

The rigidity of universal solvable Lie algebras of Iwasawa subalgebras.

By Kiyoshi Ito

Suzuka College of Technology, Shiroko, 510-02, Japan

Introduction.

An n -dimensional Lie algebra $\mathfrak{g}$ is called rigid if all n -dimensional Lie algebras near $\mathfrak{g}$ are isomorphic to $\mathfrak{g}$. Nijenhuis & Richardson proved that $\mathfrak{g}$ is rigid if its Chevalley cohomology group $H^2(\mathfrak{g}, \mathfrak{g}) = 0$ ([N-R]). As necessary conditions, Carles proved that rigid Lie algebras over the complex numbers $\mathbb{C}$ have to be algebraic and satisfy several other conditions ([Ca] Prop 4.1).

In this paper we treat the rigidity of solvable Lie algebras $\mathfrak{g}$ over $\mathbb{C}$. In § 1 we show that rigid $\mathfrak{g}$ are isomorphic to the universal solvable Lie algebras $u(\mathfrak{n}) = \mathfrak{n} \ltimes T$. Here $\mathfrak{n}$ are the nilradicals of $\mathfrak{g}$, and T are maximal abelian subalgebras of $\text{Der}(\mathfrak{n})$ composed of semi-simple elements. Although the rigidity of low dimensional solvable Lie algebras were obtained([C-D], [Be]), in general dimensions there are not many known examples of rigid solvable Lie algebras except for Borel subalgebras of semi-simple Lie algebras([L-L]). From § 2 we try to compute $H^2(u(\mathfrak{n}^{\mathbb{C}}), u(\mathfrak{n}^{\mathbb{C}}))$ and check their rigidity when $\mathfrak{n}$ are nilpotent parts of Iwasawa decompositions of semi-simple Lie algebras over $\mathbb{R}$. When the semi-simple Lie algebras are normal real forms, $u = u(\mathfrak{n}^{\mathbb{C}})$ are isomorphic to Borel subalgebras, and $H^2(u, u) = 0$ by [L-L]. In this paper we determine $H^2(u, u)$ and the rigidity of u when $\mathfrak{n}$ are nilpotent parts of real simple Lie algebras of real rank 1. Those results are shown in Proposition 3.2 and 3.4. For other several cases, $H^2(u, u)$ are given in Remark 3.3 without their proofs. We can conclude that not full but partial generalization of [L-L] is

possible.

§ 1 Universal solvable Lie algebras

Let $\mathfrak{n}$ be a nilpotent Lie algebra over $\mathbb{C}$, and let $\text{Der}(\mathfrak{n})$ be its all derivations. Choosing a maximal abelian subalgebra T of $\text{Der}(\mathfrak{n})$ consisting of semi-simple elements (another T' and T are conjugate), we define the universal solvable Lie algebra $\mathfrak{u} = \mathfrak{u}(\mathfrak{n})$ by the semi-direct product $\mathfrak{n} \rtimes T$, where T acts on $\mathfrak{n}$ naturally.

Proposition 1.1. *Let $\mathfrak{g}$ be a non-nilpotent solvable Lie algebra. If $\mathfrak{g}$ is rigid, then $\mathfrak{g}$ is isomorphic to $\mathfrak{u}(\mathfrak{n})$, where $\mathfrak{n}$ is the largest nilpotent ideal of $\mathfrak{g}$.*

proof). If $\mathfrak{g}$ is rigid, then $\mathfrak{g}$ is algebraic and splittable ([Ca] Proposition 4.1). Then $\mathfrak{g}$ is isomorphic to a subalgebra of $\mathfrak{u}(\mathfrak{n})$ ([Ma] Theorem 7), that is to say, there exists a non-zero subspace T_1 of T and $\mathfrak{g} \simeq \mathfrak{n} \rtimes T_1$. If $T_1 \neq T$, then we can choose a continuous family of subspaces $T_t \subset T$ ($0 \leq t \leq 1$) such that $\mathfrak{n} \rtimes T_t$ ($0 \leq t < 1$) is not isomorphic to $\mathfrak{g}$, because there are only finite number of subspaces $T' \subset T$ such that $\mathfrak{n} \rtimes T' \simeq \mathfrak{n} \rtimes T_1$ ([Ma] Theorem 7). Hence we get a non-trivial deformation of $\mathfrak{g}$. This is a contradiction, whence $T_1 = T$ and $\mathfrak{g} \simeq \mathfrak{u}(\mathfrak{n})$.

Remark. In Proposition 1.1 we can remove the word "non-nilpotent" because of Colloraire 4.4 (ii) of [Ca]. In [C-D] and [Be] rigid solvable Lie algebras were determined completely when their dimensions are not more than 8. There is a conjecture "No nilpotent Lie algebra is rigid".

§ 2 Reduction of the computation of $H^2(\mathfrak{u}, \mathfrak{u})$.

Our purpose is to find rigid solvable Lie algebras, and we compute 2-cohomology groups of $\mathfrak{u} = \mathfrak{u}(\mathfrak{n})$ for several types of $\mathfrak{n}$. Let $\mathfrak{u} = \mathfrak{n} \rtimes T$ be a universal Lie algebra for a given $\mathfrak{n}$. Since $\mathfrak{u}$ is the semi-direct product of $\mathfrak{n}$ and T , and the action of T on $\mathfrak{n}$ is

semi-simple, we can use the following:

Lemma 2.1 (Hochschild-Serre [H-S]).

$$H^i(u, u) = \sum_{j+k=i} H^j(T, \mathbb{C}) \otimes H^k(\mathfrak{n}, u)^T \quad (i \geq 0),$$

where $H^i(u, u)$ and $H^k(\mathfrak{n}, u)^T$ are the cohomology groups with respect to the adjoint representations, and $H^j(T, \mathbb{C})$ are the ones with respect to the trivial representations.

Remark. By this lemma, as necessary conditions for $H^2(u, u) = 0$, we get the followings :

$$H^1(u, u) = H^0(u, u) = 0 \text{ when } \dim T \geq 2, \text{ and}$$

$$\dim H^1(u, u) = \dim H^0(u, u) \text{ when } \dim T = 1,$$

because $H^j(T, \mathbb{C}) \simeq \Lambda^j(T) = 0$ if and only if $j > \dim T$. Here the latter condition is equivalent to the condition $\dim \text{Der}(\mathfrak{g}) = \dim \mathfrak{g}$ since $H^1(\mathfrak{g}, \mathfrak{g}) = \text{Der}(\mathfrak{g}) / \text{ad}(\mathfrak{g})$, $H^0(\mathfrak{g}, \mathfrak{g}) = \mathfrak{z}$, and $\mathfrak{g} / \mathfrak{z} \simeq \text{ad}(\mathfrak{g})$. It is remarkable that the necessary conditions for the rigidity in [Ca] Proposition 4.1(i) are no more than the necessary conditions for $H^2(u, u) = 0$.

In order to compute $H^k(\mathfrak{n}, u)^T$, we use the weight space decomposition of $\mathfrak{n}$ with respect to T : $\mathfrak{n} = \bigoplus_{\lambda \in W} \mathfrak{n}_\lambda$, ($W \subset T^*$). Then we have the following:

Lemma 2.2. For a positive integer i , assume $\lambda_1 + \lambda_2 + \dots + \lambda_i \neq 0$ (not necessarily distinct $\lambda_1, \dots, \lambda_i \in W$), then we get $C^i(\mathfrak{n}, u)^T = C^i(\mathfrak{n}, \mathfrak{n})^T$. For $i \geq 2$, we have $H^i(\mathfrak{n}, u)^T = H^i(\mathfrak{n}, \mathfrak{n})^T$, and for $i = 1$ we have the followings:

$$H^1(\mathfrak{n}, u)^T = \{ D \in \text{Der}(\mathfrak{n}) \mid D \text{ is nilpotent and } D \mathfrak{n}_\lambda \subset \mathfrak{n}_\lambda \ (\lambda \in W) \} \text{ and } H^0(\mathfrak{n}, u)^T = 0.$$

proof). Let us write $c \in C^i(\mathfrak{n}, u)^T$ as $c = \varphi + \psi$ ($\varphi \in C^i(\mathfrak{n}, \mathfrak{n})$, $\psi \in C^i(\mathfrak{n}, T)$), we get $\psi = 0$ by writing down the conditions

$$\{Y \cdot (\varphi + \psi)\}(X_{\lambda_1}, \dots, X_{\lambda_i}) = 0 \quad (Y \in T, X_{\lambda_1} \in \mathfrak{n}_{\lambda_1}, \dots, X_{\lambda_i} \in \mathfrak{n}_{\lambda_i}),$$

and using the assumption. Therefore $C^i(\mathfrak{n}, u)^T = C^i(\mathfrak{n}, \mathfrak{n})^T$, and for $i \geq 2$ we have $H^i(\mathfrak{n}, u)^T = Z^i(\mathfrak{n}, \mathfrak{n})^T / dC^{i-1}(\mathfrak{n}, \mathfrak{n})^T = H^i(\mathfrak{n}, \mathfrak{n})^T$.

When $i = 1$, $\mathfrak{n}^T = 0$ since $0 \notin W$. Then $C^0(\mathfrak{n}, \mathfrak{u})^T = \mathfrak{u}^T = \mathfrak{n}^T \rtimes T = T$, hence $H^1(\mathfrak{n}, \mathfrak{u})^T = Z^1(\mathfrak{n}, \mathfrak{n})^T / dT$. Here $Z^1(\mathfrak{n}, \mathfrak{n})^T = \{ D \in \text{Der}(\mathfrak{n}) \mid DY = YD \ (Y \in T) \}$ and $dT = T$. Since $\text{Der}(\mathfrak{n})$ is algebraic, for $D \in Z^1(\mathfrak{n}, \mathfrak{n})^T$ D_S and $D_N \in Z^1(\mathfrak{n}, \mathfrak{n})^T$ where $D = D_S + D_N$ is the Jordan decomposition of D . Here $D_S \in T$ by the definition of T . Hence we get

$$\begin{aligned} H^1(\mathfrak{n}, \mathfrak{u})^T &= \{ D \in \text{Der}(\mathfrak{n}) \mid D \text{ is nilpotent and } DY = YD \ (Y \in T) \} \\ &= \{ D \in \text{Der}(\mathfrak{n}) \mid D \text{ is nilpotent and } D \mathfrak{n}_\lambda \subset \mathfrak{n}_\lambda \ (\lambda \in W) \}. \end{aligned}$$

Next $H^0(\mathfrak{n}, \mathfrak{u})^T = \{ c \in C^0(\mathfrak{n}, \mathfrak{u})^T = \mathfrak{u}^T \mid dc = 0 \}$. Here $dc = c$ because $c \in \mathfrak{u}^T = \mathfrak{n}^T \rtimes T = T$, and we get $H^0(\mathfrak{n}, \mathfrak{u})^T = 0$.

Remark. About the vanishing of $H^i(\mathfrak{u}, \mathfrak{u})$ ($i = 0, 1$), there is a similar result in Proposition 4.1 of [L-L] which is obtained by a different method.

Corollary 2.3. *If $\lambda \neq 0$, $\lambda + \mu \neq 0$, and $\dim \mathfrak{n}_\lambda = 1$ ($\lambda, \mu \in W$), then $H^i(\mathfrak{u}, \mathfrak{u}) = 0$ ($i = 0, 1$) and $H^2(\mathfrak{u}, \mathfrak{u}) = H^2(\mathfrak{n}, \mathfrak{n})^T$.*

proof). By the assumptions $H^i(\mathfrak{n}, \mathfrak{u})^T = 0$ ($i = 0, 1$), then we get $H^i(\mathfrak{u}, \mathfrak{u}) = 0$ ($i = 0, 1$) and $H^2(\mathfrak{u}, \mathfrak{u}) = H^2(\mathfrak{n}, \mathfrak{n})^T$ using Lemma 2.1.

Definition. Let $\bar{\mathfrak{n}}$ be a nilpotent Lie algebra over $\mathbb{C}$. We call $\bar{\mathfrak{n}}$ is a *Iwasawa subalgebra* when there exists a semi-simple Lie algebra $\mathfrak{s}$ over $\mathbb{R}$ and its Iwasawa decomposition; $\mathfrak{s} = \mathfrak{f} \oplus \mathfrak{a} \oplus \mathfrak{n}$ such that $\bar{\mathfrak{n}} = \mathfrak{n}^{\mathbb{C}}$.

Proposition 2.4. *Let $\mathfrak{n}$ be an Iwasawa subalgebra, then $\lambda \neq 0$, $\lambda + \mu \neq 0$, $\dim \mathfrak{n}_\lambda = 1$ ($\lambda, \mu \in W$), therefore $H^i(\mathfrak{u}, \mathfrak{u}) = 0$ ($i = 0, 1$), and $H^2(\mathfrak{u}, \mathfrak{u}) = H^2(\mathfrak{n}, \mathfrak{n})^T$.*

proof). There exists an Iwasawa decomposition $\mathfrak{s} = \mathfrak{f} \oplus \mathfrak{a} \oplus \mathfrak{n}$ (over $\mathbb{C}$), and let $\mathfrak{h}$ be a Cartan subalgebra containing $\mathfrak{a}$. Then $(\text{ad } \mathfrak{h})|_{\mathfrak{n}} \subset \text{Der}(\mathfrak{n})$ and we can choose T such that $(\text{ad } \mathfrak{h})|_{\mathfrak{n}} \subset T$ since $\mathfrak{n}$ are direct sum of some positive root spaces of $\mathfrak{n}$ with respect to $\text{ad } \mathfrak{h}$ (see e.g. [He]). Therefore each root space $\mathfrak{s}_\alpha$ in $\mathfrak{n}$ ($\alpha \in \Delta_+$) is decomposed into some weight spaces $\mathfrak{n}_\lambda$ ($\lambda \in W$). Since $\dim \mathfrak{s}_\alpha = 1$, for any $\lambda \in W$ there exists $\alpha \in \Delta_+$ such that $\mathfrak{n}_\lambda = \mathfrak{s}_\alpha$ and $\lambda|_{\text{ad } \mathfrak{h}} = \alpha$.

Since $\alpha \neq 0$, $\alpha + \beta \neq 0$ ($\alpha, \beta \in \Delta_+$), and $\dim s_\alpha = 1$ ($\alpha, \beta \in \Delta_+$), Proposition 2.4 follows.

Remark 2.5. For an Iwasawa subalgebra $\mathfrak{n}$, we have

$[\mathfrak{n}_\lambda, \mathfrak{n}_\mu] = \mathfrak{n}_{\lambda+\mu}$ ($\lambda, \mu, \lambda + \mu \in W$) because there exist $\alpha, \beta \in \Delta_+$ such that $\mathfrak{n}_\lambda = s_\alpha$, $\mathfrak{n}_\mu = s_\beta$, and $[s_\alpha, s_\beta] \neq 0$.

To compute $H^2(\mathfrak{n}, \mathfrak{n})^T$ we use the following:

Lemma 2.6. If $\dim \mathfrak{n}_\lambda = 1$ ($\lambda \in W$), then we have

$$(1). \quad \dim B^2(\mathfrak{n}, \mathfrak{n})^T = \dim \mathfrak{n} - \dim T,$$

$$(2). \quad \dim C^2(\mathfrak{n}, \mathfrak{n})^T = \# \{ (\lambda, \mu) \in W \times W \mid \lambda + \mu \in W \text{ and } \lambda < \mu \}.$$

Moreover if an Iwasawa subalgebra $\mathfrak{n}$ is 2-step (i.e. $[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] = 0$), then we have

$$(3). \quad C^2(\mathfrak{n}, \mathfrak{n})^T = Z^2(\mathfrak{n}, \mathfrak{n})^T.$$

proof). To prove (1), we use the surjective homomorphism $d : C^1(\mathfrak{n}, \mathfrak{n})^T \longrightarrow B^2(\mathfrak{n}, \mathfrak{n})^T$. Since $C^1(\mathfrak{n}, \mathfrak{n})^T = \{ c \in \text{End } \mathfrak{n} \mid c \mathfrak{n}_\lambda \subset \mathfrak{n}_\lambda \text{ } (\lambda \in W) \}$, and $\dim \mathfrak{n}_\lambda = 1$, we get $\dim C^1(\mathfrak{n}, \mathfrak{n})^T = \dim \mathfrak{n}$. And $\ker d = T$, therefore we have the equation (1). By the definition we have

$C^2(\mathfrak{n}, \mathfrak{n})^T = \{ c \in C^2(\mathfrak{n}, \mathfrak{n}) \mid c(\mathfrak{n}_\lambda, \mathfrak{n}_\mu) \subset \mathfrak{n}_{\lambda+\mu} \text{ } (\lambda, \mu, \lambda + \mu \in W) \}$, and we get the equation (2) because $\dim \mathfrak{n}_\lambda = 1$ ($\lambda \in W$).

Next for $c \in C^2(\mathfrak{n}, \mathfrak{n})^T$, using Remark 2.5 we can prove

$$dc(\mathfrak{n}_\lambda, \mathfrak{n}_\mu, \mathfrak{n}_\nu) \subset [\mathfrak{n}_\lambda, [\mathfrak{n}_\mu, \mathfrak{n}_\nu]] + [\mathfrak{n}_\nu, [\mathfrak{n}_\lambda, \mathfrak{n}_\mu]] + [\mathfrak{n}_\mu, [\mathfrak{n}_\nu, \mathfrak{n}_\lambda]].$$

Therefore $dc = 0$ if $\mathfrak{n}$ is 2-step, hence we get (3).

§ 3 The rigidity of $\mathfrak{u}$ when $\mathfrak{n}$ are some Iwasawa subalgebras.

We compute $H^2(\mathfrak{u}, \mathfrak{u})$ for the 2-step Iwasawa subalgebras that appear in the simple Lie algebras of real rank 1. Those simple Lie algebras are $\mathfrak{so}(n+1, 1)$, $\mathfrak{su}(n+1, 1)$, $\mathfrak{sp}(n+1, 1)$, and $\mathfrak{f}_{4,(-20)}$. Then those nilpotent parts are known to be $(\mathbb{K}^n \oplus \text{Im } \mathbb{K})^\mathbb{C}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, $n = 1$ and $\mathfrak{C}$, and $\text{Im } \mathbb{K}$ are imaginary parts of $\mathbb{K}$), where the brackets of $\mathbb{K}^n \oplus \text{Im } \mathbb{K}$ are given by

$$[(\alpha, \beta), (\alpha', \beta')] = 2 \text{Im} \left(\sum_{i=1}^n \bar{\alpha}'_i \alpha_i \right) \quad (\alpha, \alpha' \in \mathbb{K}^n, \beta, \beta' \in \text{Im } \mathbb{K})$$

(see e.g. [Mo]). For those 2-step nilpotent Lie algebras, we compute T , W , and $H^2(u, u)$ using Lemma 2.6.

Proposition 3.1. *In the derivation algebras $\text{Der}(\mathbb{K}^n \oplus \text{Im } \mathbb{K})^{\mathbb{C}}$, we can choose T as follows:*

$$\begin{aligned}
 \text{(i)} \quad \mathbb{K} = \mathbb{R}, \quad T &= \left\{ \begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix} \right\}, \\
 \text{(ii)} \quad \mathbb{K} = \mathbb{C}, \quad T &= \left\{ \begin{pmatrix} D + sI & \vdots & 0 & \vdots \\ \vdots & \ddots & \vdots & 0 \\ 0 & \vdots & -D + sI & \vdots \\ \vdots & 0 & \vdots & 2s \end{pmatrix} \mid D = \begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix} \right\}, \\
 \text{(iii)} \quad \mathbb{K} = \mathbb{H}, \quad T &= \left\{ \begin{pmatrix} sI & \vdots & -D + tI & 0 & \vdots \\ \vdots & \ddots & \vdots & 0 & \vdots \\ D + tI & \vdots & sI & \vdots & 0 \\ \vdots & 0 & \vdots & 2s & \vdots \\ 0 & \vdots & \vdots & 0 & 2t \\ \vdots & \vdots & \vdots & -2t & 0 & 2s \end{pmatrix} \mid D: \text{diag.} \right\},
 \end{aligned}$$

$$\text{(iv)} \quad \mathbb{K} = \mathbb{C} \quad (n = 1),$$

$$T = \left\{ \begin{pmatrix} sI + R(\theta) & \vdots & 0 \\ sI + R(\theta_1) & \vdots & \\ sI + R(\theta_2) & \vdots & \\ \vdots & \vdots & sI + R(\theta_3) \\ 0 & \vdots & 2s & \vdots & 2sI - R(\theta_2 + \theta_3) \\ & \vdots & 2sI - R(\theta_1 + \theta_3) \\ & \vdots & 2sI - R(\theta_1 + \theta_2) \end{pmatrix} \right\},$$

$$\text{where } \theta = \theta_1 + \theta_2 + \theta_3, \text{ and } R(\theta) = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}.$$

proof). (i) being trivial, in order to prove (ii) we see the fact

$$\begin{aligned}
 \text{Der}(\mathbb{C}^n \oplus \text{Im } \mathbb{C})^{\mathbb{C}} &= \left\{ \begin{pmatrix} sI + A & \vdots & 0 \\ \vdots & \ddots & \vdots \\ * & \vdots & 2s \end{pmatrix} \mid A \in \mathfrak{sp}(n, \mathbb{R}), s \in \mathbb{R} \right\}^{\mathbb{C}} \\
 &\simeq \{ \mathbb{R}^{2n} \rtimes (\mathbb{R} \oplus \mathfrak{sp}(n, \mathbb{R})) \}^{\mathbb{C}}.
 \end{aligned}$$

Using a standard Cartan subalgebra of $\mathfrak{sp}(n, \mathbb{R})$, we have the above expression of T . Next let us prove (iii). We use the fact

$$\begin{aligned}
 \text{Der}(\mathbb{H}^n \oplus \text{Im } \mathbb{H})^{\mathbb{C}} &= \left\{ \begin{pmatrix} X + Y & \vdots & 0 \\ \vdots & \ddots & \vdots \\ * & \vdots & 2a & 2b & 2c \\ & \vdots & -2b & 2a & -2d \\ & \vdots & -2c & 2d & 2a \end{pmatrix} \mid X = \begin{pmatrix} A & -B & -C & -D \\ B & A & -D & C \\ C & D & A & -B \\ D & -C & B & A \end{pmatrix}, \right. \\
 Y &= \left. \begin{pmatrix} aI & dI & cI & -bI \\ -dI & aI & bI & cI \\ -cI & -bI & aI & -dI \\ bI & -cI & dI & aI \end{pmatrix}, A \in \text{Skew}(n), B, C, D \in \text{Symm}(n), a, b, c, d \in \mathbb{R} \right\}^{\mathbb{C}},
 \end{aligned}$$

$$\simeq \{(\mathbb{R}^{4n} \otimes \mathbb{R}^3) \rtimes (\mathbb{H} \oplus \mathfrak{sp}(n))\}^{\mathbb{C}}.$$

The dimension of maximal abelian subalgebra of $\mathbb{H} \oplus \mathfrak{sp}(n) \simeq \mathbb{R} \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$ is $n + 2$. Here the above expression of T is abelian, consisting of semi-simple elements, and $n + 2$ dimension too. Hence we get (iii). In order to prove (iv), we use the fact

$$\text{Der}(\mathfrak{C} \oplus \text{Im } \mathfrak{C})^{\mathbb{C}} = \left\{ \begin{pmatrix} sI + A & \vdots & 0 \\ \dots & \ddots & \dots \\ * & \vdots & 2sI + B \end{pmatrix} \mid A = (a_{ij})_{0 \leq i, j \leq 7} \in \mathfrak{so}(8), \right.$$

$$s \in \mathbb{R}, B = (b_{ij})_{1 \leq i, j \leq 7} \in \mathfrak{so}(7), \text{ such that}$$

$$\begin{aligned} 2a_{08} &= b_{21} + b_{65} + b_{74}, \\ 2a_{47} &= b_{21} + b_{65} - b_{74}, \\ 2a_{56} &= b_{21} - b_{65} + b_{74}, \\ 2a_{12} &= -b_{21} + b_{65} + b_{74}, \\ -2a_{57} &= b_{31} + b_{64} + b_{75}, \quad 2a_{01} = b_{32} + b_{54} + b_{76}, \quad -2a_{14} = b_{41} + b_{63} + b_{72}, \\ -2a_{02} &= b_{31} + b_{64} - b_{75}, \quad 2a_{67} = b_{32} + b_{54} - b_{76}, \quad -2a_{36} = b_{41} + b_{63} - b_{72}, \\ -2a_{13} &= b_{31} - b_{64} + b_{75}, \quad 2a_{45} = b_{32} - b_{54} + b_{76}, \quad -2a_{27} = b_{41} - b_{63} + b_{72}, \\ -2a_{46} &= -b_{31} + b_{64} + b_{75}, \quad 2a_{23} = -b_{32} + b_{54} + b_{76}, \quad -2a_{05} = -b_{41} + b_{63} + b_{72}, \\ -2a_{35} &= b_{42} + b_{53} + b_{71}, \quad 2a_{07} = b_{43} + b_{52} + b_{61}, \quad -2a_{26} = b_{51} + b_{62} + b_{73} \\ -2a_{24} &= b_{42} + b_{53} - b_{71}, \quad 2a_{16} = b_{43} + b_{52} - b_{61}, \quad -2a_{15} = b_{51} + b_{62} - b_{73} \\ -2a_{06} &= b_{42} - b_{53} + b_{71}, \quad 2a_{25} = b_{43} - b_{52} + b_{61}, \quad -2a_{04} = b_{51} - b_{62} + b_{73} \\ -2a_{17} &= -b_{42} + b_{53} + b_{71}, \quad 2a_{34} = -b_{43} + b_{52} + b_{61}, \quad -2a_{37} = -b_{51} + b_{62} + b_{73} \end{aligned} \right\}^{\mathbb{C}},$$

$$\simeq \{(\mathbb{R}^8 \otimes \mathbb{R}^7) \rtimes (\mathbb{R} \oplus \mathfrak{so}(7))\}^{\mathbb{C}}.$$

Using a standard Cartan subalgebra of $\mathfrak{so}(7)$, we get T in (iv).

Remark. In the above expression of $\text{Der}(\mathfrak{C} \oplus \text{Im } \mathfrak{C})$, we can prove that the map $\mathfrak{so}(7) \ni B \longrightarrow A \in \mathfrak{so}(8)$ is the spin representation of $\mathfrak{so}(7)$.

Proposition. 3.2. When $\mathfrak{n} = (\mathbb{K}^n \oplus \text{Im } \mathbb{K})^{\mathbb{C}}$, $H^2(u, u)$ are given as follows:

	(i) $\mathbb{K} = \mathbb{R}$	(ii) $\mathbb{K} = \mathbb{C}$	(iii) $\mathbb{K} = \mathbb{H}$	(iv) $\mathbb{K} = \mathfrak{C}$
$H^2(u, u)$	0	0	$\mathbb{C}^{n-1}$	$\mathbb{C}^5$

proof). First we compute W using T in Proposition 3.1.

$$(i) \quad W = \{d_i\}_{1 \leq i \leq n}.$$

$$(ii) \quad W = \{s \pm d_i, 2s\}_{1 \leq i \leq n}.$$

$$(iii) \quad W = \{s \pm (d_i + t)\sqrt{-1}, s \pm (d_i - t)\sqrt{-1}, 2s, 2s \pm 2t\sqrt{-1}\}_{1 \leq i \leq n}.$$

$$(iv) \quad W = \{s \pm \theta\sqrt{-1}, s \pm \theta_1\sqrt{-1}, s \pm \theta_2\sqrt{-1}, s \pm \theta_3\sqrt{-1}, \\ 2s, 2s \pm (\theta_2 + \theta_3)\sqrt{-1}, 2s \pm (\theta_1 + \theta_3)\sqrt{-1}, 2s \pm (\theta_1 + \theta_2)\sqrt{-1}\}.$$

Next we compute $\dim Z^2(\mathfrak{n}, \mathfrak{n})^T$ using Lemma 2.6 (2) and (3).

$$(i) \quad \dim Z^2(\mathfrak{n}, \mathfrak{n})^T = 0, \text{ because } \lambda + \mu \notin W \ (\lambda, \mu \in W).$$

(ii) $\dim Z^2(\mathfrak{n}, \mathfrak{n})^T = n$, because $2s = (s + d_i) + (s - d_i)$.

(iii) $\dim Z^2(\mathfrak{n}, \mathfrak{n})^T = 4n$, because

$$2s = (s + (d_i \pm t)\sqrt{-1}) + (s - (d_i \pm t)\sqrt{-1}),$$

$$2s \pm 2t\sqrt{-1} = (s + (d_i \pm t)\sqrt{-1}) + (s - (d_i \mp t)\sqrt{-1}).$$

(iv) $\dim Z^2(\mathfrak{n}, \mathfrak{n})^T = 16$, because

$$2s = (s + \theta\sqrt{-1}) + (s - \theta\sqrt{-1}) = (s + \theta_1\sqrt{-1}) + (s - \theta_1\sqrt{-1}),$$

$$2s \pm (\theta_2 + \theta_3)\sqrt{-1} = (s \pm \theta_2\sqrt{-1}) + (s \pm \theta_3\sqrt{-1}) = (s \pm \theta\sqrt{-1}) + (s \mp \theta_1\sqrt{-1}),$$

$$2s \pm (\theta_1 + \theta_3)\sqrt{-1} = (s \pm \theta_1\sqrt{-1}) + (s \pm \theta_3\sqrt{-1}) = (s \pm \theta\sqrt{-1}) + (s \mp \theta_2\sqrt{-1}),$$

and

$$2s \pm (\theta_1 + \theta_2)\sqrt{-1} = (s \pm \theta_1\sqrt{-1}) + (s \pm \theta_2\sqrt{-1}) = (s \pm \theta\sqrt{-1}) + (s \mp \theta_3\sqrt{-1}).$$

Last we use the equation :

$$\dim H^2(\mathfrak{n}, \mathfrak{n})^T = \dim Z^2(\mathfrak{n}, \mathfrak{n})^T - \dim \mathfrak{n} + \dim T.$$

Computing the right hand side, we get Proposition 3.2.

Remark 3.3. In the above proof we have used $C^2(\mathfrak{n}, \mathfrak{n})^T = Z^2(\mathfrak{n}, \mathfrak{n})^T$. This is not true for any Iwasawa nilpotent Lie algebra $\mathfrak{n}$ such that $\text{step } \mathfrak{n} \geq 3$. Then we must compute the rank of system of linear equations : $dc(X_\lambda, X_\mu, X_\nu) = 0$ ($X_\lambda \in \mathfrak{n}_\lambda, X_\mu \in \mathfrak{n}_\mu, X_\nu \in \mathfrak{n}_\nu$). We report the results $H^2(u, u) = 0$ for the nilpotent parts of $\mathfrak{so}(n+k, k)$ and $\mathfrak{su}(n+k, k)$, (for any $k \in \mathbb{N}$).

Since the condition $H^2(u, u) \neq 0$ does not mean the rigidity of u ([Ri]), we need the following:

Proposition 3.4. $u(\mathfrak{n})$ is not rigid when $\mathfrak{n} = (\mathbb{H}^n \oplus \text{Im } \mathbb{H})^{\mathbb{C}}$ ($n \geq 2$) or $\mathfrak{n} = (\mathbb{C} \oplus \text{Im } \mathbb{C})^{\mathbb{C}}$.

proof). We give the proof when $\mathfrak{n} = (\mathbb{H}^n \oplus \text{Im } \mathbb{H})$ ($n \geq 2$). Let us choose weight vectors of T ; $\{X_i, Y_i, Z_i, W_i, A, B, C\}_{1 \leq i \leq n}$ corresponding to the weights ; $\{s + (d_i + t)\sqrt{-1}, s - (d_i + t)\sqrt{-1}, s + (d_i - t)\sqrt{-1}, s - (d_i - t)\sqrt{-1}, 2s, 2s + 2t\sqrt{-1}, 2s - 2t\sqrt{-1}\}$. Let μ be the Lie bracket of u , and $\varphi \in C^2(\mathfrak{n}, \mathfrak{n})^T$ ($\subset C^2(u, u)^T$) defined by

$$\begin{cases} \varphi(X_i, Y_i) = p_i A \\ \varphi(Z_i, W_i) = q_i A \\ \varphi = 0 \quad (\text{other cases}) \end{cases} \quad (p_i, q_i \in \mathbb{C}).$$

We can check that $\mu + \varepsilon\varphi$ ($\varepsilon \in \mathbb{C}$) is also a Lie algebra, so $\mu + \varepsilon\varphi$ is a deformation of μ . Assume that μ is rigid, then the tangent vector $\varphi \in B^2(u, u)$ ([N-R]). Since φ is T -invariant $\varphi \in B^2(u, u)^T = d(C^1(u, u)^T)$, therefore there exists $f \in C^1(u, u)^T$ such that $\varphi = df$. As $f(\pi_\lambda) \subset \pi_\lambda$ ($\lambda \in W$), and $\varphi = 0$ on $\pi \times T$ and $T \times T$, $\varphi = d(f|_\pi)$ and

$$f|_\pi = \text{diag}(x_i, y_i, z_i, w_i, a, b, c) \\ \text{with respect to } \{X_i, Y_i, Z_i, W_i, A, B, C\}_{1 \leq i \leq n}.$$

Since

$$\begin{cases} df(X_i, W_i) = 0 \\ df(Y_i, Z_i) = 0 \end{cases} \quad \text{and} \quad \begin{cases} df(X_i, Y_i) = p_i A \\ df(Z_i, W_i) = q_i A \end{cases},$$

we have

$$\begin{cases} x_i + w_i - b = 0 \\ y_i + z_i - c = 0 \end{cases} \quad \text{and} \quad \begin{cases} r_i(x_i + y_i - a) = p_i \\ s_i(z_i + w_i - a) = q_i \end{cases},$$

where r_i and s_i are non-zero number defined by $\begin{cases} \mu(X_i, Y_i) = r_i A \\ \mu(Z_i, W_i) = s_i A \end{cases}$.

Computing $x_i + y_i + z_i + w_i$, we get

$$2a + \frac{p_i}{r_i} + \frac{q_i}{s_i} = b + c \quad (1 \leq i \leq n).$$

As $n \geq 2$, this equation has no solution when we put $p_i = i r_i$ and $q_i = i s_i$. This is a contradiction, hence μ is not rigid.

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補足 ($\mathcal{U}$ と $H^2(\mathcal{U}, \mathcal{U})$ のより効率的な計算法)

初めに

「論文」では、実単純 Lie 環 $\mathfrak{G}$ の巾零部 $\mathcal{U}$ が $\text{Rrank } \mathfrak{G} = 1$ の時、見易い形で知られている事を用いて $\mathcal{U}$ を定義通りに構成している。即ち、 $\text{Der } \mathcal{U}^{\mathbb{C}}$ を求め、その中から T を選び、 $\mathcal{U} = \mathcal{U}^{\mathbb{C}} \rtimes T$ を構成している。そして $\mathcal{U}$ が 2step である事を利用し $H^2(\mathcal{U}, \mathcal{U})$ を求めている。

この方法をこのまま一般の実単純 Lie 環 $\mathfrak{G}$ の巾零部 $\mathcal{U}$ に適用するのは困難である。そこで次の3段階を踏んでより効率的に $\mathcal{U}$ と $H^2(\mathcal{U}, \mathcal{U})$ を計算する方法を示す。具体例として

例1 $\mathfrak{G} = \mathfrak{su}(n+k, k)$ の場合

例2 $\mathfrak{G} = \mathfrak{so}(n+k, k)$ “

をあげ、Remark 3.3 で触れた $H^2(\mathcal{U}, \mathcal{U}) = 0$ を示す。

これから示す方法では、佐武図形の性質(ルート系と Cartan 対合の性質)から $\text{Der } \mathcal{U}^{\mathbb{C}}$ を経由せずに $\mathcal{U}$ が求められるのだが、「論文」では $\mathcal{U}$ が、 $\mathbb{K}^n$ の $\text{Im } \mathbb{K}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ ($n=1$)) と同じような表示であるにもかかわらず、 $\mathcal{U}$ の表示や $H^2(\mathcal{U}, \mathcal{U})$ がまるで異なる事がより具体的に調べられている。また「論文」中の $\mathcal{U}$ についての $\text{Der } \mathcal{U}$ の表示 (Prop. 3.1 の証明中) には、C. Riehm の論文 "Explicit spin representations and Lie algebras of Heisenberg type. J. London

Math. Soc. (2) 29 (1984) 49-62." 等との関連も有るようなので、「論文」をこれから述べる方法で書き直す事はしなかった。

オ一段階 佐武図形からの Iwasawa 部分環の構成

教科書 [He] に従って $\mathfrak{nc}^e$ を ルート部分空間の直和で表す。

$\mathfrak{G}$: 実単純 Lie 環

$\mathfrak{G} = \mathfrak{h} \oplus \mathfrak{m}$: Cartan 分解 (θ をその Cartan 対合とする)

$\mathfrak{a}$: $\mathfrak{m}$ 内の 1 つの極大可換部分環

$\mathfrak{g}_\alpha$: α を含む $\mathfrak{G}$ の $\mathfrak{h}$ (Cartan 部分環)

Δ_+ : $\mathfrak{g}^e$ についての $\mathfrak{G}^e$ の正ルート全体 (ある順序について)

$$P_+ = \{ \alpha \in \Delta_+ \mid \alpha \cdot \theta \neq \alpha \ (\Leftrightarrow \alpha|_{\mathfrak{a}} \neq 0) \}$$

$$P_- = \{ \alpha \in \Delta_+ \mid \alpha \cdot \theta = \alpha \ (\Leftrightarrow \alpha|_{\mathfrak{a}} = 0) \}$$

すると

$$\mathfrak{nc}^e = \bigoplus_{\alpha \in P_+} (\mathfrak{G}^e)_\alpha$$

佐武図形の黒丸単純ルートだけを使って表せるルートが P_- で、白丸単純ルートを 1 つ以上使って表せるルートが P_+ になる。というのは $\alpha, \beta \in \Delta_+$ s.t. $\alpha + \beta \in \Delta_+$ に対し

$$\alpha \text{ or } \beta \in P_+ \Leftrightarrow \alpha + \beta \in P_+$$

$$(\alpha \text{ and } \beta \in P_- \Leftrightarrow \alpha + \beta \in P_-)$$

が成立するからである。そして $\mathfrak{nc}^e / [\mathfrak{nc}^e, \mathfrak{nc}^e]$ の基底として、 P_+ の元を単純ルートの和として表した場合の白丸が 1 つ

であるルートのルートベクトルが選べ、 $\pi^{\mathbb{C}}$ のStep数は最高ルートを単純ルートの和として表した場合の白丸の個数で与えられる。

例1

$$\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_R \quad \alpha_{R+1} \quad \cdots \quad \alpha_{n+R-1} \quad \alpha_{n+R} \quad \cdots \quad \alpha_{n+2R-1}$$

$\mathfrak{g}^{\mathbb{C}} \simeq \mathfrak{sl}(n+2R, \mathbb{C})$, $\mathfrak{h}^{\mathbb{C}} \simeq \text{diag}(\lambda_1, \dots, \lambda_{n+2R})$ で同一視すると

$$\Delta_+ = \{ \lambda_i - \lambda_j \mid 1 \leq i < j \leq n+2R \} \quad (\alpha_i = \lambda_i - \lambda_{i+1})$$

$$\Delta_- = \{ \quad \quad \quad \mid R+1 \leq i < j \leq n+R-1 \}$$

$\lambda_i - \lambda_j$ のルートベクトルは E_{ij} に取れるから

$$\pi^{\mathbb{C}} \simeq \left\{ \begin{pmatrix} 0 & * & & * & \\ 0 & 0 & & & \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & & 0 & * \\ 0 & 0 & & 0 & 0 \end{pmatrix} \right\}$$

$\leftarrow R \rightarrow \leftarrow n \rightarrow \leftarrow R \rightarrow$

例2

① $n = 2n' + 1$ の時

$$\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_R \quad \cdots \quad \alpha_{n+R-1} \quad \alpha_{n+R} \quad \cdots \quad \alpha_{n'+R}$$

$$\mathfrak{g}^{\mathbb{C}} \simeq \mathfrak{so}(n+2R, \mathbb{C}) \simeq \left\{ \begin{pmatrix} 0 & a & b \\ -{}^t b & X & Y \\ -{}^t a & Z & -{}^t X \end{pmatrix} \right\} \quad \begin{matrix} a, b \in \mathbb{C}^{n'+R} \\ {}^t Y = -Y, {}^t Z = -Z \end{matrix}$$

$\leftarrow n'+R \rightarrow \leftarrow \quad \rightarrow$

$$\mathfrak{h}^{\mathbb{C}} \simeq \text{diag}(0, \lambda_1, \dots, \lambda_{n'+R}, -\lambda_1, \dots, -\lambda_{n'+R})$$

$$\Delta_+ = \{ \lambda_i \pm \lambda_j, \lambda_i \mid 1 \leq i < j \leq n'+R, 1 \leq i \leq n'+R \}$$

$$\alpha_i = \lambda_i - \lambda_{i+1} \quad (1 \leq i \leq n'+R-1), \quad \alpha_{n+R} = \lambda_{n'+R}$$

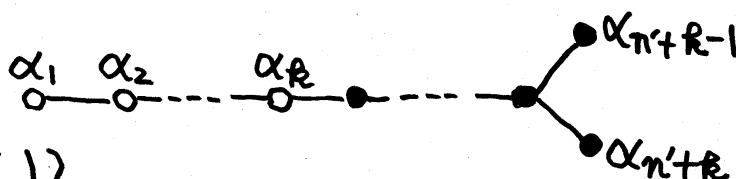
$$P_+ = \{\lambda_i \pm \lambda_j, \lambda_i\} \mid 1 \leq i \leq k$$

$$\begin{matrix} \leftarrow k \times n' \rightarrow \\ \left\{ \begin{pmatrix} 0 & * & * \\ -b & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \right\} \end{matrix}$$

$$\mathfrak{r}^c \simeq \left\{ \begin{pmatrix} 0 & 0 & b \\ -b & X & Y - {}^t Y \\ 0 & 0 & -{}^t X \end{pmatrix} \right\} \quad b_{k+1} = \dots = b_{n'+k} = 0$$

$$X, Y \in \mathfrak{r}_{k, n'}$$

② $n = 2n'$ の時



$$\mathfrak{g}^c \simeq \left\{ \begin{pmatrix} X & Y \\ -Z & -{}^t X \end{pmatrix} \right\} \quad {}^t Y = -Y, {}^t Z = -Z$$

$$\Delta_+ = \{\lambda_i \pm \lambda_j\} \mid 1 \leq i < j \leq n' + k$$

$$\alpha_i = \lambda_i - \lambda_{i+1} \quad (1 \leq i \leq n' + k - 1), \quad \alpha_{n'+k} = \lambda_{n'+k-1} + \lambda_{n'+k}$$

$$P_+ = \{\lambda_i \pm \lambda_j\} \mid 1 \leq i \leq k$$

$$\mathfrak{r}^c \simeq \left\{ \begin{pmatrix} X & Y - {}^t Y \\ -b & -{}^t X \end{pmatrix} \right\} \quad X, Y \in \mathfrak{r}_{k, n'}$$

注意 例2の計算には「Lie群(II)」(岩堀長慶 岩波書店)の中の古典単純Lie環の表示を使った。

オ2段階 $\text{ad } \mathfrak{h}$ を利用する T の求め方

Prop. 2.4 で触れたように、 $T \supset \text{ad } \mathfrak{h}^c|_{\mathfrak{r}^c}$ に選べる。すると T と $\text{ad } \mathfrak{h}^c|_{\mathfrak{r}^c}$ は可換だから、 $\mathfrak{r}^c$ を構成する各ルート空間 ($\mathfrak{g}_\alpha^c: 1$ 次元) を保つ。

$\therefore D \in T \Rightarrow \exists! d_\alpha \in \mathbb{C} \text{ s.t. } D|_{\mathfrak{g}_\alpha^c} = d_\alpha 1_{\mathfrak{g}_\alpha^c} \quad (\alpha \in P_+)$
 ここで、 $\{d_\alpha\}_{\alpha \in P_+}$ は $D \in \text{Der } \mathfrak{r}^c$ より連立方程式

$$d\alpha + d\beta = d(\alpha + \beta) \quad (\forall \alpha, \beta, \alpha + \beta \in P_+) \quad \text{-----} (*)$$

を満たす。 $\mathfrak{r}^e$ は中零だから $D \in T$ は $\mathfrak{r}^e / [\mathfrak{r}^e, \mathfrak{r}^e]$ の上だけで決まる。従って $(*)$ は $\{d\alpha \mid \alpha \in P_+ \text{ は白丸を1つだけ使うルート}\}$ を使って表せる。この連立方程式を $(*)'$ とすると

$$\dim T = \dim \mathfrak{r}^e - \text{rank}(*') = \dim \mathfrak{r}^e / [\mathfrak{r}^e, \mathfrak{r}^e] - \text{rank}(*')$$

他方 $\text{ad } \mathfrak{g}^e |_{\mathfrak{r}^e}$ の次元については次が成立

命題 $\dim \text{ad } \mathfrak{g}^e |_{\mathfrak{r}^e} = \dim \mathfrak{g} \quad (\because \text{rank } \mathfrak{G} \leq \dim T)$

証明 $\mathfrak{g} \ni H \longrightarrow \text{ad } H |_{\mathfrak{r}^e} \in \text{Der } \mathfrak{r}^e$ が単射、を言う。

$$[H, \mathfrak{r}^e] = 0 \Rightarrow [H, \bigoplus_{\alpha \in P_+} (\mathfrak{G}^e)_\alpha] = 0$$

P_+ には $\dim \mathfrak{G}$ 個の一次独立なルートがあるから

$$\alpha(H) = 0 \quad (\alpha \in P_+) \quad \text{より} \quad H \in \mathfrak{g} \cap \mathfrak{h}$$

$$\therefore [H, \mathfrak{r}^e] = [H, \theta \mathfrak{r}^e] = 0$$

$$\therefore [H, \mathfrak{r}^e] = 0 \quad (\because \mathfrak{r}^e \oplus \theta \mathfrak{r}^e \supset \mathfrak{r}^e)$$

$\mathfrak{h}$ の中には $\mathfrak{G}$ の ideal は 0 しかないのだから $H = 0$ //

これらによって殆どどの場合 $T = \text{ad } \mathfrak{g}^e |_{\mathfrak{r}^e}$ が示せる。

例1

$D \in T$ なら $\mathfrak{G}^e_{\lambda_i - \lambda_j}$ の上で定数 (= d_{ij} とおく) 倍、 D は

$$\begin{cases} d_{i+1} & (1 \leq i \leq k-1 \text{ or } n+k+1 \leq i \leq n+2k-1) \\ d_{ki}, d_{n+k+1} & (k+1 \leq i \leq n+k) \end{cases}$$

で決まるか

$$d_{ki} + d_{i, n+k+1} = d_{k, n+k+1} \quad (k+1 \leq i \leq n+k)$$

を満たす。よって $\text{rank}(\ast)' \geq n-1$

$$\therefore \dim \text{ad } \mathfrak{g}^e|_{\text{rec}} \leq \dim T \leq \underbrace{2n+2(k-1)-(n-1)}_{\text{共に } n+2k-1}$$

$$\therefore \text{ad } \mathfrak{g}^e|_{\text{rec}} = T$$

例2.

$$D \in T \Rightarrow \begin{cases} \mathfrak{G}^e_{\lambda_i - \lambda_j} \text{の上で } d_{ij} \text{倍}, \mathfrak{G}^e_{\lambda_i + \lambda_j} \text{の上で } d'_{ij} \text{倍} \\ \mathfrak{G}^e_{\lambda_i} \text{の上で } d_i \text{倍} \quad (n=2n'+1 \text{の時のみ}) \end{cases}$$

とかける。

① $n=2n'+1$ の時 D は

$$\begin{cases} d_{i+1} \quad (1 \leq i \leq k-1), d_{kj} \quad (k+1 \leq j \leq n'+k) \\ d'_{kj} \quad (k+1 \leq j \leq n'+k), d_k \end{cases}$$

で決まるが

$k=1$ の時 ----- $\mathfrak{r}^e$:可換 $\begin{matrix} n'+1\text{次元} & 2n'+1\text{次元} \\ \downarrow & \downarrow \end{matrix}$

$$\therefore \text{rank}(\ast)' = 0 \quad \text{ad } \mathfrak{g}^e|_{\text{rec}} \subsetneq T$$

$k \geq 2$ の時 ----- $d'_{k-1,k}$ の表し方より

$$d_{k-1,j} + d'_{kj} = (d_{k-1,k} + d_k) + d_k \quad (k+1 \leq j \leq n'+k)$$

$$\therefore \text{rank}(\ast)' \geq n'$$

$$\therefore n'+k \leq \dim T \leq 2n'+k-n'$$

$$\therefore \text{ad } \mathfrak{g}^e|_{\text{rec}} = T$$

② $n=2n'$ の時 D は

$d_{i,i+1}$ ($1 \leq i \leq k-1$), $d_{k,j}$, $d'_{k,j}$ ($k+1 \leq j \leq n'+k$)
で決まるが①の場合と同様に

$k=1$ の時 $\text{ad } \mathfrak{g}^{\mathbb{C}}|_{\mathfrak{re}} \subsetneq T$

$k \geq 2$ の時

$$d_{k-1,j} + d'_{k,j} = d'_{k-1,k} \quad (k+1 \leq j \leq n'+k)$$

$$\therefore \text{rank}(\ast)' \geq n'-1$$

$$\therefore n'+k \leq \dim T \leq 2n'+k-1 - (n'-1)$$

$$\therefore \text{ad } \mathfrak{g}^{\mathbb{C}}|_{\mathfrak{re}} = T$$

注意 古典型の実単純 Lie 環では、 $\mathfrak{so}(n+1, 1)$ 以外では $\text{ad } \mathfrak{g}^{\mathbb{C}}|_{\mathfrak{re}} = T$ に取れる事が上の方法で証明できる。例外型でも可能と思われる。

オ3段階 $H^2(\mathfrak{u}, \mathfrak{u}) = H^2(\mathfrak{re}, \mathfrak{re})^T$ の計算

$\mathfrak{re}$ の T 不変な部分 Lie 環 $\mathcal{O}$ で、 $\text{Der } \mathcal{O}$ の同時対角化可能極大集合が同じ T に取れるものを考える。すると T 不変コホモロジーの定義と Lemma 2.6(1) より、

$$\dim H^2_{\uparrow}(\mathfrak{re})^T = \dim H^2(\mathcal{O})^T + \dim C^2(\mathfrak{re})^T / C^2(\mathcal{O})^T - \dim \mathfrak{re} / \mathcal{O}$$

$H^2(\mathfrak{re}, \mathfrak{re})^T$ の略 $-\text{rank}\{\mathcal{O}$ のに含まれぬ $\mathfrak{re}$ のコサイクル条件\}

が成り立つ。 $\mathcal{O}$ をうまく取り帰納的に $H^2(\mathfrak{re})^T$ を計算する。

例1 そのままでも示せるが、次のように一般化して示す。

$n \in \mathbb{N}$ と $k, l \in \mathbb{Z}$ s.t. $k+l \geq 1$ に対し

$$\pi_{k,n,l} = \left\{ \begin{pmatrix} 0 & * & * & * \\ & 0 & & \\ 0 & 0 & * & \\ 0 & 0 & 0 & * \end{pmatrix} \right\} \begin{matrix} \uparrow k \\ \times n \\ \downarrow l \end{matrix} \text{ とおく.}$$

対角行列の adjoint 表現が共通の T に取れ, Corollary 2.3 の仮定が満たされ $H^2(\mathcal{U}) = H^2(\pi)^T$ である。

$\pi_{1,n,0}$ は可換なので $C^2(\pi)^T = 0 \quad \therefore H^2(\pi)^T = 0$

$\pi_{k,n,0} (k \geq 2)$ では $\alpha = \pi_{k-1,n+1,0}$ に取る。

Lemma 2.6(2) より

$$\dim C^2(\pi)^T / C^2(\alpha)^T = \# \{ (E_{ik}, E_{kj}) \}_{\substack{1 \leq i \leq k-1 \\ k+1 \leq j \leq n+k}} \\ = (k-1)n$$

そして α のに含まれぬ π のコサイクル条件の rank は

$$\begin{aligned} &\geq \# \{ (E_{i,k-1}, E_{k-1,k}, E_{kj}) \}_{\substack{1 \leq i \leq k-2 \\ k+1 \leq j \leq n+k}} \\ &\geq (k-2)n \end{aligned}$$

$$\therefore \dim H^2(\pi)^T \leq 0 + (k-1)n - n - (k-2)n (=0)$$

$$\therefore H^2(\pi)^T = 0$$

$\pi_{k,n,l} (l \geq 1)$ では $\alpha = \pi_{k,n+1,l-1}$ に取る。同様に

$$\begin{aligned} \dim C^2(\pi)^T / C^2(\alpha)^T &= \# \{ (E_{ij}, E_{j,n+k+1}) \}_{\substack{1 \leq i \leq k \\ k+1 \leq j \leq n+k}} \\ &\quad + \# \{ (E_{j,n+k+1}, E_{n+k+1,i}) \}_{\substack{k+1 \leq j \leq n+k \\ n+k+2 \leq i \leq n+k+l}} \\ &= (k+l-1)n \end{aligned}$$

$$\begin{aligned} (\alpha \text{ のに含まれぬ } \pi \text{ の rank}) &\geq \# \{ (E_{ik}, E_{kj}, E_{j,n+k+1}) \}_{\substack{1 \leq i \leq k-1 \\ k+1 \leq j \leq n+k}} \\ &\quad + \# \{ (E_{kj}, E_{j,n+k+1}, E_{n+k+1,i}) \}_{\substack{k+1 \leq j \leq n+k \\ n+k+2 \leq i \leq n+k+l}} \\ &\geq (k+l-2)n \end{aligned}$$

$$\therefore \dim H^2(\pi)^T \leq 0 + (k+l-1)n - n - (k+l-2)n (=0)$$

$$\therefore H^2(\mathcal{U}) = H^2(\mathcal{V})^T = 0$$

例2 $n = 2n' + 1$ の時だけ示す。オ1段階より

$$\mathcal{V}^c = \left\{ \begin{pmatrix} 0 & 0 & b \\ -b & X & Y \\ 0 & 0 & -X \end{pmatrix} \right\} \quad \begin{matrix} b_{k+1} = \dots = b_{k+n'} = 0 \\ X, Y \in \mathcal{V}_{k, n'} \end{matrix}$$

を $\mathcal{V}(k, n')$ と書く。 $k \geq 2$ の時 T が共通に取れるので帰納法で示す。 $k=2$ では Lemma 2.6 (1) と $\begin{matrix} \text{上の表示で } Y \text{ の } (2, j) \text{ 成分} \\ \text{だけが1の行列} \end{matrix}$

$$\dim C^2(\mathcal{V})^T = \# \left\{ \begin{matrix} (X_{12}, X_{2j}), (X_{12}, Y_{2j}), (X_{1j}, Y_{2j}) \\ (X_{2j}, Y_{1j}), (X_{12}, b_2), (b_1, b_2) \end{matrix} \right\}_{3 \leq j \leq n'+2} = 4n' + 2$$

$$\text{rank} \{ T\text{-不変サイクル条件} \} \geq \# \{ (X_{12}, X_{2j}, Y_{2j}) \}_{3 \leq j \leq n'+2} \geq n'$$

より $\dim H^2(\mathcal{V})^T \leq 0$ が示せ $H^2(\mathcal{U}) = 0$ 。 $k \geq 3$ では

$\mathcal{V}(k, n') \supset \mathcal{O} = \mathcal{V}(k-1, n'+1)$ とすると

$$\begin{aligned} \dim C^2(\mathcal{V})^T / C^2(\mathcal{O})^T &= \# \left\{ \begin{matrix} (X_{ik}, Y_{kj}), (X_{ij}, Y_{kj}) \\ (X_{ik}, b_k) \end{matrix} \right\}_{\substack{1 \leq i \leq k-1 \\ k+1 \leq j \leq n'+k}} \\ &= (k-1)(2n'+1) \end{aligned}$$

$$\dim \mathcal{V} / \mathcal{O} = 2n' + 1$$

$$\begin{aligned} (\mathcal{O} \text{ の } \dots \text{ rank}) &\geq \# \left\{ \begin{matrix} (X_{i, k-1}, X_{k-1, k}, Y_{kj}) \\ (X_{i, k-1}, X_{k-1, j}, Y_{kj}) \end{matrix} \right\}_{\substack{1 \leq i \leq k-2 \\ k+1 \leq j \leq n'+k}} \\ &\quad + \# \{ (X_{i, k-1}, X_{k-1, k}, b_k) \}_{1 \leq i \leq k-2} \\ &\geq (k-2)(2n'+1) \end{aligned}$$

これら3式と $H^2(\mathcal{O})^T = 0$ より 例1 と同様に $H^2(\mathcal{U}) = 0$