

# Stable transition layers in a balanced bistable equation with degeneracy

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## 1 Introduction and Main Results

In this paper, we consider steady-state solutions for the following problem:

$$\begin{cases} u_t - \varepsilon^2 u_{xx} = f(x, u), & (x, t) \in (0, 1) \times (0, \infty), \\ u_x(0, t) = u_x(1, t) = 0, & t \in (0, \infty), \\ u(x, 0) = u_0(x), & x \in (0, 1), \end{cases}$$

where  $\varepsilon$  is a positive number and  $f(x, u)$  is given by

$$f(x, u) = -u(u - \alpha(x))(u + \alpha(x)).$$

Here  $\alpha : [0, 1] \rightarrow \mathbb{R}$  is a positive  $C^1$  function and a  $C^2$  function except for a finite number of points on  $[0, 1]$ . Such  $f(x, u)$  is a typical example of the so-called bistable nonlinearity and we note that  $f(x, u)$  satisfies that

$$\int_{-\alpha(x)}^{\alpha(x)} f(x, u) du = 0.$$

In this sense, we call the bistable function  $f$  to be *balanced*.

Since we are interested in the stationary problem, we consider the following problem:

$$(P_\varepsilon) \begin{cases} -\varepsilon^2 u_{xx} = f(x, u) & \text{in } (0, 1), \\ u_x(0) = u_x(1) = 0. \end{cases}$$

It is easily shown that there exist stable solutions  $u_\varepsilon^{(+)}$ ,  $u_\varepsilon^{(-)}$  for  $(P_\varepsilon)$  such that  $\lim_{\varepsilon \rightarrow 0} u_\varepsilon^{(+)}(x) = \alpha(x)$ ,  $\lim_{\varepsilon \rightarrow 0} u_\varepsilon^{(-)}(x) = -\alpha(x)$  uniformly in  $x \in [0, 1]$  (see [7, Proposition 2.2]). The aim of this paper is to find stable solutions  $u_\varepsilon$  with transition layers.

Nakashima [7] has studied the problem  $(P_\varepsilon)$  when  $\alpha$  is smooth and nondegenerate, i.e.  $\alpha'' \neq 0$  at each local minimum of  $\alpha$ . In this paper we consider the case that  $\alpha$  degenerates on an interval  $I$  of positive measure where  $\alpha$  takes its local minimum, that is,  $\alpha'(x) = 0$  on  $I$ . Nakashima and Tanaka [9] also have studied such a degenerate case and obtain solutions with a single layer and multi-layers (clustering layers) by using a variational method. However the stability of these solutions were not discussed. In this paper we obtain a stable solution with transition layers by a sub-supersolution method of Brezis and Nirenberg type (see [2]) and precise profile of the solution near the interval where  $\alpha$  degenerates by using a blow up argument inspired by the arguments in Dancer and Shusen Yan [3].

Now we state precise conditions on  $\alpha$ .

**Conditions .** (C1)  $\alpha$  is a positive function on  $[0, 1]$  and  $\alpha \in C^1[0, 1]$ .

(C2) There exist a finite number of points  $x_1, x_2, \dots, x_{2m} \in (0, 1)$  ( $m \geq 1$ ) such that

(i)  $\alpha'(x) = 0$  on  $I_i := [x_{2i-1}, x_{2i}]$  for  $i = 1, \dots, m$ ;

(ii)  $\alpha \in C^2((x_{2i}, x_{2i+1}))$  for each  $i = 0, 1, \dots, m$ , and there exist limits

$$\alpha''(x_{2i-1} - 0) = \lim_{h \downarrow 0} \frac{\alpha'(x_{2i-1}) - \alpha'(x_{2i-1} - h)}{h}$$

and

$$\alpha''(x_{2i} + 0) = \lim_{h \downarrow 0} \frac{\alpha'(x_{2i-1} + h) - \alpha'(x_{2i})}{h}$$

for each  $i = 1, \dots, m$ ;

(iii)  $\alpha''(x_{2i-1} - 0) > 0$  and  $\alpha''(x_{2i} + 0) > 0$  for  $i = 1, \dots, m$ .

Hereafter we denote  $\alpha''(x_{2i-1})$ ,  $\alpha''(x_{2i})$  instead of  $\alpha''(x_{2i-1} - 0)$ ,  $\alpha''(x_{2i} + 0)$ .

**Remark .** The condition (i) of (C2) implies that  $\alpha(x) = \text{const.}$  on  $I_i$  and if  $x_{2i-1} = x_{2i}$  for  $i = 1, \dots, m$ , this is the case as in Nakashima [7].

We set  $L = \{x_1, x_2, \dots, x_{2m}\}$  and  $K = \{I_1, I_2, \dots, I_m\}$ . We choose any subset  $\hat{K}$  of  $K$ . We denote  $\hat{K} = \{\hat{I}_1, \hat{I}_2, \dots, \hat{I}_l\}$  with  $1 \leq l \leq m$  and  $\hat{I}_i = [x'_{2i-1}, x'_{2i}]$  for each  $i = 1, \dots, l$  where we use the notation  $x'_0 = 0$ ,  $x'_{2l+1} = 1$ . We consider the following two cases:

$$\begin{aligned} \text{(I)} \quad \Omega_1 &= \bigcup_{i=0}^{\lfloor \frac{2l-1}{2} \rfloor} (x'_{4i}, x'_{4i+1}), \quad \Omega_2 = \bigcup_{i=0}^{\lfloor \frac{2l-3}{2} \rfloor} (x'_{4i+2}, x'_{4i+3}) \\ \text{(II)} \quad \Omega_1 &= \bigcup_{i=0}^{\lfloor \frac{2l-3}{2} \rfloor} (x'_{4i+2}, x'_{4i+3}), \quad \Omega_2 = \bigcup_{i=0}^{\lfloor \frac{2l-1}{2} \rfloor} (x'_{4i}, x'_{4i+1}) \end{aligned}$$

and we set

$$\Omega_i^\delta = \{x \in (0, 1) | \text{dist}(x, \partial\Omega_i \setminus \{0, 1\}) > \delta\}.$$

First, we construct a solution to  $(P_\varepsilon)$  that may have transition layers.

**Theorem 1.1.** Assume that (C1) and (C2) hold. Then for sufficiently small  $\varepsilon > 0$ , there exists a family of stable solutions  $\{u_\varepsilon\}$  of  $(P_\varepsilon)$  such that

$$\begin{aligned} |-\alpha(x) - u_\varepsilon(x)| &< \sigma \text{ in } \Omega_1^\delta, \\ |\alpha(x) - u_\varepsilon(x)| &< \sigma \text{ in } \Omega_2^\delta, \end{aligned}$$

where  $\sigma = \sigma(\varepsilon) = o_\varepsilon(1)$ ,  $\delta = \delta(\varepsilon) = o_\varepsilon(1)$ .

Moreover  $u_\varepsilon$  is a local minimizer of the functional

$$J_\varepsilon(u) = \int_0^1 \frac{\varepsilon^2}{2} |u_x|^2 - F(x, u) dx,$$

where  $F(x, u) = \int_0^u f(x, s) ds$ .

Next theorem describes the precise profile of  $u_\varepsilon$  near the intervals where  $\alpha$  degenerates.

**Theorem 1.2.** Consider the case (I). Let  $u_\varepsilon$  be the solution of  $(P_\varepsilon)$  obtained in Theorem 1.1. Then  $u_\varepsilon$  has exactly one layer in  $[x'_{2i-1} - 2\varepsilon^{1-\rho}, x'_{2i} + 2\varepsilon^{1-\rho}]$  for any small  $0 < \rho < 1$  and for each  $i = 1, 2, \dots, l$ . That is for any small  $\eta > 0$ , there exists  $\varepsilon_0 > 0$ , such that for any  $\varepsilon \in (0, \varepsilon_0]$ , the followings hold.

- (1) For each  $i = 1, 2, \dots, l$ , there exists the unique pair of numbers  $\{t_{\varepsilon,1,i}, t_{\varepsilon,2,i}\}$  such that  $x'_{2i-1} - 2\varepsilon^{1-\rho} < t_{\varepsilon,1,i} < t_{\varepsilon,2,i} < x'_{2i} + 2\varepsilon^{1-\rho}$  and the followings hold.

(a) If  $i$  is odd number, the followings hold;

$$\begin{cases} u_{\varepsilon} < -\bar{\alpha}_i + \eta \text{ on } [x'_{2i-1} - 2\varepsilon^{1-\rho}, t_{\varepsilon,1,i}), \\ u_{\varepsilon}(t_{\varepsilon,1,i}) = -\bar{\alpha}_i + \eta, \\ u_{\varepsilon}(t_{\varepsilon,2,i}) = \bar{\alpha}_i - \eta, \\ u_{\varepsilon} > \bar{\alpha}_i - \eta \text{ on } (t_{\varepsilon,2,i}, x'_{2i} + 2\varepsilon^{1-\rho}]. \end{cases}$$

(b) If  $i$  is even number, the followings hold;

$$\begin{cases} u_{\varepsilon} > \bar{\alpha}_i - \eta \text{ on } [x'_{2i-1} - 2\varepsilon^{1-\rho}, t_{\varepsilon,1,i}), \\ u_{\varepsilon}(t_{\varepsilon,1,i}) = \bar{\alpha}_i - \eta, \\ u_{\varepsilon}(t_{\varepsilon,2,i}) = -\bar{\alpha}_i + \eta, \\ u_{\varepsilon} < -\bar{\alpha}_i + \eta \text{ on } (t_{\varepsilon,2,i}, x'_{2i} + 2\varepsilon^{1-\rho}]. \end{cases}$$

Here  $\bar{\alpha}_i = \alpha(x_{2i-1}) = \alpha(x_{2i})$ .

- (2) If  $i$  is odd number,  $u_{\varepsilon}$  is increasing on  $(t_{\varepsilon,1,i}, t_{\varepsilon,2,i})$  and if  $i$  is even number,  $u_{\varepsilon}$  is decreasing on  $(t_{\varepsilon,1,i}, t_{\varepsilon,2,i})$ .

- (3)  $0 < R_1 \leq \frac{t_{\varepsilon,2,i} - t_{\varepsilon,1,i}}{\varepsilon} \leq R_2$ , where  $R_1$  and  $R_2$  are two constants independent of  $\varepsilon > 0$ .

**Remark .** If we take the case (II), the statement (a) of (1) holds if  $i$  is odd number and statement (b) of (1) holds if  $i$  is even number. And if  $i$  is odd number,  $u_{\varepsilon}$  is increasing on  $(t_{\varepsilon,1,i}, t_{\varepsilon,2,i})$  and if  $i$  is even number,  $u_{\varepsilon}$  is decreasing on  $(t_{\varepsilon,1,i}, t_{\varepsilon,2,i})$ .

**Remark .** Since  $\{t_{\varepsilon,1,i}\}_{0 < \varepsilon < \varepsilon_0}$  and  $\{t_{\varepsilon,2,i}\}_{0 < \varepsilon < \varepsilon_0}$  are bounded sequences, from the part (3) of Theorem 1.2, we may assume that there exists  $t_i \in [x'_{2i-1}, x'_{2i}]$  such that  $t_{\varepsilon,1,i}, t_{\varepsilon,2,i} \rightarrow t_i$  as  $\varepsilon \rightarrow 0$ . But the exact location of  $t_i$  is not yet known when  $x'_{2i} - x'_{2i-1} > 0$  and this is an open problem.

This paper is organized as follows. To prove Theorem 1.1 and 1.2, we take a sub-supersolution method of Brezis and Nirenberg Type ([2]). Hence in section 2, we prepare the sub-supersolution method. In section 3 we construct a subsolution and a supersolution and prove Theorem 1.1. In section 4 we prove Theorem 1.2.

## 2 Preliminaries

In this section we prepare the sub-supersolution method of Brezis and Nirenberg type under the Neumann boundary condition. In [2], Brezis and Nirenberg developed such method under the Dirichlet boundary condition. Nacimento [6] pointed out their method also works under the Neumann boundary condition without a precise proof. Now we give the definition of a subsolution and a supersolution in the form suitable for our problem.

Consider the following problem:

$$\begin{cases} u_{xx} + g(x, u) = 0, & 0 < x < 1, \\ u_x(0) = u_x(1) = 0, \end{cases} \quad (2.1)$$

where  $g(x, s)$  is a  $C^1$  function with respect to  $(x, s)$  and we assume the following growth condition

$$|g(x, s)| \leq C(1 + |s|^p)$$

for some  $1 < p < +\infty$  and for some  $C > 0$ . Moreover we assume that for some  $k \geq 0$  the function  $g(x, s) + ks$  is nondecreasing in  $s$  for each  $x$ . We remark that  $\bar{g}(x, s) := g(x, s) + s$  also satisfies

$$\begin{aligned} |\bar{g}(x, s)| &\leq C(1 + |s|^p) + |s| \\ &\leq C(1 + |s|^p) + \frac{|s|^p}{p} + \frac{1^q}{q} \\ &\leq C'(1 + |s|^p) \end{aligned}$$

for some  $C' > 0$ . We define the subsolution and the supersolution for (2.1) as follows.

**Definition 2.1.** Let  $u^*$  (resp.  $u_*$ ) :  $[0, 1] \rightarrow \mathbb{R}$  be a continuous function. The function  $u^*$  (resp.  $u_*$ ) is called a *supersolution* (resp. *subsolution*) of (2.1) if

(S1) there exists  $\delta_0 > 0$  such that  $u^*$  (resp.  $u_*$ )  $\in C^2((0, \delta_0) \cup (1 - \delta_0, 1)) \cap C^1([0, \delta_0] \cup [1 - \delta_0, 1])$ ,

(S2) for all  $\varphi \in C_0^\infty(0, 1)$  with  $\varphi \geq 0$ , we have

$$\begin{aligned} &\int_0^1 (-u^* \varphi_{xx} - g(x, u^*) \varphi) dx \geq 0 \\ &\left( \text{resp. } \int_0^1 (-u_* \varphi_{xx} - g(x, u_*) \varphi) dx \leq 0 \right), \end{aligned} \quad (2.2)$$

(S3)  $u_x^*(0) \leq 0$  and  $u_x^*(1) \geq 0$  (resp.  $u_{*x}(0) \geq 0$  and  $u_{*x}(1) \leq 0$ ).

Before we state the existence of a solution to (2.1), we have to define the energy functional  $I$  of (2.1):

$$I(u) = \int_0^1 \frac{1}{2} |u_x|^2 - G(x, u) dx, \quad G(x, u) = \int_0^u g(x, s) ds.$$

Note that if we define

$$\bar{G}(x, u) = \int_0^u \bar{g}(x, s) ds$$

the energy functional  $I$  can be written as follows

$$I(u) = \int_0^1 \frac{1}{2} |u_x|^2 + \frac{1}{2} u^2 - \bar{G}(x, u) dx.$$

The next proposition is the existence result of a solution to (2.1) between a subsolution and a supersolution for the Neumann boundary condition.

**Proposition 2.2.** *If there exists a supersolution  $u^*$  to (2.1) and a subsolution  $u_*$  to (2.1) with  $u_* < u^*$  and neither  $u_*$  nor  $u^*$  is a solution of (2.1). Then there exists a solution  $u_0$  to (2.1) such that  $u_* \leq u_0 \leq u^*$  and  $u_0$  is a local minimizer of  $I$  on  $H^1(0, 1)$ . Moreover  $u_0$  is a global minimizer of the following functional  $\tilde{I}$ :*

$$\tilde{I}(u) = \int_0^1 \frac{1}{2} |u_x|^2 - \tilde{G}(x, u) dx, \quad \tilde{G}(x, u) = \int_0^u \tilde{g}(x, s) ds$$

where

$$\tilde{g}(x, s) = \begin{cases} g(x, u_*(x)), & s < u_*(x), \\ g(x, s), & u_*(x) \leq s \leq u^*(x), \\ g(x, u^*(x)), & u^*(x) < s. \end{cases}$$

We need some lemmas to prove Proposition 2.2.

**Lemma 2.3.** (cf.[2, Theorem 1]) Assume that  $u_0 \in H^1(0, 1)$  is a local minimizer of  $I$  in the  $C^1$  topology; this means that there exists some  $r > 0$  such that

$$I(u_0) \leq I(u_0 + v) \text{ for } v \in C^1[0, 1] \text{ with } \|v\|_{C^1} \leq r. \quad (2.3)$$

Then  $u_0$  is a local minimizer of  $I$  in the  $H^1$  topology; i.e. there exists  $\varepsilon_0 > 0$  such that

$$I(u_0) \leq I(u_0 + v) \text{ for } v \in H^1(0, 1) \text{ with } \|v\|_{H^1} \leq \varepsilon_0.$$

*Proof.* The proof is divided in 2 steps.

**Step 1.** We claim that  $u_0 \in C^{1,\gamma}[0, 1]$  for all  $0 < \gamma < 1$ .

Recall that  $u_0$  is a weak solution of

$$\begin{cases} -u_0'' = g(x, u_0), & \text{in } (0, 1), \\ u_0'(0) = u_0'(1) = 0, \end{cases}$$

where ' denote the derivative in  $x$ . First by the Sobolev imbedding we have  $u_0 \in C[0, 1]$  and hence  $u_0 \in L^p(0, 1)$  for any  $1 < p < \infty$ . Next by the standard regularity result, we have  $u_0 \in W^{2,p}(0, 1)$  for any  $1 < p < \infty$ . Again by the Sobolev imbedding, we have  $u_0 \in C^{1,\gamma}[0, 1]$  for any  $0 < \gamma < 1$ .

**Step 2.** Without loss of generality we may now assume that  $u_0 = 0$ . Suppose that the conclusion does not hold. Then for every  $\varepsilon > 0$ , there exists  $v_\varepsilon \in B_\varepsilon(0) := \{w \in H^1(0, 1) \mid \|w\|_{H^1} \leq \varepsilon\}$  such that

$$I(v_\varepsilon) < I(0). \quad (2.4)$$

By a standard lower semicontinuity argument  $\min_{B_\varepsilon} I$  is attained at some point which we may still denote by  $v_\varepsilon$ . We shall prove that  $v_\varepsilon \rightarrow 0$  in  $C^1$  and this contradict to (2.3) and (2.4). The corresponding Euler equation for  $v_\varepsilon$  involves a Lagrange multiplier  $\mu_\varepsilon \leq 0$ , namely,  $v_\varepsilon$  satisfies

$$\langle DI(v_\varepsilon), \zeta \rangle_{(H^1)^*, H^1} = \mu_\varepsilon \langle v_\varepsilon, \zeta \rangle_{H^1} \text{ for } \zeta \in H^1(0, 1)$$

i.e.

$$\int_0^1 v_\varepsilon' \zeta' + v_\varepsilon \zeta - \bar{g}(x, v_\varepsilon) \zeta dx = \mu_\varepsilon \int_0^1 v_\varepsilon' \zeta' + v_\varepsilon \zeta dx \text{ for } \zeta \in H^1(0, 1),$$

where  $DI(v_\varepsilon)$  denotes the Frechet derivative of  $I$  at  $v_\varepsilon$ . This means

$$-v_\varepsilon'' + v_\varepsilon = \frac{1}{1 - \mu_\varepsilon} \bar{g}(x, v_\varepsilon). \quad (2.5)$$

Using (2.5) together with the remark  $|\bar{g}(x, u)| \leq C(1 + |u|^p)$  and the essential fact  $\mu_\varepsilon \leq 0$ , one may obtain from the bound  $\|v\|_{H^1} \leq C$  to  $\|v\|_{C^{1,\gamma}} \leq C$  by the bootstrap argument as in step 1 (independent of  $\varepsilon > 0$ ). Since  $v_\varepsilon \rightarrow 0$  in  $H^1$ ,  $v_\varepsilon \rightarrow 0$  in  $C^1$ . The proof is completed.  $\square$

Next lemma is due to [2].

**Lemma 2.4.** ([2, Theorem 2]) Let  $u \in L_{loc}^1(0, 1)$  and assume that for some  $k \geq 0$ ,  $u$  satisfies

$$\begin{cases} -u'' + ku \geq 0 & \text{in } \mathcal{D}'(0, 1), \\ u \geq 0 & \text{in } (0, 1). \end{cases}$$

Then either  $u \equiv 0$ , or there exists  $\varepsilon > 0$  such that

$$u(x) \geq \varepsilon \text{dist}(x, \partial(0, 1)).$$

*Proof.* See [2]. □

Next lemma is the strong maximum principle for a subharmonic function in the distribution sense.

**Lemma 2.5.** *Let  $a < b$  and  $u \in C[a, b]$  be a subharmonic function, i.e.*

$$\int_a^b -u\varphi'' dx \leq 0$$

*for  $\varphi \in C_0^\infty(a, b)$  with  $\varphi \geq 0$ . Then if  $u$  is not a constant function, the maximum of  $u$  over  $[a, b]$  is attained at  $x = a$  or  $x = b$ . Moreover we have*

$$u(x) < \max_{[a, b]} u = \max\{u(a), u(b)\} \quad \text{for } x \in (a, b).$$

*Proof.* See for example [4]. □

Now we are ready to prove Proposition 2.2.

*Proof of Proposition 2.2.* Let  $u_0$  be a minimizer of  $\tilde{I}$  on  $H^1(0, 1)$ , it is easily seen the minimum is achieved and satisfies

$$-u_0'' = \tilde{g}(x, u_0) \quad \text{in } (0, 1).$$

By the bootstrap argument, we have that  $u_0 \in C^{1,\gamma}[0, 1]$ . We claim that  $u_* \leq u_0 \leq u^*$ . We will just prove the  $u_* \leq u_0$ . Set  $A = \{x \in (0, 1) | u_0(x) < u_*(x)\}$  and we will show  $A = \emptyset$ . First we have

$$-(u_* - u_0)'' \leq g(x, u_*) - \tilde{g}(x, u_0) \quad (2.6)$$

and in particular

$$-(u_* - u_0)'' \leq 0 \quad \text{in } \mathcal{D}'(A).$$

First we assume that  $u_* - u_0 \leq 0$  at  $x = 0, 1$ . Then  $A \subset (0, 1)$  and since  $u_* - u_0 \leq 0$  on  $\partial A$ , it follows from Lemma 2.5 that  $u_* - u_0 \leq 0$  in  $A$ . Hence we can conclude  $A = \emptyset$ . Next we prove that  $u_* - u_0 \leq 0$  at  $x = 0, 1$ . Let us assume that  $u_*(0) - u_0(0) > 0$  and  $u_*(1) - u_0(1) \leq 0$ . Similarly  $w := u_* - u_0$  satisfies

$$-w'' \leq 0 \quad \text{in } \mathcal{D}'(A)$$

and  $w$  attain its strict maximum on  $x = 0$  by Lemma 2.5. Note that since  $u_*$  is not solution for (2.1),  $w$  is not constant. Indeed if  $w$  is constant, i.e.,  $u_* - u_0 = C$  for some constant  $C > 0$ , then  $u_*$  satisfies that

$$-u_*'' = -u_0'' = \tilde{g}(x, u_* - C) = g(x, u_*)$$

and this contradict to the assumption that  $u_*$  is not a solution to (2.2). Since

$$-u_0'' = g(x, u_*(x))$$

and  $u_*$  is  $C^2$  near  $x = 0$  and  $g$  is  $C^1$ , we have  $u_0 \in C^2$  and  $w = u_* - u_0$  is  $C^2$  near  $x = 0$ . Moreover  $w$  satisfies

$$\begin{aligned} -w'' &\leq 0 \quad \text{for } x > 0 \text{ small,} \\ w(0) &> w(x) \quad \text{for } x > 0 \text{ small.} \end{aligned}$$

Hence by Hopf's Lemma we have  $w'(0) = u_*'(0) - u_0'(0) < 0$  and this contradict to the (S3) in the Definition 2.1. Similarly we can obtain the contradiction if we assume that  $u_*(0) - u_0(0) > 0$  and  $u_*(1) - u_0(1) > 0$  or  $u_*(0) - u_0(0) \leq 0$  and  $u_*(1) - u_0(1) > 0$ .

Returning to (2.6) we have

$$-(u_* - u_0)'' + k(u_* - u_0) \leq (g(x, u_*) + ku_*) - (g(x, u_0) + ku_0) \leq 0.$$

Since  $u_*$  is not a solution, it follows from Proposition 2.4 that there is some  $\varepsilon > 0$  such that

$$u_*(x) - u_0(x) \leq -\varepsilon \text{dist}(x, \partial(0, 1)) \quad \text{for } x \in (0, 1).$$

Similarly for  $u^*$  we have

$$u_*(x) + \varepsilon \text{dist}(x, \partial(0, 1)) \leq u_0(x) \leq u^*(x) - \varepsilon \text{dist}(x, \partial(0, 1)) \quad \text{for } x \in (0, 1).$$

It follows that if  $u \in C^1[0, 1]$  and  $\|u - u_0\|_{C^1} \leq \varepsilon$  then

$$u_* \leq u \leq u^* \quad \text{in } (0, 1).$$

By the remark following this proof,  $I(u) - \tilde{I}(u)$  is constant for  $\|u - u_0\|_{C^1} \leq \varepsilon$ . Hence  $u_0$  is a local minimizer for  $I$  in the  $C^1$  topology (since it is global minimizer for  $\tilde{I}$ ). Now, we invoke Proposition 2.3 to claim that  $u_0$  is also a local minimizer of  $I$  in the  $H^1$  topology. This completes the proof of Proposition 2.2.  $\square$

**Remark .** If we take a function  $u \in H^1(0, 1)$  satisfies  $u_* \leq u \leq u^*$ , we have

$$\begin{aligned} \tilde{G}(x, u) &= \int_0^u \tilde{g}(x, s) ds \\ &= \int_{u_*}^u \tilde{g}(x, s) ds + \int_0^{u_*} \tilde{g}(x, s) ds \\ &= \int_{u_*}^u g(x, s) ds + \int_0^{u_*} \tilde{g}(x, s) ds \\ &= G(x, u) - G(x, u_*(x)) + \tilde{G}(x, u_*(x)). \end{aligned}$$

Thus the functional  $\tilde{I}$  is

$$\begin{aligned} \tilde{I}(u) &= \int_0^1 \frac{1}{2} |u_x|^2 - \tilde{G}(x, u) dx \\ &= \int_0^1 \frac{1}{2} |u_x|^2 - G(x, u) dx + \text{const.} \end{aligned}$$

and we can replace  $I$  by  $\tilde{I}$  for the function  $u \in H^1(0, 1)$  satisfies  $u_* \leq u \leq u^*$ .

Next we give the sufficient condition for functions becoming subsolutions and supersolutions. This condition is due to Nakashima [7]. First we state a notation.

Let  $u : [0, 1] \rightarrow \mathbb{R}$  be a continuous function and  $u \in C^1([0, \delta_0] \cup (1 - \delta_0, 1])$  for some  $\delta_0 > 0$  such that for a finite number of points  $a_1, a_2, \dots, a_k \in (0, 1)$

(i)  $u : [0, 1] \rightarrow \mathbb{R}$  is class  $C^2$  in  $(a_0, a_1) \cup (a_1, a_2) \cup \dots \cup (a_k, a_{k+1})$  with  $a_0 = 0, a_{k+1} = 1$ .

(ii) There exists  $\lim_{x \rightarrow a_i + 0} u_x(x), \lim_{x \rightarrow a_i - 0} u_x(x)$  for each  $i = 1, 2, \dots, k$ .

We denote  $P(a_1, a_2, \dots, a_k)$  the set of function  $u$  satisfies (i) and (ii).

**Proposition 2.6.** Let  $u^* \in P(a_1, a_2, \dots, a_k)$  satisfies the following conditions:

(S1)' For each  $i = 0, 1, \dots, k$

$$-u_{xx}^* - g(x, u^*) \geq 0 \text{ in } (a_i, a_{i+1}).$$

(S2)' For each  $i = 1, 2, \dots, k$

$$\lim_{x \rightarrow a_i + 0} u_x^*(x) \leq \lim_{x \rightarrow a_i - 0} u_x^*(x).$$

(S3)'  $u_x^*(0) \leq 0$  and  $u_x^*(1) \geq 0$ .

Then  $u^*$  is a supersolution for (2.1). If  $u_* \in P(a_1, a_2, \dots, a_k)$  satisfies (S1)', (S2)' and (S3)' which reversed the inequality sign, then  $u_*$  is a subsolution for (2.1).

*Proof.* It suffices to show that

$$\int_0^1 (-u^* \varphi_{xx} - g(x, u^*) \varphi) dx \geq 0$$

for any  $\varphi \in C_0^\infty(0, 1)$ , with  $\varphi \geq 0$ . Indeed, using the integration by parts, we have

$$\begin{aligned} & \int_0^1 (-u^* \varphi_{xx} - g(x, u^*) \varphi) dx \\ &= \sum_{i=0}^k \int_{a_i}^{a_{i+1}} (-u^* \varphi_{xx} - g(x, u^*) \varphi) dx \\ &= \sum_{i=0}^k [u_x^*(a_{i+1} - 0) \varphi(a_{i+1}) - u_x^*(a_i + 0) \varphi(a_i)] \\ & \quad + \sum_{i=0}^k \int_{a_i}^{a_{i+1}} (-u_{xx}^* - g(x, u^*)) \varphi dx \\ &= [u_x^*(1) \varphi(1) - u_x^*(0) \varphi(0)] + \sum_{i=1}^k [u_x^*(a_i - 0) - u_x^*(a_i + 0)] \varphi(a_i) \\ & \quad + \sum_{i=0}^k \int_{a_i}^{a_{i+1}} (-u_{xx}^* - g(x, u^*)) \varphi dx \geq 0. \end{aligned}$$

Here we have used the assumption (S1)', (S2)' and (S3)' and  $a_0 = 0$  and  $a_{k+1} = 1$ . □

Finally, we consider an auxiliary problem for each positive number  $\gamma$ :

$$u_{zz} + u(\gamma - u)(\gamma + u) = 0, u(-\infty) = -\gamma, u(+\infty) = \gamma. \quad (2.7)$$

By the phase plane method we can obtain some properties of the solution for (2.7).

**Lemma 2.7.** For each  $\gamma > 0$ , there exists a unique solution  $U(z; \gamma)$  of (2.7) with  $U(0; \gamma) = 0$ . Moreover, it has the following properties:

(1)  $\frac{d}{dz} U(z, \gamma) > 0$  for  $z \in \mathbb{R}$ .

(2) There exist positive constants  $C_1$  and  $C_2$  independent of  $\gamma$  such that

$$\begin{aligned} |U(z; \gamma) - \gamma| &< C_1 \gamma \exp(-C_2 \gamma z) \quad z \geq 0, \\ |U(z; \gamma) + \gamma| &< C_1 \gamma \exp(C_2 \gamma z) \quad z \leq 0. \end{aligned}$$



(3) There exist positive constants  $C_3$  and  $C_4$  independent of  $\gamma$  such that

$$|U'(z; \gamma)| < C_3 \gamma^2 \exp(-C_4 \gamma |z|) \quad z \in \mathbb{R}.$$

(4)  $\frac{d^2}{dz^2} U(z; \gamma) \geq 0$  for  $z \leq 0$  and  $\frac{d^2}{dz^2} U(z; \gamma) \leq 0$  for  $z \geq 0$ .

(5)  $U(z; \gamma) = \gamma U(\gamma z; 1)$ .

Although the following lemma is elementary, this is very important in our argument.

**Lemma 2.8.** Let  $\gamma > 0$  and  $0 < \eta < \gamma$  be fixed constants and  $v$  satisfies

$$\begin{cases} -v_{zz} = v(\gamma^2 - v^2) & \text{on } \mathbb{R}, \\ v(0) = -\gamma + \eta, \\ v(z) \leq -\gamma + \eta & \text{for } z \leq 0, \\ v \text{ is bounded on } \mathbb{R}. \end{cases}$$

Then  $v$  is a unique solution of

$$\begin{cases} -v_{zz} = v(\gamma^2 - v^2) & \text{on } \mathbb{R}, \\ v(0) = -\gamma + \eta, \\ v'(z) > 0 & z \in \mathbb{R}, \\ v(z) \rightarrow \pm\gamma & \text{as } z \rightarrow \pm\infty. \end{cases}$$

*Proof.* Since  $v$  is bounded, by using the phase plane analysis,  $v$  is a periodic solution or a unique heteroclinic solution joining  $-\gamma$  and  $\gamma$ . Since  $v(z) \leq -\gamma + \eta < 0$  for  $z \leq 0$ , we can conclude that  $v$  is the unique heteroclinic solution joining  $-\gamma$  and  $\gamma$ .  $\square$

Making use of Lemma 2.7, we will construct a supersolution and a subsolution for  $(P_\varepsilon)$  and we obtain a solution  $u_\varepsilon$  to  $(P_\varepsilon)$  by using Proposition 2.2 in the following sections.

### 3 Proof of Theorem 1.1

In this section we prove Theorem 1.1. Although the construction of a subsolution and a supersolution is almost same as in Nakashima [7], we give the proof of Theorem 1.1 in details for reader's convenience.

For the sake of simplicity, we first assume  $L = \{x_1, x_2\}$  and  $K = \hat{K} = \{I_1\}$ ; so that  $\alpha'(x) = 0$  for  $x \in [x_1, x_2]$  and  $\alpha''(x_1) > 0$ ,  $\alpha''(x_2) > 0$  and we set  $\alpha(x_1) = \alpha(x_2) = \bar{\alpha}$ . First we construct a subsolution and a supersolution in the case when  $\alpha'(0) \geq 0$  and  $\alpha'(1) \geq 0$ .

**Proposition 3.1.** Assume  $\alpha'(0) \geq 0$  and  $\alpha'(1) \geq 0$ . Let  $\sigma_1$  and  $\sigma_2$  be positive numbers satisfying  $\sigma_1 < 1$  and  $\sigma_1 < \sigma_2 < 2\sigma_1$ . For sufficiently small  $\varepsilon > 0$ , there exist  $\zeta_1 = \zeta_1(\varepsilon)$  and  $\zeta_2 = \zeta_2(\varepsilon)$  such that  $-2\varepsilon^{1-\sigma_1} < \zeta_1 < -\varepsilon^{1-\sigma_1} < \zeta_2 < 0$  and the following function

$$u_\varepsilon^*(x) = \begin{cases} -\alpha(x) + \varepsilon^{2-\sigma_2}, & 0 \leq x \leq x_1 + \zeta_1, \\ U(\varepsilon^{-1}(x - x_1) + \varepsilon^{-\sigma_1}; \alpha(x_1 - \varepsilon^{1-\sigma_1})), & x_1 + \zeta_1 \leq x \leq x_1 + \zeta_2, \\ \alpha(x) + \varepsilon^{2-\sigma_2}, & x_1 + \zeta_2 \leq x \leq 1 \end{cases}$$

is a supersolution solution for  $(P_\varepsilon)$ .

In the following proposition we give a subsolution for  $(P_\varepsilon)$ . From the condition (S3)' of Proposition 2.6 for subsolutions, we need slight modification near  $x = 0$  and 1.

**Proposition 3.2.** Assume that  $\alpha'(0) > 0$  and  $\alpha'(1) > 0$ . Let  $\sigma_1$  and  $\sigma_2$  be the same numbers as in Proposition 3.1 and let  $\sigma_3$  and  $\sigma_4$  be positive numbers satisfying  $\sigma_4 < \sigma_3 < 2\sigma_4$  and  $\sigma_3 < \sigma_2$ . For sufficiently small  $\varepsilon > 0$ , there exist  $\zeta_3 = \zeta_3(\varepsilon)$  and  $\zeta_4 = \zeta_4(\varepsilon)$  such that the following function  $u_{*,\varepsilon}$  is a subsolution for  $(P_\varepsilon)$ .

$$u_{*,\varepsilon}(x) = \begin{cases} -\alpha(x) - \varepsilon^{2-\sigma_2} - \varepsilon^{-\sigma_3}(x - \varepsilon^{\sigma_4})^2, & 0 \leq x \leq \varepsilon^{\sigma_4}, \\ -\alpha(x) - \varepsilon^{2-\sigma_2}, & \varepsilon^{\sigma_4} \leq x \leq x_2 + \zeta_3, \\ U(\varepsilon^{-1}(x - x_2) - \varepsilon^{-\sigma_1}; \alpha(x_2 + \varepsilon^{1-\sigma_1})), & x_2 + \zeta_3 \leq x \leq x_2 + \zeta_4, \\ \alpha(x) - \varepsilon^{2-\sigma_2}, & x_2 + \zeta_4 \leq x \leq 1 - \varepsilon^{\sigma_4}, \\ \alpha(x) - \varepsilon^{2-\sigma_2} - \varepsilon^{-\sigma_3}(x - 1 + \varepsilon^{\sigma_4})^2, & 1 - \varepsilon^{\sigma_4} \leq x \leq 1. \end{cases}$$

In Propositions 3.1 and 3.2 we should say that numbers  $\zeta_1$ ,  $\zeta_2$ ,  $\zeta_3$  and  $\zeta_4$  are chosen so that  $u_\varepsilon^* \in P(x_1 + \zeta_1, x_1 + \zeta_2)$  and  $u_{*,\varepsilon} \in P(x_2 + \zeta_3, x_2 + \zeta_4)$  satisfies the condition  $(S1)'$  and  $(S2)'$  of Proposition 2.6. More precisely,  $\zeta_1$  is determined from

$$\zeta_1 = \min \{ \zeta \in (-x_1, -\varepsilon^{1-\sigma_1}) : -\alpha(x_1 + \zeta) + \varepsilon^{2-\sigma_2} = U(\varepsilon^{-1}\zeta + \varepsilon^{-\sigma_1}; \alpha(x_1 - \varepsilon^{1-\sigma_1})) \} \quad (3.1)$$

and  $\zeta_2$  is a negative number satisfying  $\zeta_2 \in (-\varepsilon^{1-\sigma_1}, 0)$  and

$$\alpha(\zeta_2 + x_1) + \varepsilon^{2-\sigma_2} = U(\varepsilon^{-1}\zeta_2 + \varepsilon^{-\sigma_1}; \alpha(x_1 - \varepsilon^{1-\sigma_1})) \quad (3.2)$$

and  $\zeta_3$  is a positive number satisfying

$$-\alpha(x_2 + \zeta_3) + \varepsilon^{2-\sigma_2} = U(\varepsilon^{-1}\zeta_3 - \varepsilon^{\sigma_1}; \alpha(x_2 + \varepsilon^{1-\sigma_1})), \quad (3.3)$$

and  $\zeta_4$  is satisfying  $\zeta_4 \in (\varepsilon^{1-\sigma_1}, 1 - x_2)$  and

$$\zeta_4 = \max \{ \zeta \in (\varepsilon^{1-\sigma_1}, 1 - x_2) : \alpha(x_2 + \zeta) - \varepsilon^{2-\sigma_2} = U(\varepsilon^{-1}\zeta - \varepsilon^{-\sigma_1}; \alpha(x_2 + \varepsilon^{1-\sigma_1})) \}. \quad (3.4)$$

The following two lemmas assure the unique existence of such  $\zeta_1$ ,  $\zeta_2$ ,  $\zeta_3$  and  $\zeta_4$ .

**Lemma 3.3.** For sufficiently small  $\varepsilon > 0$ ,  $\zeta_1$  and  $\zeta_4$  are uniquely determined from (3.1) and (3.4). Moreover, they satisfy;

- (1) (i)  $-\varepsilon^{1+\sigma_1-\sigma_2-\delta(\varepsilon)} < \zeta_1 + \varepsilon^{1-\sigma_1} < -\varepsilon^{1+\sigma_1-\sigma_2+\delta(\varepsilon)}$ ,  
(ii)  $\varepsilon^{1+\sigma_1-\sigma_2-\delta(\varepsilon)} < \zeta_4 - \varepsilon^{1-\sigma_1} < \varepsilon^{1+\sigma_1-\sigma_2+\delta(\varepsilon)}$ ,

where  $\delta$  is a positive number such that  $\delta = \delta(\varepsilon) = o_\varepsilon(1)$ .

- (2) (i)  $-\alpha'(\zeta_1 + x_1) \geq \varepsilon^{-1}U_z(\varepsilon^{-1}\zeta_1 + \varepsilon^{-\sigma_1}; \alpha(x_1 - \varepsilon^{1-\sigma_1}))$ .  
(ii)  $\alpha'(x_2 + \zeta_4) \leq \varepsilon^{-1}U_z(\varepsilon^{-1}\zeta_4 - \varepsilon^{-\sigma_1}; \alpha(x_2 + \varepsilon^{1-\sigma_1}))$ .

*Proof.* We only prove for  $\zeta_1$ . Set  $F_1(x) := U(\varepsilon^{-1}(x - x_1) + \varepsilon^{-\sigma_1}; \alpha(x_1 - \varepsilon^{1-\sigma_1})) + \alpha(x) - \varepsilon^{2-\sigma_2}$ . It follows from (4) of Lemma 2.7 that  $U(\varepsilon^{-1}(x - x_1) + \varepsilon^{-\sigma_1}; \alpha(x_1 - \varepsilon^{1-\sigma_1}))$  is convex for  $x \leq x_1 - \varepsilon^{1-\sigma_1}$ , so that  $F_1(x)$  is also convex in  $(x_1 - 2\varepsilon^{1-\sigma_1}, x_1 - \varepsilon^{1-\sigma_1})$ . Therefore,  $F_1(x)$  has at most two zero-points in  $(x_1 - 2\varepsilon^{1-\sigma_1}, x_1 - \varepsilon^{1-\sigma_1})$ .

We will prove that a zero-point of  $F_1(x)$  indeed exists. Let  $k$  be any number such that  $k < \sigma_2 - \sigma_1$  and  $|k| < 2\sigma_1 - \sigma_2$ . Setting  $x_k = x_1 - \varepsilon^{1-\sigma_1} - \varepsilon^{1+\sigma_1-\sigma_2+k}$ , we get  $F_1(x_k) = U(-\varepsilon^{\sigma_1-\sigma_2+k}; \alpha(x_1 - \varepsilon^{1-\sigma_1})) + \alpha(x_k) - \varepsilon^{2-\sigma_2}$ . Using the Taylor expansion of  $\alpha$  extended for  $x > x_1$  in  $C^2$  at  $x_1$  we obtain

$$\begin{aligned} \alpha(x_1 - \varepsilon^{1-\sigma_1}) &= \alpha(x_1) - \alpha'(x_1)\varepsilon^{1-\sigma_1} + \frac{\alpha''(x_1)}{2}\varepsilon^{2-2\sigma_1} + o(\varepsilon^{2-2\sigma_1}) \\ &= \bar{\alpha} + \frac{\alpha''(x_1)}{2}\varepsilon^{2-2\sigma_1} + o(\varepsilon^{2-2\sigma_1}). \end{aligned}$$

If  $\varepsilon > 0$  is sufficiently small, then  $\varepsilon^{\sigma_1 - \sigma_2 + k}$  becomes large; so that (2) and (5) of Lemma 2.7 implies

$$\begin{aligned}
 & U(-\varepsilon^{\sigma_1 - \sigma_2 + k}; \alpha(x_1 - \varepsilon^{1 - \sigma_1})) \\
 = & \alpha(x_1 - \varepsilon^{1 - \sigma_1}) U(-\alpha(x_1 - \varepsilon^{1 - \sigma_1}) \varepsilon^{\sigma_1 - \sigma_2 + k}; 1) \\
 = & \alpha(x_1 - \varepsilon^{1 - \sigma_1}) U\left(-\left(\bar{\alpha} + \frac{\alpha''(x_1)}{2} \varepsilon^{2 - 2\sigma_1} + o(\varepsilon^{2 - 2\sigma_1})\right) \varepsilon^{\sigma_1 - \sigma_2 + k}; 1\right) \\
 = & \alpha(x_1 - \varepsilon^{1 - \sigma_1}) (-1 + o(\varepsilon^2)).
 \end{aligned}$$

Hence we have

$$F_1(x_k) = -\alpha(x_1 - \varepsilon^{1 - \sigma_1}) + \alpha(x_1 - \varepsilon^{1 - \sigma_1} - \varepsilon^{1 + \sigma_1 - \sigma_2 + k}) - \varepsilon^{2 - \sigma_2} + o(\varepsilon^2).$$

Using the Taylor expansion of  $\alpha$  at  $x = x_1 - \varepsilon^{1 - \sigma_1}$  we can show

$$F_1(x_k) = -\alpha'(x_1 - \varepsilon^{1 - \sigma_1}) \varepsilon^{1 + \sigma_1 - \sigma_2 + k} - \varepsilon^{2 - \sigma_2} + o(\varepsilon^{2 + 2\sigma_1 - 2\sigma_2 + 2k}).$$

Here we should note

$$-\alpha'(x_1 - \varepsilon^{1 - \sigma_1}) \varepsilon^{1 + \sigma_1 - \sigma_2 + k} = \alpha''(x_1) \varepsilon^{2 - \sigma_2 + k} + o(\varepsilon^{2 - \sigma_2 + k}).$$

Therefore, in view of  $|k| < 2\sigma_1 - \sigma_2$ , one can deduce

$$F_1(x_k) = \alpha''(x_1) \varepsilon^{2 - \sigma_2 + k} - \varepsilon^{2 - \sigma_2} + o(\varepsilon^{2 - \sigma_2 + k}).$$

Let  $k > 0$  be fixed. Since  $\alpha''(x_1) > 0$ , it is easy to see that

$$F_1(x_k) < 0 \text{ and } F_1(x_{-k}) > 0 \quad (3.5)$$

with  $x_{-k} < x_k$ . Hence we can find  $\zeta_1$  such that  $F_1(x_1 + \zeta_1) = 0$  and  $x_1 + \zeta_1 \in (x_{-k}, x_k)$ . From (3.5), we can see that  $\frac{dF_1}{dx}(\zeta_1 + x_1) \leq 0$ , which means that  $x_1 + \zeta_1$  is the smallest zero-point of  $F_1$  in  $(x_1 - 2\varepsilon^{1 - \sigma_1}, x_1 - \varepsilon^{1 - \sigma_1})$ . Clearly  $\zeta_1$  satisfies (2). The proof of Lemma 3.3 is completed.  $\square$

**Lemma 3.4.** For sufficiently small  $\varepsilon > 0$ , there exist a unique  $\zeta_2 \in (-\varepsilon^{1 - \sigma_1}, 0)$  and a unique  $\zeta_3 \in (0, \varepsilon^{1 - \sigma_1})$  satisfying (3.2) and (3.3).

*Proof.* We only prove for  $\zeta_2$ . Setting  $F_2(x) = \alpha(x) + \varepsilon^{2 - \sigma_2} - U(\varepsilon^{-1}(x - x_1) + \varepsilon^{-\sigma_1}; \alpha(x_1 - \varepsilon^{1 - \sigma_1}))$ , we have only to show  $F_2(x_1) < 0$ ,  $F_2(x_1 - \varepsilon^{1 - \sigma_1}) > 0$  and  $\frac{dF_2}{dx} < 0$  in  $(x_1 - \varepsilon^{1 - \sigma_1}, x_1)$  to get conclusion. By (2) of Lemma 2.7,

$$\begin{aligned}
 F_2(x_1) &= \alpha(x_1) + \varepsilon^{2 - \sigma_2} - U(\varepsilon^{-\sigma_1}; \alpha(x_1 - \varepsilon^{1 - \sigma_1})) \\
 &= \alpha(x_1) + \varepsilon^{2 - \sigma_2} - \alpha(x_1 - \varepsilon^{1 - \sigma_1}) U(\alpha(x_1 - \varepsilon^{1 - \sigma_1}) \varepsilon^{-\sigma_1}; 1) \\
 &= \alpha(x_1) + \varepsilon^{2 - \sigma_2} - \alpha(x_1 - \varepsilon^{1 - \sigma_1}) + O(\exp(-C'_2 \varepsilon^{-\sigma_1}))
 \end{aligned} \quad (3.6)$$

for some  $C'_2 > 0$  if  $\varepsilon > 0$  is sufficiently small. The Taylor expansion of  $\alpha$  extended in  $C^2$  for  $x_1 > 0$  at  $x_1$  gives

$$F_2(x_1) = \varepsilon^{2 - \sigma_2} - \frac{1}{2} \alpha''(x_1) \varepsilon^{2 - 2\sigma_1} + o(\varepsilon^{2 - 2\sigma_1}) < 0$$

for sufficiently small  $\varepsilon > 0$  because  $2\sigma_1 > \sigma_2$  and  $\alpha''(x_0) > 0$ . And we have

$$\begin{aligned}
 F_2(x_1 - \varepsilon^{1 - \sigma_1}) &= \alpha(x_1 - \varepsilon^{1 - \sigma_1}) + \varepsilon^{2 - \sigma_2} \\
 &= \alpha(x_1) + \frac{\alpha''(x_1)}{2} \varepsilon^{2 - 2\sigma_2} + \varepsilon^{2 - \sigma_1} + o(\varepsilon^{2 - 2\sigma_1}) > 0.
 \end{aligned}$$

Finally it follows from (1) of Lemma 2.7 that

$$\frac{d}{dx}F_2(x) = -\alpha'(x) - \varepsilon^{-1}U_z(\varepsilon^{-1}(x - x_1) + \varepsilon^{-\sigma_1}; \alpha(x_1 - \varepsilon^{1-\sigma_1})) < 0 \quad (3.7)$$

for  $0 < x < x_1$ . Thus the proof is completed.  $\square$

**Proof of Proposition 3.1.** First we will show that  $u_\varepsilon^*$  satisfies the condition (S1)' of Proposition 2.6 in each interval  $J_1 := (0, \zeta_1 + x_1)$ ,  $J_2 := (\zeta_1 + x_1, \zeta_2 + x_1)$ ,  $J_3 := (\zeta_2 + x_1, x_1)$ ,  $J_4 := (x_1, x_2)$ ,  $J_5 := (x_2, 1)$ . We set  $\Phi_1(u)(x) := \varepsilon^2 u_{xx} + u(\alpha(x)^2 - u^2)$ . In  $J_1$  we have

$$\Phi_1(u_\varepsilon^*)(x) = -\varepsilon^2 \alpha''(x) - 2\alpha(x)^2 \varepsilon^{2-\sigma_2} + O(\varepsilon^{4-2\sigma_2}) < 0$$

if  $\varepsilon > 0$  is sufficiently small. The similar inequality holds in  $J_3, J_4$  and  $J_5$ . In the remaining interval  $J_2$  we have

$$\Phi_1(u_\varepsilon^*) = U(\varepsilon^{-1}(x - x_1) + \varepsilon^{-\sigma_1}; \alpha(x_1 - \varepsilon^{1-\sigma_1}))\{\alpha(x)^2 - \alpha(x_1 - \varepsilon^{1-\sigma_1})^2\}. \quad (3.8)$$

Here we observe

$$\begin{cases} U(\varepsilon^{-1}(x - x_1) + \varepsilon^{-\sigma_1}; \alpha(x_1 - \varepsilon^{1-\sigma_1})) > 0 & \text{and } \alpha(x) < \alpha(x_1 - \varepsilon^{1-\sigma_1}), \\ & \text{if } x_1 - \varepsilon^{1-\sigma_1} < x < x_1 + \zeta_2, \\ U(\varepsilon^{-1}(x - x_1) + \varepsilon^{-\sigma_1}; \alpha(x_1 - \varepsilon^{1-\sigma_1})) < 0 & \text{and } \alpha(x) > \alpha(x_1 - \varepsilon^{1-\sigma_1}), \\ & \text{if } x_1 + \zeta_1 < x < x_1 - \varepsilon^{1-\sigma_1}, \end{cases}$$

for sufficiently small  $\varepsilon > 0$ . Therefore, the right-hand side of (3.8) is negative, so that (S1)' is verified in each  $J_i$  ( $i = 1, 2, 3, 4, 5$ ). By (2) of Lemma 3.3 and (3.7), it is easy to verify (S2)' at  $x = x_1 + \zeta_1$  and  $x = x_1 + \zeta_2$  from Lemmas 3.3 and 3.4 and at  $x = x_1$  and  $x = x_2$  since  $\alpha$  is  $C^1$  function. Finally, (S3)' comes from the assumption that  $\alpha'(0) \geq 0$  and  $\alpha'(1) \geq 0$ . Thus we have proved that  $u_\varepsilon^*$  is a supersolution for  $(P_\varepsilon)$ .  $\square$

When  $\alpha$  satisfies  $\alpha'(0) < 0$  or  $\alpha'(1) < 0$ ,  $u_\varepsilon^*$  in Proposition 3.1 does not satisfy (S3) or (S3)'. Therefore, we have to modify  $u_\varepsilon^*$  near  $x = 0$  or  $x = 1$  as in Proposition 3.2. The following proposition deals with typical case  $\alpha'(0) < 0$  and  $\alpha'(1) < 0$ .

**Proposition 3.5.** Assume  $\alpha'(0) < 0$  and  $\alpha'(1) < 0$ . Let  $\sigma_1$  and  $\sigma_2$  be the same numbers as in Proposition 3.1 and let  $\sigma_3$  and  $\sigma_4$  be positive numbers satisfying  $\sigma_4 < \sigma_3 < 2\sigma_4$  and  $\sigma_3 < \sigma_2$ . For sufficiently small  $\varepsilon > 0$ , the following function

$$u_\varepsilon^*(x) = \begin{cases} -\alpha(x) + \varepsilon^{2-\sigma_2} + \varepsilon^{-\sigma_3}(x - \varepsilon^{\sigma_4})^2, & 0 \leq x \leq \varepsilon^{\sigma_4}, \\ -\alpha(x) + \varepsilon^{2-\sigma_2}, & \varepsilon^{\sigma_4} \leq x \leq x_1 + \zeta_1, \\ U(\varepsilon^{-1}(x - x_1) + \varepsilon^{-\sigma_1}; \alpha(x_1 - \varepsilon^{1-\sigma_1})), & x_1 + \zeta_1 \leq x \leq x_1 + \zeta_2, \\ \alpha(x) + \varepsilon^{2-\sigma_2}, & x_1 + \zeta_2 \leq x \leq 1 - \varepsilon^{\sigma_4}, \\ \alpha(x) + \varepsilon^{2-\sigma_2} + \varepsilon^{-\sigma_3}(x - 1 + \varepsilon^{\sigma_4})^2, & 1 - \varepsilon^{\sigma_4} \leq x \leq 1 \end{cases}$$

is a supersolution for  $(P_\varepsilon)$ . Here  $\zeta_1$  and  $\zeta_2$  are the same constants as in Proposition 3.1.

**Remark .** The same conclusion is valid with obvious modification in case  $\alpha'(0) < 0$  and  $\alpha'(1) \geq 0$  or  $\alpha'(0) \geq 0$  and  $\alpha'(1) < 0$ .

Since for the subsolution, we can prove similarly, we omit the proof for the subsolution. Now we are ready to prove Theorem 1.1.

*Proof of Theorem 1.1.* We begin with the proof for the case  $L = \{x_1, x_2\}$ ,  $K = \hat{K} = \{I_1\}$ . Let  $\Omega_1 = (0, x_1)$ ,  $\Omega_2 = (x_2, 1)$  (Case(I)). By Proposition 3.1 and Proposition 3.5, we can show that there exists a supersolution  $u_\varepsilon^*$  with the following properties.

$$\begin{cases} \alpha(x) < u_\varepsilon^*(x) < \alpha(x) + \varepsilon^\theta & \text{for } x_2 \leq x \leq 1, \\ -\alpha(x) < u_\varepsilon^*(x) < -\alpha(x) + \varepsilon^\theta & \text{for } 0 \leq x \leq x_1 - 2\varepsilon^{1-\sigma_1}, \end{cases} \quad (3.9)$$

for sufficiently small  $\varepsilon > 0$  with  $\theta = \min\{2 - \sigma_2, 2\sigma_4 - \sigma_3\}$ . Similarly, Proposition 3.2 allows us to find a subsolution  $u_{*,\varepsilon}$  satisfying

$$\begin{cases} \alpha(x) - \varepsilon^\theta < u_{*,\varepsilon}(x) < \alpha(x) & \text{for } x_2 + 2\varepsilon^{1-\sigma_1} \leq x \leq 1, \\ -\alpha(x) - \varepsilon^\theta < u_{*,\varepsilon}(x) < -\alpha(x) & \text{for } 0 \leq x \leq x_1. \end{cases} \quad (3.10)$$

Since Proposition 2.2 assures the existence of a solution  $u_\varepsilon$  such that  $u_{*,\varepsilon} < u_\varepsilon < u_\varepsilon^*$ , it follows that  $u_\varepsilon$  satisfies

$$\begin{cases} |u_\varepsilon(x) - \alpha(x)| < \varepsilon^\theta & \text{for } x_2 + 2\varepsilon^{1-\sigma_1} \leq x \leq 1, \\ |u_\varepsilon(x) + \alpha(x)| < \varepsilon^\theta & \text{for } 0 \leq x \leq x_1 - 2\varepsilon^{1-\sigma_1}. \end{cases} \quad (3.11)$$

Moreover  $u_\varepsilon$  is a local minimizer of the functional  $J_\varepsilon$  on  $H^1(0, 1)$ . Thus the proof is completed for the case (I).

For the case (II), we have only to consider  $(P_\varepsilon)$  with  $u$  and  $\alpha$  replaced by  $v(x) := u(1 - x)$  and  $\beta(x) := \alpha(1 - x)$ . Repeating the above procedure we see the existence of solution  $v$  for

$$\begin{cases} \varepsilon^2 v_{xx}(x) + v(\beta(x)^2 - v^2) = 0 & \text{in } (0, 1), \\ v_x(0) = v_x(1) = 0, \end{cases}$$

such that  $v$  satisfies (3.11) with  $\alpha$ ,  $x_1$  and  $x_2$  replaced by  $\beta$ ,  $1 - x_1$  and  $1 - x_2$ . Therefore  $u(x) = v(1 - x)$  becomes the desired solution. Moreover  $u$  is a local minimizer of the functional  $J_\varepsilon$  on  $H^1(0, 1)$ .

Finally we study the case when the number of elements of the set  $\hat{K}$  is larger than or equal to 2. Set  $\hat{K} = \{I'_1, I'_2, \dots, I'_l\}$ ,  $I'_i = [x'_{2i}, x'_{2i+1}]$  and  $x'_0 = 0$  and  $x'_{2l+1} = 1$ . We prove for the case (I), that is the case when

$$\Omega_1 = \bigcup_{i=0}^{\lfloor \frac{2l-1}{2} \rfloor} (x'_{4i}, x'_{4i+1}), \quad \Omega_2 = \bigcup_{i=0}^{\lfloor \frac{2l-3}{2} \rfloor} (x'_{4i+2}, x'_{4i+3}).$$

The preceding arguments enable us to construct a suitable supersolution  $u_\varepsilon^*$  and a suitable subsolution  $u_{*,\varepsilon}$ . For example, in  $(x'_1, x'_2)$ , the supersolution  $u_\varepsilon^*$  is defined by  $u_\varepsilon^*(x) = \alpha(x) + \varepsilon^{2-\sigma_2}$  and the subsolution  $u_{*,\varepsilon}$  is defined by  $u_{*,\varepsilon} = -\alpha(x) - \varepsilon^{2-\sigma_2}$  and in  $(x'_2, x'_3)$  the supersolution  $u_\varepsilon^*$  is defined by  $u_\varepsilon^*(x) = \alpha(x) + \varepsilon^{2-\sigma_2}$  and the subsolution  $u_{*,\varepsilon}$  is defined by  $u_{*,\varepsilon}(x) = \alpha(x) - \varepsilon^{2-\sigma_2}$  in  $(x'_2 + 2\varepsilon^{1-\sigma_1}, x'_3 - 2\varepsilon^{1-\sigma_1})$  with some  $\sigma_1 > 0$  and  $\sigma_2 > 0$ . We observe that, if  $x$  lies in the neighborhood of  $x'_{4i}$  ( $i \geq 1$ ) and  $x'_{4i+1}$  ( $i \geq 0$ ), then  $u_\varepsilon^*$  is defined with use of suitable dilation, translation or reflection of the solution  $U$  of (2.7) and if  $x$  lies in a neighborhoods of  $x'_{4i+2}$  and  $x'_{4i+3}$  ( $i \geq 0$ ),  $u_{*,\varepsilon}$  is defined with use of suitable dilation, translation or reflection of solution  $U$  of (2.7). Thus the proof is completed.  $\square$

## 4 Proof of Theorem 1.2

In this section we prove Theorem 1.2.

*Proof of Theorem 1.2.* For the sake of simplicity, we only prove for the case when  $K = \hat{K} = \{I_1\}$  and  $I_1 = [x_1, x_2]$  and  $\Omega_1 = (0, x_1)$  (Case(I)).

Let  $u_\varepsilon$  be the solution obtained by Theorem 1.1 and fix  $\eta > 0$  sufficiently small. We put  $\bar{\alpha} = \alpha(x_1) = \alpha(x_2)$ . Since  $u_\varepsilon(x) < -\alpha(x) + \varepsilon^{2-\sigma_2} < -\bar{\alpha} + \varepsilon^{2-\sigma_2}$  for  $x \in [0, x_1 - 2\varepsilon^{1-\sigma_1}]$ , for sufficiently small  $\varepsilon > 0$  we have  $u_\varepsilon(x) < -\bar{\alpha} + \eta$  for  $x \in [0, x_1 - 2\varepsilon^{1-\sigma_1}]$ . Similarly we have  $u_\varepsilon(x) > \bar{\alpha} - \eta$  for  $x \in [x_2 + 2\varepsilon^{1-\sigma_1}, 1]$ . Hence we can define the followings

$$\begin{aligned}\bar{x}_\varepsilon &= \inf\{x > x_1 - 2\varepsilon^{1-\sigma_1} | u_\varepsilon(x) = -\bar{\alpha} + \eta\}, \\ \tilde{x}_\varepsilon &= \sup\{x < x_2 + 2\varepsilon^{1-\sigma_1} | u_\varepsilon(x) = \bar{\alpha} - \eta\}.\end{aligned}$$

We may assume that  $\bar{x}_\varepsilon \rightarrow \bar{x} \in [x_1, x_2]$  and  $\tilde{x}_\varepsilon \rightarrow \tilde{x} \in [x_1, x_2]$ . Now we let  $v_\varepsilon(t) = u_\varepsilon(\bar{x}_\varepsilon + \varepsilon t)$ . Then we have

$$\begin{aligned}-v_\varepsilon'' &= v(\alpha(\bar{x}_\varepsilon + \varepsilon t)^2 - v^2), \\ v_\varepsilon(0) &= -\bar{\alpha} + \eta.\end{aligned}$$

Since  $\{v_\varepsilon\}$  are uniformly bounded in  $L^\infty$  and  $\bar{x}_\varepsilon \rightarrow \bar{x} \in [x_1, x_2]$ , it is easy to see that  $v_\varepsilon \rightarrow v$  in  $C^1_{\text{loc}}(\mathbb{R})$ , and

$$-v'' = v(\bar{\alpha}^2 - v^2) \quad t \in \mathbb{R}.$$

Since it is easily seen that for any  $t \leq 0$ ,  $u_\varepsilon(\bar{x}_\varepsilon + \varepsilon t) \leq -\bar{\alpha} + \eta$  for sufficiently small  $\varepsilon > 0$ , we can obtain  $v \leq -\bar{\alpha} + \eta$  for  $t \leq 0$ . Hence by Lemma 2.8,  $v$  satisfies  $v'(t) > 0$  and  $v(t) \rightarrow \pm\bar{\alpha}$  as  $t \rightarrow \pm\infty$ . As a result, we can find a  $R > 0$  large, such that  $v(R) = \bar{\alpha} - \eta$ . Thus, there is a  $R_\varepsilon \in (R-1, R+1)$ , such that  $v'_\varepsilon(t) < 0$  if  $t \in [0, R_\varepsilon]$  and  $v_\varepsilon(R_\varepsilon) = -\bar{\alpha} - \eta$ . Indeed, since  $v(R+1) > \bar{\alpha} - \eta$ ,  $v(R-1) < \bar{\alpha} - \eta$ ,  $v'(t) > 0$  on  $\mathbb{R}$  and  $v_\varepsilon \rightarrow v$  in  $C^1([R-1, R+1])$ , for sufficiently small  $\varepsilon > 0$ ,  $v_\varepsilon(R+1) > \bar{\alpha} - \eta$ ,  $v_\varepsilon(R-1) < \bar{\alpha} - \eta$  and  $v'_\varepsilon > 0$  on  $[R-1, R+1]$ . Hence there exists the desired  $R_\varepsilon \in (R-1, R+1)$ . We may assume that  $R_\varepsilon \rightarrow R$ . Therefore,  $u'_\varepsilon(x) > 0$  if  $x \in [\bar{x}_\varepsilon, \bar{x}_\varepsilon + \varepsilon R_\varepsilon]$  and  $u_\varepsilon(\bar{x}_\varepsilon + \varepsilon R_\varepsilon) = \bar{\alpha} - \eta$ .

**Claim.**  $\tilde{x}_\varepsilon = \bar{x}_\varepsilon + \varepsilon R_\varepsilon$ .

Suppose that the claim is not true. Then we can find a  $t_\varepsilon > \bar{x}_\varepsilon + \varepsilon R_\varepsilon$ , such that  $u_\varepsilon(x) > \bar{\alpha} - \eta$  for  $x \in (\bar{x}_\varepsilon + \varepsilon R_\varepsilon, t_\varepsilon)$ ,  $u_\varepsilon(t_\varepsilon) = \bar{\alpha} - \eta$  and  $u'_\varepsilon(t_\varepsilon) \leq 0$ . Note that  $\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1}(t_\varepsilon - (\bar{x}_\varepsilon + \varepsilon R_\varepsilon)) = +\infty$ . Indeed if it is not satisfied, there exist  $R'_\varepsilon \geq 0$  such that  $t_\varepsilon - (\bar{x}_\varepsilon + \varepsilon R_\varepsilon) = \varepsilon R'_\varepsilon$  and  $\sup_\varepsilon R'_\varepsilon < +\infty$ . We may assume that  $R'_\varepsilon \rightarrow R'$  for some  $R' \geq 0$ . Since  $v' > 0$  on  $\mathbb{R}$  and  $v_\varepsilon \rightarrow v$  in  $C^1([0, R+R'+1])$ , for sufficiently small  $\varepsilon > 0$ ,  $v_\varepsilon(R_\varepsilon + R'_\varepsilon) = \bar{\alpha} - \eta$  and  $v'_\varepsilon(R_\varepsilon + R'_\varepsilon) > 0$ . Hence we have  $u'_\varepsilon(t_\varepsilon) = u'_\varepsilon(\bar{x}_\varepsilon + \varepsilon(R_\varepsilon + R'_\varepsilon)) = \varepsilon^{-1}v'_\varepsilon(R_\varepsilon + R'_\varepsilon) > 0$ . This contradict to  $u'_\varepsilon(t_\varepsilon) \leq 0$ . Let  $\bar{v}_\varepsilon(t) = u_\varepsilon(t_\varepsilon + \varepsilon t)$ . It is easy to check that  $\bar{v}_\varepsilon \rightarrow \bar{v}$  in  $C^1_{\text{loc}}(\mathbb{R})$  and  $\bar{v}$  satisfies

$$\begin{cases} -\bar{v}'' = \bar{v}(\bar{\alpha}^2 - \bar{v}^2) & \text{in } \mathbb{R}, \\ \bar{v}(0) = \bar{\alpha} - \eta. \end{cases}$$

Let  $t \leq 0$ . Since  $\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1}(t_\varepsilon - (\bar{x}_\varepsilon + \varepsilon R_\varepsilon)) = +\infty$ ,  $t_\varepsilon + \varepsilon t > \bar{x}_\varepsilon + \varepsilon R_\varepsilon$  for sufficiently small  $\varepsilon > 0$ . Hence we obtain  $\bar{v}(t) \geq \bar{\alpha} - \eta$  for  $t \leq 0$  and  $\bar{v} \rightarrow \mp\bar{\alpha}$  as  $t \rightarrow \pm\infty$  by Lemma 2.8. Hence there exists  $\tilde{R} > 0$  such that  $\bar{v}(\tilde{R}) = -\bar{\alpha} + \eta$ . Thus, there is  $\tilde{R}_\varepsilon \in (R-1, R+1)$ , such that  $\bar{v}'(t) < 0$  if  $t \in [0, \tilde{R}_\varepsilon]$  and  $\bar{v}(\tilde{R}_\varepsilon) = -\bar{\alpha} + \eta$ . Therefore,  $u'_\varepsilon(x) < 0$  if  $x \in [t_\varepsilon, t_\varepsilon + \varepsilon \tilde{R}_\varepsilon]$  and  $u_\varepsilon(t_\varepsilon + \varepsilon \tilde{R}_\varepsilon) = -\bar{\alpha} + \eta$ . We will analyze the energy of  $u_\varepsilon$  on  $[\bar{x}_\varepsilon, \tilde{x}_\varepsilon]$  to lead to a contradiction.

Since the energy functional correspond to the problem (P<sub>ε</sub>) is

$$J_\varepsilon(u) = \int_0^1 \frac{\varepsilon^2}{2} |u'|^2 + \frac{(\alpha(x)^2 - u^2)^2}{4} dx - \int_0^1 \frac{\alpha(x)^4}{4} dx$$

and the term  $\int_0^1 \frac{\alpha(x)^4}{4} dx$  is independent of  $u$ , we can replace the energy functional  $J_\varepsilon$  by

$$J_\varepsilon(u) = \int_0^1 \frac{\varepsilon^2}{2} |u'|^2 + \frac{(\alpha(x)^2 - u^2)^2}{4} dx.$$

Since by Proposition 2.2, we have

$$J_\varepsilon(u_\varepsilon) \leq J_\varepsilon(\varphi)$$

for  $\varphi \in H^1(0, 1)$  with  $u_{*,\varepsilon} \leq \varphi \leq u_\varepsilon^*$ , for any  $y_2 > y_1$ ,  $u_\varepsilon$  is a minimizer of the following problem

$$\inf\{J_\varepsilon(u, (y_1, y_2)) : u - u_\varepsilon \in H_0^1(y_1, y_2), u_{*,\varepsilon} \leq u \leq u_\varepsilon^*\}, \quad (4.1)$$

where

$$J_\varepsilon(u, M) = \int_M \frac{\varepsilon^2}{2} |u'|^2 + \frac{(\alpha(x)^2 - u^2)^2}{4} dx$$

for any open set  $M$ . Let  $m_{\varepsilon, y_1, y_2}$  denote the minimum value of the problem (4.1). We will obtain the lower bound and the upper bound for  $m_{\varepsilon, \bar{x}_\varepsilon, \bar{x}_\varepsilon}$ . At first we obtain the lower bound. First we have

$$\begin{aligned} & J_\varepsilon(u_\varepsilon, (\bar{x}_\varepsilon, \bar{x}_\varepsilon + \varepsilon R_\varepsilon)) \\ &= \int_{\bar{x}_\varepsilon}^{\bar{x}_\varepsilon + \varepsilon R_\varepsilon} \left( \frac{\varepsilon^2}{2} |u'_\varepsilon|^2 + \frac{(\alpha(x)^2 - u_\varepsilon^2)^2}{4} \right) dx \\ &= \varepsilon \int_0^{R_\varepsilon} \left( \frac{1}{2} |v'_\varepsilon|^2 + \frac{(\alpha(\bar{x}_\varepsilon + \varepsilon t)^2 - v_\varepsilon^2)^2}{4} \right) dt \\ &= (\beta + O(\eta) + o_\varepsilon(1))\varepsilon \end{aligned} \quad (4.2)$$

where

$$\beta = \int_{-\infty}^{+\infty} \left( \frac{1}{2} |v'(t)|^2 + \frac{(\bar{\alpha}^2 - v^2)^2}{4} \right) dt.$$

Indeed, we have

$$\begin{aligned} & \int_0^{R_\varepsilon} \frac{1}{2} |v'_\varepsilon|^2 + \frac{(\alpha(\bar{x}_\varepsilon + \varepsilon t)^2 - v_\varepsilon^2)^2}{4} dt \\ &= \int_0^{R_\varepsilon} \frac{1}{2} |v'|^2 + \frac{(\bar{\alpha}^2 - v^2)^2}{4} dt + o_\varepsilon(1). \end{aligned} \quad (4.3)$$

Next we remark that

$$\int_0^R \frac{1}{2} |v'|^2 + \frac{(\bar{\alpha}^2 - v^2)^2}{4} dt = \int_{-R(\eta)}^{R(\eta)} \frac{1}{2} U'(t; \bar{\alpha})^2 + \frac{(\bar{\alpha}^2 - U(t; \bar{\alpha})^2)^2}{4} dt,$$

where  $R(\eta)$  is the unique positive number satisfies  $U(R(\eta); \bar{\alpha}) = \bar{\alpha} - \eta$ . We claim that

$$\lim_{\eta \rightarrow 0} \frac{1}{\eta} \int_{R(\eta)}^{+\infty} \frac{1}{2} U'(t; \bar{\alpha})^2 + \frac{(\bar{\alpha}^2 - U(t; \bar{\alpha})^2)^2}{4} dt < +\infty.$$

By using L'Hospital's rule, we only have to show is that for

$$\begin{aligned} S(\eta) &:= \frac{\partial}{\partial \eta} \int_{R(\eta)}^{+\infty} \frac{1}{2} U'(t; \bar{\alpha})^2 + \frac{(\bar{\alpha}^2 - U(t; \bar{\alpha})^2)^2}{4} dt \\ &= - \left( \frac{U'(R(\eta); \bar{\alpha})^2}{2} + \frac{(\bar{\alpha}^2 - U(R(\eta); \bar{\alpha})^2)^2}{4} \right) R_\eta(\eta), \end{aligned} \quad (4.4)$$

the following limit exists

$$\lim_{\eta \rightarrow 0} S(\eta) < +\infty.$$

Multiplying the equation

$$-U''(t; \bar{\alpha}) = (\bar{\alpha}^2 U(t; \bar{\alpha}) - U(t; \bar{\alpha})^3)$$

by  $U'(t; \bar{\alpha})$  and integrating over  $[R(\eta), +\infty)$  we have

$$\frac{U'(R(\eta); \bar{\alpha})^2}{2} = \frac{(\bar{\alpha}^2 - U(R(\eta); \bar{\alpha})^2)^2}{4} = \frac{\eta^2(2\bar{\alpha} - \eta)^2}{4}.$$

Hence we obtain

$$|U'(R(\eta); \bar{\alpha})| = \frac{1}{2}\eta|2\bar{\alpha} - \eta|. \quad (4.5)$$

Next we remark that from  $U(R(\eta); \bar{\alpha}) = \bar{\alpha} - \eta$  we have

$$U'(R(\eta); \bar{\alpha})R_\eta(\eta) = -1. \quad (4.6)$$

From (4.4), (4.5) and (4.6) we obtain

$$\begin{aligned} |S(\eta)| &\leq \frac{\eta^2(2\bar{\alpha} - \eta)^2}{2} |R_\eta(\eta)| \\ &= \frac{\eta^2(2\bar{\alpha} - \eta)^2}{2} \cdot \frac{1}{|U'(R(\eta); \bar{\alpha})|} \\ &= \frac{\eta^2(2\bar{\alpha} - \eta)^2}{2} \cdot \frac{2}{\eta|2\bar{\alpha} - \eta|} \\ &= \eta|2\bar{\alpha} - \eta|. \end{aligned}$$

From (4.3) and above estimate we obtain (4.2).

Similarly we have

$$J_\varepsilon(u_\varepsilon, (\bar{t}_\varepsilon, \bar{t}_\varepsilon + \varepsilon\tilde{R}_\varepsilon)) = (\beta + O(\eta) + o_\varepsilon(1))\varepsilon \quad (4.7)$$

Hence from (4.2), (4.7), we obtain

$$\begin{aligned} m_{\varepsilon, \bar{x}_\varepsilon, \tilde{x}_\varepsilon} &= J_\varepsilon(u_\varepsilon, (\bar{x}_\varepsilon, \bar{x}_\varepsilon + \varepsilon R_\varepsilon)) + J(u_\varepsilon, (\bar{x}_\varepsilon + \varepsilon R_\varepsilon, t_\varepsilon)) \\ &\quad + J_\varepsilon(u_\varepsilon, (t_\varepsilon, t_\varepsilon + \varepsilon\tilde{R}_\varepsilon)) + J_\varepsilon(u_\varepsilon, (t_\varepsilon + \varepsilon\tilde{R}_\varepsilon, \tilde{x}_\varepsilon)) \\ &\geq 2(\beta + O(\eta) + o_\varepsilon(1))\varepsilon. \end{aligned}$$

Now we give the upper bound for  $m_{\varepsilon, \bar{x}_\varepsilon, \tilde{x}_\varepsilon}$ . We define the following function  $\bar{u}_\varepsilon$

$$\bar{u}_\varepsilon(x) := \begin{cases} u_\varepsilon(x) & \text{if } x \in [\bar{x}_\varepsilon, \bar{x}_\varepsilon + \varepsilon R_\varepsilon], \\ \frac{u_\varepsilon^*(\bar{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon) - (\bar{\alpha} - \eta)}{\varepsilon} (x - (\bar{x}_\varepsilon + \varepsilon R_\varepsilon)) + \bar{\alpha} - \eta, & \text{if } x \in [\bar{x}_\varepsilon + \varepsilon R_\varepsilon, \bar{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon], \\ u_\varepsilon^*(x) & \text{if } x \in [\bar{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon, \tilde{x}_\varepsilon - \varepsilon], \\ \frac{(\bar{\alpha} - \eta) - u_\varepsilon^*(\tilde{x}_\varepsilon - \varepsilon)}{\varepsilon} (x - \tilde{x}_\varepsilon) + u_\varepsilon^*(\tilde{x}_\varepsilon - \varepsilon) & \text{if } x \in [\tilde{x}_\varepsilon - \varepsilon, \tilde{x}_\varepsilon] \end{cases}$$

We note that the function  $\bar{u}_\varepsilon$  satisfies  $u_{\varepsilon,*} \leq u_\varepsilon \leq \bar{u}_\varepsilon^*$  and  $\bar{x}_\varepsilon + \varepsilon R_\varepsilon > x_1 - \varepsilon^{1-\sigma_1}$  hold. We estimate  $J_\varepsilon(\bar{u}_\varepsilon, (\bar{x}_\varepsilon, \tilde{x}_\varepsilon))$ . We only consider the most delicate case when  $\bar{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon < x_1 + \zeta_2 < \tilde{x}_\varepsilon - \varepsilon$ . In other case, it can be estimated more easily. First we note that from (4.2) we have

$$J_\varepsilon(\bar{u}_\varepsilon, (\bar{x}_\varepsilon, \bar{x}_\varepsilon + \varepsilon R_\varepsilon)) = J_\varepsilon(u_\varepsilon, (\bar{x}_\varepsilon, \bar{x}_\varepsilon + \varepsilon R_\varepsilon)) = \varepsilon(\beta + O(\eta) + o_\varepsilon(1)). \quad (4.8)$$

Next we estimate  $J_\varepsilon(\bar{u}_\varepsilon, (\bar{x}_\varepsilon + \varepsilon R_\varepsilon, \bar{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon))$ . Since  $\bar{\alpha} - \eta < u_\varepsilon^*(\bar{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon) < \alpha(\bar{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon) + \varepsilon^{2-\sigma_2}$ , we have  $0 < u_\varepsilon^*(\bar{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon) - (\bar{\alpha} - \eta) < \eta + O(\varepsilon^{2-2\sigma_1}) + \varepsilon^{2-\sigma_2}$ . Hence



$\bar{u}'_\varepsilon(x) = \frac{\eta + O(\varepsilon^{2-2\sigma_1}) + \varepsilon^{2-\sigma_2}}{\varepsilon}$ . Thus we have

$$\begin{aligned} \int_{\bar{x}_\varepsilon + \varepsilon R_\varepsilon}^{\bar{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon} \frac{\varepsilon^2}{2} |\bar{u}'_\varepsilon|^2 dx &= O(\varepsilon\eta) + \varepsilon O(\varepsilon^{2-2\sigma_1}) + \varepsilon \cdot \varepsilon^{2-\sigma_2} \\ &= O(\varepsilon\eta) + o(\varepsilon). \end{aligned}$$

Since  $\bar{\alpha} - \eta \leq \bar{u}_\varepsilon \leq \alpha(x) + \varepsilon^{2-\sigma_2}$  on  $[\bar{x}_\varepsilon + \varepsilon R_\varepsilon, \bar{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon]$ , we have

$$\begin{aligned} (\bar{\alpha} - \eta)^2 - \alpha(x)^2 &\leq \bar{u}_\varepsilon(x)^2 - \alpha(x)^2 \leq O(\varepsilon^{2-\sigma_2}), \\ -2\eta\bar{\alpha} + \eta^2 + O(\varepsilon^{2-2\sigma_1}) &\leq \bar{u}_\varepsilon(x)^2 - \alpha(x)^2 \leq O(\varepsilon^{2-\sigma_2}). \end{aligned}$$

Hence we have

$$(\alpha(x)^2 - \bar{u}_\varepsilon(x)^2)^2 \leq O(\varepsilon^{4-2\sigma_2}) + O(\eta) + O(\varepsilon^{2-2\sigma_1})$$

and

$$\begin{aligned} \int_{\bar{x}_\varepsilon + \varepsilon R_\varepsilon}^{\bar{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon} \frac{(\alpha(x)^2 - \bar{u}_\varepsilon(x)^2)^2}{4} dx &= O(\varepsilon\eta) + \varepsilon(O(\varepsilon^{4-2\sigma_2}) + O(\varepsilon^{2-2\sigma_1})) \\ &= O(\varepsilon\eta) + o(\varepsilon). \end{aligned}$$

Thus we obtain

$$J_\varepsilon(\bar{u}_\varepsilon, (\bar{x}_\varepsilon + \varepsilon R_\varepsilon, \bar{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon)) \leq O(\varepsilon\eta) + o(\varepsilon). \quad (4.9)$$

Next we estimate  $J_\varepsilon(\bar{u}_\varepsilon, (\bar{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon, \tilde{x}_\varepsilon - \varepsilon))$ . Since we assume  $\bar{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon < x_1 + \zeta_2 < \tilde{x}_\varepsilon - \varepsilon$ , we divide the interval  $(\bar{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon, \tilde{x}_\varepsilon - \varepsilon)$  to  $(\bar{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon, x_1 + \zeta_2)$  and  $(x_1 + \zeta_2, \tilde{x}_\varepsilon - \varepsilon)$ . We set  $V(x) = U(\varepsilon^{-1}(x - x_1) + \varepsilon^{-\sigma_1} : \alpha(x_1 - \varepsilon^{1-\sigma}))$ . So we have

$$\begin{aligned} &J_\varepsilon(\bar{u}_\varepsilon, (\bar{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon, x_1 + \zeta_2)) \\ &= J_\varepsilon(u_\varepsilon^*, (\bar{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon, x_1 + \zeta_2)) \\ &= J_\varepsilon(V, (\bar{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon, x_1 + \zeta_2)) \\ &\leq \int_{x_1 - \varepsilon^{1-\sigma_1}}^{x_1} \frac{\varepsilon^2 V'(x)^2}{2} + \frac{(\alpha(x)^2 - V(x)^2)^2}{4} dx \\ &= \varepsilon \int_0^{\varepsilon^{-\sigma_1}} \frac{U'(t, \alpha(x_1 - \varepsilon^{1-\sigma_1}))^2}{2} dt \\ &\quad + \varepsilon \int_0^{\varepsilon^{-\sigma_1}} \frac{(\alpha(x_1 + \varepsilon t - \varepsilon^{1-\sigma_1}))^2 - U(t, \alpha(x_1 - \varepsilon^{1-\sigma_1}))^2}{4} dt \\ &= \varepsilon \left( \frac{\beta}{2} + o_\varepsilon(1) \right). \end{aligned} \quad (4.10)$$

Last equality follows from (5) of Lemma 2.7 if  $0 < \sigma_1 < \frac{2}{3}$ . Indeed, first we note that from (5) of Lemma 2.7 we have  $U(t, \alpha(x_1 - \varepsilon^{1-\sigma_1})) = \alpha(x_1 - \varepsilon^{1-\sigma_1})U(\alpha(x_1 - \varepsilon^{1-\sigma_1})t, 1)$  and  $U'(t, \alpha(x_1 - \varepsilon^{1-\sigma_1}))^2 = \alpha(x_1 - \varepsilon^{1-\sigma_1})^4 U'(\alpha(x_1 - \varepsilon^{1-\sigma_1})t, 1)$ . Similarly  $U'(t, \bar{\alpha})^2 = \bar{\alpha}^4 U'(\bar{\alpha}t, 1)^2$ . Hence we obtain

$$\begin{aligned} &U'(t, \alpha(x_1 - \varepsilon)) ^2 - U'(t, \bar{\alpha})^2 \\ &= \alpha(x_1 - \varepsilon^{1-\sigma_1})^4 (U'(\alpha(x_1 - \varepsilon^{1-\sigma_1})t, 1)^2 - U'(\bar{\alpha}t, 1)^2) \\ &\quad + (\alpha(x_1 - \varepsilon^{1-\sigma_1})^4 - \bar{\alpha}^4) U'(\bar{\alpha}t, 1)^2. \end{aligned}$$

From (3) of Lemma 2.7, we have

$$|U'(\alpha(x_1 - \varepsilon^{1-\sigma_1})t, 1)^2 - U'(\bar{\alpha}t, 1)^2| \leq C \exp(-C't)$$

for some  $C, C' > 0$  independent of  $\varepsilon > 0$ . Hence we obtain

$$\int_0^{\varepsilon^{-\sigma_1}} \frac{U'(t, \alpha(x_1 - \varepsilon^{1-\sigma_1}))^2}{2} dt \rightarrow \int_0^\infty \frac{U'(t, \bar{\alpha})^2}{2} dt \text{ as } \varepsilon \rightarrow 0.$$

Next we estimate

$$\int_0^{\varepsilon^{-\sigma_1}} \frac{(\alpha(x_1 + \varepsilon t - \varepsilon^{1-\sigma_1}))^2 - U(t, \alpha(x_1 - \varepsilon^{1-\sigma_1}))^2}{4} dx.$$

First we note that

$$\begin{aligned} & (\alpha(x_1 + \varepsilon t - \varepsilon^{1-\sigma_1}))^2 - U(t, \alpha(x_1 - \varepsilon^{1-\sigma_1}))^2 - (\bar{\alpha}^2 - U(t, \bar{\alpha})^2)^2 \\ &= \alpha(\#)^4 - 2\alpha(\#)^2 U(t, \alpha(\natural))^2 + U(t, \alpha(\natural))^4 \\ & \quad - (\bar{\alpha}^4 - 2\bar{\alpha}^2 U(t, \bar{\alpha})^2 + U(t, \bar{\alpha})^4) \\ &= (\alpha(\#)^4 - \bar{\alpha}^4) - 2\alpha(\#)^2 (U(t, \alpha(\natural))^2 - U(t, \bar{\alpha})^2) \\ & \quad - 2(\alpha(\#)^2 - \bar{\alpha}^2) U(t, \bar{\alpha})^2 + (U(t, \alpha(\natural))^4 - U(t, \bar{\alpha})^4), \end{aligned}$$

where  $\alpha(\#) = \alpha(x_1 + \varepsilon t - \varepsilon^{1-\sigma_1})$  and  $\alpha(\natural) = \alpha(x_1 - \varepsilon^{1-\sigma_1})$ . Since by (2) of Lemma 2.7, we only have to estimate the term  $\alpha(\#)^4 - \bar{\alpha}^4$ .

Since  $\alpha(x_1 + \varepsilon t - \varepsilon^{1-\sigma_1}) = \bar{\alpha} + O(\varepsilon^{2-2\sigma_1})$  for  $t \in [0, \varepsilon^{-\sigma_1}]$ , we have  $\alpha(x_1 + \varepsilon t - \varepsilon^{1-\sigma_1})^4 - \bar{\alpha}^4 = O(\varepsilon^{2-2\sigma_1})$  and

$$\int_0^{\varepsilon^{-\sigma_1}} |\alpha(x_1 + \varepsilon t - \varepsilon^{1-\sigma_1})^4 - \bar{\alpha}^4| dt = O(\varepsilon^{2-3\sigma_1}).$$

If we take  $0 < \sigma_1 < \frac{2}{3}$ , we obtain

$$\int_0^{\varepsilon^{-\sigma_1}} |\alpha(x_1 + \varepsilon t - \varepsilon^{1-\sigma_1})^4 - \bar{\alpha}^4| dt = o_\varepsilon(1).$$

Hence we can conclude that

$$\begin{aligned} & \int_0^{\varepsilon^{-\sigma_1}} \frac{(\alpha(x_1 + \varepsilon t - \varepsilon^{1-\sigma_1}))^2 - U(t, \alpha(x_1 - \varepsilon^{1-\sigma_1}))^2}{2} dt \\ &= \int_0^\infty \frac{(\bar{\alpha}^2 - U(t, \bar{\alpha})^2)^2}{4} dt + o_\varepsilon(1) \end{aligned}$$

and

$$\begin{aligned} & \int_0^{\varepsilon^{-\sigma_1}} \frac{U'(t, \alpha(x_1 - \varepsilon^{1-\sigma_1}))^2}{2} dt \\ & \quad + \int_0^{\varepsilon^{-\sigma_1}} \frac{(\alpha(x_1 + \varepsilon t - \varepsilon^{1-\sigma_1}))^2 - U(t, \alpha(x_1 - \varepsilon^{1-\sigma_1}))^2}{4} dt \\ &= \left( \frac{\beta}{2} + o_\varepsilon(1) \right). \end{aligned}$$

Next we estimate  $J_\varepsilon(\bar{u}_\varepsilon, (x_1 + \zeta_2, \tilde{x}_\varepsilon - \varepsilon))$ . We note that on  $(x_1 + \zeta_2, \tilde{x}_\varepsilon - \varepsilon)$ ,  $\bar{u}_\varepsilon(x) = u_\varepsilon^*(x) = \alpha(x) + \varepsilon^{2-\sigma_2}$  and  $\alpha'(x) = O(\varepsilon^{2-2\sigma_1})$ . Hence we have

$$J_\varepsilon(\bar{u}_\varepsilon, (x_1 + \zeta_2, \tilde{x}_\varepsilon - \varepsilon))$$

$$\begin{aligned}
&= J_\varepsilon(\alpha(x) + \varepsilon^{2-\sigma_2}, (x_1 + \zeta_2, \tilde{x}_\varepsilon - \varepsilon)) \\
&= \int_{x_1+\zeta_2}^{\tilde{x}_\varepsilon-\varepsilon} \frac{\varepsilon^2}{2} \alpha'(x)^2 + \frac{(\alpha(x)^2 - (\alpha(x) + \varepsilon^{2-\sigma_2})^2)^2}{4} dx \\
&= \int_{x_1+\zeta_2}^{\tilde{x}_\varepsilon-\varepsilon} O(\varepsilon^{6-4\sigma_1}) + O(\varepsilon^{4-2\sigma_2}) dx \\
&= O(\varepsilon^{6-4\sigma_1}) + O(\varepsilon^{4-2\sigma_2}) \\
&= o(\varepsilon)
\end{aligned} \tag{4.11}$$

for  $0 < \sigma_2 < \frac{3}{2}$ . Finally we can estimate  $J_\varepsilon(\bar{u}_\varepsilon, (\tilde{x}_\varepsilon - \varepsilon, \tilde{x}_\varepsilon))$  similarly as in  $J_\varepsilon(\bar{u}_\varepsilon, (\bar{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon))$ , that is we can obtain

$$J_\varepsilon(\bar{u}_\varepsilon, (\tilde{x}_\varepsilon - \varepsilon, \tilde{x}_\varepsilon)) \leq O(\varepsilon\eta) + o(\varepsilon). \tag{4.12}$$

As a result, from (4.8), (4.9), (4.10), (4.11) and (4.12) we obtain

$$m_{\varepsilon, \bar{x}_\varepsilon, \tilde{x}_\varepsilon} \leq \varepsilon \left( \beta + \frac{\beta}{2} \right) + O(\varepsilon\eta) + o(\varepsilon). \tag{4.13}$$

Combining (4.8) and (4.13), we are led to

$$2\beta\varepsilon + O(\varepsilon\eta) + o(\varepsilon) \leq \frac{3}{2}\beta\varepsilon + O(\varepsilon\eta) + o(\varepsilon)$$

and

$$\frac{\beta}{2} \leq \frac{O(\varepsilon\eta)}{\varepsilon\eta} \eta + \frac{o(\varepsilon)}{\varepsilon}.$$

Since  $O(\varepsilon\eta)/\varepsilon\eta$  is bounded, we can take  $\eta > 0$  so small that

$$\frac{O(\varepsilon\eta)}{\varepsilon\eta} \eta < \frac{\beta}{3}.$$

This is a contradiction for  $\varepsilon > 0$  small. So we can conclude  $\bar{x}_\varepsilon + \varepsilon R_\varepsilon = \tilde{x}_\varepsilon$  and we can set  $t_{\varepsilon,1,1} = \bar{x}_\varepsilon$  and  $t_{\varepsilon,2,1} = \tilde{x}_\varepsilon$ . In other case, it can be estimated more easily. Thus, the proof is completed.  $\square$

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