varphi_i(N, v) = \varphi_i(\{i\}, v) - \sum_{i \in R \subseteq N, |R| \geq 2} \sum_{i \in S \subseteq R} (-1)^{|R|+1-|S|} \varphi_i(S, v) + \frac{\Delta_N v(N)}{n}.$$

Here

$$\sum_{i \in R \subseteq N, |R| \geq 2} \sum_{i \in S \subseteq R} (-1)^{|R|+1-|S|} \varphi_i(S, v) = \sum_{i \in S \subseteq N} (-1)^{1-|S|} \varphi_i(S, v) \sum_{S \subseteq R \subseteq N, |R| \geq 2} (-1)^{|R|}.$$

Furthermore we see

$$\sum_{S \subseteq R \subset N, |R| \geq 2} (-1)^{|R|} = \begin{cases} (-1)^{n+1}, & \text{if } |S| \geq 2; \\ 1 - (-1)^n, & \text{if } S = \{i\}. \end{cases}$$

So,

$$\begin{aligned} \varphi_i(N, v) &= \varphi_i(\{i\}, v) - [1 - (-1)^n] \varphi_i(\{i\}, v) \\ &\quad - \sum_{i \in S \subset N, |S| \geq 2} (-1)^{1-|S|} \varphi_i(S, v) (-1)^{n+1} + \frac{\Delta_N v(N)}{n} \\ &= \sum_{i \in S \subset N} (-1)^{n+1-|S|} \varphi_i(S, v) + \frac{\Delta_N v(N)}{n}. \quad \square \end{aligned}$$

3. Marginal worth vector and the potential function

In this section we see a relation between the potential functions for subgames. Proposition 3.1 and Theorem 2.6 could be compared.

Define the *potential* function (See [Hart/Mas-Colell 1989]) for a TU game (N, v) by

$$P(N, v) = \sum_{\emptyset \neq R \subset N} \frac{c_R(v)}{|R|}. \quad (13)$$

In the same way the potential function $P(S, v)$ for each subgame (S, v) , $\emptyset \neq S \subset N$, is defined.

Proposition 3.1. For a TU game (N, v) ,

$$\sum_{\emptyset \neq S \subset N} (-1)^{n+1-|S|} P(S, v) + \frac{\Delta_N v(N)}{n} = 0. \quad (14)$$

Proof: By induction on n . Assume that it holds for all (R, v) such that $|R| \leq n-1$. Then

$$\sum_{\emptyset \neq S \subset R} (-1)^{|R|+1-|S|} P(S, v) + \frac{\Delta_R v(R)}{|R|} = 0.$$

From this and (13),

$$\begin{aligned} P(N, v) &= - \sum_{\emptyset \neq R \subset N} \sum_{\emptyset \neq S \subset R} (-1)^{|R|+1-|S|} P(S, v) + \frac{\Delta_N v(N)}{n} \\ &= - \sum_{\emptyset \neq S \subset N} P(S, v) \sum_{S \subseteq R \subset N} (-1)^{|R|+1-|S|} + \frac{\Delta_N v(N)}{n}. \end{aligned}$$

Here, letting $s = |S|$,

$$\begin{aligned} \sum_{S \subseteq R \subseteq N} (-1)^{|R|+1-|S|} &= \sum_{r=s}^{n-1} (-1)^{r+1-s} \binom{n-s}{r-s} \\ &= -\binom{n-s}{0} + \binom{n-s}{1} - \dots + (-1)^{n-s} \binom{n-s}{n-s} \\ &= -(-1)^{n-s+1}. \end{aligned}$$

So,

$$P(N, v) = \sum_{\emptyset \neq S \subseteq N} P(S, v) (-1)^{n+1-|S|} + \frac{\Delta_N v(N)}{n}. \quad \square$$

4. Multilinear extension and the Shapley value

In this section we state a relation between the marginal worth and derivative of multilinear extension ([Owen 1972]).

For (N, v) , let

$$f(x_1, \dots, x_n) = \sum_{S \subseteq N} \{\prod_{k \in S} x_k \prod_{k \notin S} (1 - x_k)\} v(S), \quad (15)$$

where

$$0 \leq x_i \leq 1, \quad \forall i = 1, \dots, n.$$

Proposition 4.1 Let $R = \{i_1, \dots, i_r\}$ where $r = |R|$.

$$\frac{\partial^r f}{\partial x_{i_1} \dots \partial x_{i_r}} = \sum_{R \subseteq S} \{\prod_{k \in S \setminus R} x_k \prod_{k \notin S} (1 - x_k)\} (\Delta_R v)(S). \quad (16)$$

Proof: By induction on $r \geq 1$. Let $r = 1$ and $R = \{i\}$. Then

$$\begin{aligned} \frac{\partial f}{\partial x_i} &= \sum_{i \in S} \{\prod_{k \in S \setminus \{i\}} x_k \prod_{k \notin S} (1 - x_k)\} v(S) - \sum_{i \notin S} \{\prod_{k \in S} x_k \prod_{k \notin S, k \neq i} (1 - x_k)\} v(S) \\ &= \sum_{i \in S} \{\prod_{k \in S \setminus \{i\}} x_k \prod_{k \notin S} (1 - x_k)\} v(S) - \sum_{i \in S} \{\prod_{k \in S \setminus \{i\}} x_k \prod_{k \notin S} (1 - x_k)\} v(S \setminus \{i\}) \\ &= \sum_{i \in S} \{\prod_{k \in S \setminus \{i\}} x_k \prod_{k \notin S} (1 - x_k)\} (\Delta_i v)(S). \end{aligned}$$

Next assume it holds for $r - 1$. Then

$$\begin{aligned}
\frac{\partial^r f}{\partial x_{i_1} \cdots \partial x_{i_r}} &= \frac{\partial f}{\partial x_{i_r}} \left(\sum_{R \setminus \{i_r\} \subseteq S} \{ \Pi_{k \in S \setminus (R \setminus \{i_r\})} x_k \Pi_{k \notin S} (1 - x_k) \} (\Delta_{i_1, \dots, i_{r-1}} v)(S) \right) \\
&= \frac{\partial f}{\partial x_{i_r}} \left(\sum_{R \subseteq S} \{ \Pi_{k \in S \setminus (R \setminus \{i_r\})} x_k \Pi_{k \notin S} (1 - x_k) \} (\Delta_{i_1, \dots, i_{r-1}} v)(S) \right. \\
&\quad \left. + \sum_{R \setminus \{i_r\} \subseteq S, i_r \notin S} \{ \Pi_{k \in S \setminus R} x_k \Pi_{k \notin S} (1 - x_k) \} (\Delta_{i_1, \dots, i_{r-1}} v)(S) \right) \\
&= \sum_{R \subseteq S} \{ \Pi_{k \in S \setminus R} x_k \Pi_{k \notin S} (1 - x_k) \} (\Delta_{i_1, \dots, i_{r-1}} v)(S) \\
&\quad - \sum_{R \setminus \{i_r\} \subseteq S, i_r \notin S} \{ \Pi_{k \in S \setminus R} x_k \Pi_{k \notin S, k \neq i_r} (1 - x_k) \} (\Delta_{i_1, \dots, i_{r-1}} v)(S) \\
&= \sum_{R \subseteq S} \{ \Pi_{k \in S \setminus R} x_k \Pi_{k \notin S} (1 - x_k) \} (\Delta_{i_1, \dots, i_{r-1}} v)(S) \\
&\quad - \sum_{R \subseteq S} \{ \Pi_{k \in S \setminus R} x_k \Pi_{k \notin S} (1 - x_k) \} (\Delta_{i_1, \dots, i_{r-1}} v)(S \setminus \{i_r\}) \\
&= \sum_{R \subseteq S} \{ \Pi_{k \in S \setminus R} x_k \Pi_{k \notin S} (1 - x_k) \} (\Delta_{i_1, \dots, i_r} v)(S). \quad \square
\end{aligned}$$

5. Another expression of the Shapley value

In this section we characterize the Shapley value as a sum of marginal worths of coalitions at N which is a pre-imputation, that is, it satisfies the efficiency.

Theorem 5.1. For any TU game (N, v) , define a payoff vector $y(N, v)$ by $y_i(N, v) = \sum_{i \in S \subseteq N} \alpha_{|S|} (\Delta_S v)(N)$ for all $i \in N$. Then $y(N, v)$ is a pre-imputation for all TU games (N, v) if and only if it is the Shapley value.

Before proving this theorem, we need lemmas.

Lemma 5.2. For $\emptyset \neq S \subseteq N$,

$$(\Delta_S v)(N) = \sum_{R \subseteq S} (-1)^{|R|} v(N \setminus R). \quad (17)$$

Proof: Let $S = \{i_1, \dots, i_s\}$.

$$\begin{aligned} (\Delta_{i_1, \dots, i_s} v)(N) &= (\Delta_{i_1, \dots, i_{s-1}} v)(N) - (\Delta_{i_1, \dots, i_{s-1}} v)(N \setminus \{i_s\}) \\ &= \sum_{R \subseteq S \setminus \{i_s\}} (-1)^{|R|} v(N \setminus R) - \sum_{R \subseteq S \setminus \{i_s\}} (-1)^{|R|} v(N \setminus \{i_s\} \setminus R) \\ &= \sum_{R \subseteq S \setminus \{i_s\}} (-1)^{|R|} v(N \setminus R) - \sum_{R \subseteq S, i_s \in R} (-1)^{|R|-1} v(N \setminus R) \\ &= \sum_{R \subseteq S, i_s \notin R} (-1)^{|R|} v(N \setminus R) + \sum_{R \subseteq S, i_s \in R} (-1)^{|R|} v(N \setminus R) \\ &= \sum_{R \subseteq S} (-1)^{|R|} v(N \setminus R). \quad \square \end{aligned}$$

Lemma 5.3. For a TU game (N, v) and $i \in N$,

$$\varphi_i(N, v) = \sum_{i \in S \subseteq N} (-1)^{|S|+1} \frac{\Delta_S v(N)}{|S|}. \quad (18)$$

Proof: From Lemma 5.2,

$$\begin{aligned} \sum_{i \in S \subseteq N} (-1)^{|S|+1} \frac{\Delta_S v(N)}{|S|} &= \sum_{i \in S \subseteq N} \frac{(-1)^{|S|+1}}{|S|} \sum_{R \subseteq S} (-1)^{|R|} v(N \setminus R) \\ &= \sum_{i \in S \subseteq N} \frac{(-1)^{|S|+1}}{|S|} \left[\sum_{i \in R \subseteq S} (-1)^{|R|} v(N \setminus R) + \sum_{i \notin R \subseteq S} (-1)^{|R|} v(N \setminus R) \right] \\ &= \sum_{i \in S \subseteq N} \sum_{i \in R \subseteq S} \frac{(-1)^{|R|+|S|+1}}{|S|} v(N \setminus R) + \sum_{i \in S \subseteq N} \sum_{i \notin R \subseteq S} \frac{(-1)^{|R|+|S|+1}}{|S|} v(N \setminus R) \end{aligned}$$

$$\begin{aligned} \text{the 1st term} &= \sum_{i \in R \subseteq N} (-1)^{|R|} v(N \setminus R) \sum_{S: R \subseteq S} \frac{(-1)^{|S|+1}}{|S|} \\ &= \sum_{i \in R \subseteq N} (-1)^{|R|} v(N \setminus R) \sum_{s=|R|}^n \frac{(-1)^{s+1}}{s} \binom{n-|R|}{s-|R|} \\ &= \sum_{i \in R \subseteq N} (-1)^{|R|} v(N \setminus R) \frac{(-1)^{|R|+1}}{n \binom{n-1}{|R|-1}} = \sum_{i \in R \subseteq N} \frac{v(N \setminus R)}{n \binom{n-1}{|R|-1}}. \end{aligned}$$

$$\begin{aligned}
\text{the 2nd term} &= \sum_{i \notin R \subseteq N} (-1)^{|R|} v(N \setminus R) \sum_{S: i \in S, R \subseteq S} \frac{(-1)^{|S|+1}}{|S|} \\
&= \sum_{i \notin R \subseteq N} (-1)^{|R|} v(N \setminus R) \sum_{S: R \cup \{i\} \subseteq S} \frac{(-1)^{|S|+1}}{|S|} \\
&= \sum_{i \notin R \subseteq N} (-1)^{|R|} v(N \setminus R) \sum_{s=|R|+1}^n \frac{(-1)^{s+1}}{s} \binom{n-|R|-1}{s-|R|-1} \\
&= \sum_{i \notin R \subseteq N} (-1)^{|R|} v(N \setminus R) \frac{(-1)^{|R|}}{n \binom{n-1}{|R|}} = \sum_{i \notin R \subseteq N} \frac{v(N \setminus R)}{n \binom{n-1}{|R|}}.
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{i \in S \subseteq N} (-1)^{|S|+1} \frac{\Delta_S v(N)}{|S|} &= - \sum_{i \in R \subseteq N} \frac{v(N \setminus R)}{n \binom{n-1}{|R|-1}} + \sum_{i \notin R \subseteq N} \frac{v(N \setminus R)}{n \binom{n-1}{|R|}} \\
&= - \sum_{i \notin S} \frac{v(S)}{n \binom{n-1}{n-|S|-1}} + \sum_{i \in S \subseteq N} \frac{v(S)}{n \binom{n-1}{n-|S|}} \\
&= - \sum_{i \in S \subseteq N} \frac{v(S \setminus \{i\})}{n \binom{n-1}{n-|S|}} + \sum_{i \in S \subseteq N} \frac{v(S)}{n \binom{n-1}{n-|S|}},
\end{aligned}$$

which is the Shapley value, $\varphi_i(N, v)$. $\square$

Lemma 5.4. Let $y_i^N = \sum_{i \in S \subseteq N} \alpha_{|S|} (\Delta_S v)(N)$. If for all TU games (N, v) ,

$$\sum_{i \in N} y_i^N = v(N) \quad (19)$$

then

$$\alpha_s = \frac{(-1)^{s+1}}{s}, \text{ for all } s = 1, \dots, n. \quad (20)$$

Proof:

$$\begin{aligned}
v(N) &= \sum_{i \in N} y_i^N = \sum_{i \in N} \sum_{i \in S \subseteq N} \alpha_{|S|} (\Delta_S v)(N) \\
&= \sum_{S \neq \emptyset} \sum_{i \in S} \alpha_{|S|} (\Delta_S v)(N) \\
&= \sum_{S \neq \emptyset} |S| \alpha_{|S|} (\Delta_S v)(N) \\
&= \sum_{S \neq \emptyset} |S| \alpha_{|S|} \sum_{R \subseteq S} (-1)^{|R|} v(N \setminus R) \\
&= \sum_{R \subseteq N} (-1)^{|R|} v(N \setminus R) \sum_{S \neq \emptyset, R \subseteq S} |S| \alpha_{|S|}.
\end{aligned}$$

Since this is an identity, we compare coefficients $v(N)$ of both sides.

$$1 = \sum_{S \neq \emptyset} |S| \alpha_{|S|} = \sum_{s=1}^n \binom{n}{s} s \alpha_s. \quad (21)$$

For $R \neq \emptyset, N$ we compare coefficients of both sides.

$$0 = \sum_{R \subseteq S} |S| \alpha_{|S|} = \sum_{s=|R|}^n \binom{n-|R|}{s-|R|} s \alpha_s. \quad (22)$$

For $R = N$, by comparing coefficients of both sides, we have $n\alpha_n v(\emptyset) = 0$, which holds since $v(\emptyset) = 0$. In the equation (22) on $\{\alpha_s\}$, let R be such that $|R| = n - k$ for $k = 1, 2, 3$. Then

$$\begin{aligned} (n-1)\alpha_{n-1} + n\alpha_n &= 0, \text{ for } k = 1, \\ (n-2)\alpha_{n-2} + 2(n-1)\alpha_{n-1} + n\alpha_n &= 0, \text{ for } k = 2, \\ (n-3)\alpha_{n-3} + 3(n-2)\alpha_{n-2} + 3(n-1)\alpha_{n-1} + n\alpha_n &= 0, \text{ for } k = 3. \end{aligned}$$

From these,

$$n\alpha_n = -(n-1)\alpha_{n-1} = (n-2)\alpha_{n-2} = -(n-3)\alpha_{n-3}.$$

Assume for $\ell \leq k-1$,

$$(n-\ell)\alpha_{n-\ell} = n\alpha_n, \text{ if } \ell \text{ is even, and } = -n\alpha_n, \text{ if } \ell \text{ is odd.}$$

The equation (22) for $|R| = n - k$ becomes to

$$\begin{aligned} 0 &= \binom{k}{0} (n-k)\alpha_{n-k} + \begin{cases} n\alpha_n(-\binom{k}{1} + \binom{k}{2} - \cdots + \binom{k}{k}), & \text{if } k \text{ is even;} \\ n\alpha_n(+\binom{k}{1} - \binom{k}{2} + \cdots - \binom{k}{k}), & \text{if } k \text{ is odd;} \end{cases} \\ &= \binom{k}{0} (n-k)\alpha_{n-k} + \begin{cases} n\alpha_n(-\binom{k}{0}), & \text{if } k \text{ is even;} \\ n\alpha_n(+\binom{k}{0}), & \text{if } k \text{ is odd.} \end{cases} \end{aligned}$$

Inductive consideration implies

$$(n-k)\alpha_{n-k} = \begin{cases} n\alpha_n, & \text{if } k \text{ is even;} \\ -n\alpha_n, & \text{if } k \text{ is odd;} \end{cases} \quad (23)$$

From this, the equation (21) becomes to

$$n\alpha_n(-\binom{n}{1} + \binom{n}{2} - \cdots + \binom{n}{n}) = 1,$$

if n is even. If n is odd, the equation (21) becomes to

$$n\alpha_n\left(\binom{n}{1} - \binom{n}{2} + \cdots + \binom{n}{n}\right) = 1.$$

So $n\alpha_n = (-1)^{n+1}$. From this and (23), we have the lemma. $\square$

From Lemmas 5.2-5.4, we have the theorem. $\square$

References

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