

On the Norm of Block Products of Matrices

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1. Introduction and Preliminaries

Let $M_{m,n}$ be the space of all $m \times n$ complex matrices, and set $M_n = M_{n,n}$. For each $A \in M_{m,n}$ the vector of singular values of A (i.e. eigenvalues of $|A| = (A^*A)^{1/2} \in M_n$) arranged in decreasing order is denoted by

$$\sigma(A) = (\sigma_1(A), \sigma_2(A), \dots, \sigma_n(A)).$$

For $1 \leq p < \infty$, we denote the p -norm of A by $\|A\|_p$, i.e.

$$\|A\|_p = [\operatorname{tr}(|A|^p)]^{1/p} = \left[\sum_{i=1}^n \sigma_i(A)^p \right]^{1/p},$$

and the spectral norm (or operator norm) by $\|A\|_\infty = \sigma_1(A)$.

It is well-known that for $A, B \in M_n$ the following Hölder-type norm inequality holds:

$$\|AB\|_r \leq \|A\|_p \|B\|_q \quad \text{whenever} \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}. \quad (1)$$

This can be implied from the inequalities

$$\sum_{i=1}^k \sigma_i(AB) \leq \sum_{i=1}^k \sigma_i(A) \sigma_i(B) \quad \text{for } k = 1, 2, \dots, n. \quad (2)$$

Furthermore, stronger inequalities hold:

$$\prod_{i=1}^k \sigma_i(AB) \leq \prod_{i=1}^k \sigma_i(A) \sigma_i(B) \quad \text{for } k = 1, 2, \dots, n. \quad (3)$$

For $A = [a_{ij}], B = [b_{ij}] \in M_n$, their Schur product (or Hadamard product) $A \circ B$ is defined by the entrywise multiplication

$$A \circ B = [a_{ij} b_{ij}]_{i,j=1}^n.$$

Recently it has shown that the following similar inequalities hold ([3], [5]):

$$\sum_{i=1}^k \sigma_i(A \circ B) \leq \sum_{i=1}^k \sigma_i(A) \sigma_i(B) \quad \text{for } k = 1, 2, \dots, n. \quad (4)$$

These imply the Hölder-type norm inequality

$$\|A \circ B\|_r \leq \|A\|_p \|B\|_q \quad \text{whenever} \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}. \quad (5)$$

(See [1], [2] and [6] for related results.)

In the present article, we are interested in the problem to find a product (of two matrices) which unifies the ordinary matrix product and the Schur product and satisfies the Hölder-type norm inequalities. There are two quite natural candidates called box products: let $A, B \in M_n$ be partitioned into N^2 blocks; $A = [A_{ij}]_{i,j=1}^N$, $B = [B_{ij}]_{i,j=1}^N$ with $A_{ij}, B_{ij} \in M_p$ ($n = Np$). We define block products $A \square B$ and $A \blacksquare B$ by

$$A \square B = [A_{ij} B_{ij}]_{i,j=1}^N \quad \text{and} \quad A \blacksquare B = [\sum_{k=1}^N A_{ik} \circ B_{kj}]_{i,j=1}^N.$$

If we consider the trivial partition $N = n, p = 1$, then $A \square B = A \circ B$ and $A \blacksquare B = AB$, while if $N = 1, p = n$, then $A \square B = AB$ and $A \blacksquare B = A \circ B$. We investigate these products in the next section.

For later use, we explain a notion and elementary facts of majorization. Let $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ and $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ be vectors in $\mathbb{R}^n$. We denote the decreasing rearrangements of the components of ξ by $\xi_{[1]} \geq \xi_{[2]} \geq \dots \geq \xi_{[n]}$. ξ is said to be submajorized by η (in symbols $\xi \prec_w \eta$) if

$$\sum_{i=1}^k \xi_{[i]} \leq \sum_{i=1}^k \eta_{[i]} \quad \text{for } k = 1, 2, \dots, n.$$

If in addition $\sum_{i=1}^n \xi_i = \sum_{i=1}^n \eta_i$ holds, then ξ is said to be majorized by η (in symbols $\xi \prec \eta$). Inequalities (2) and (4) can be expressed by submajorization

$$\sigma(AB) \prec_w \sigma(A) \cdot \sigma(B) \quad \text{and} \quad \sigma(A \circ B) \prec_w \sigma(A) \cdot \sigma(B),$$

where we denotes the coordinatewise product of vectors $\sigma(A)$ and $\sigma(B)$ by $\sigma(A) \cdot \sigma(B)$. Submajorization for the sum of matrices is also known:

$$\sigma(A + B) \prec_w \sigma(A) + \sigma(B). \quad (6)$$

It is a basic fact that submajorization is preserved by the increasing convex functions: if $\xi \prec_w \eta$, then $f(\xi) \prec_w f(\eta)$ for all increasing convex function f , where $f(\xi)$ denotes the vector $(f(\xi_1), f(\xi_2), \dots, f(\xi_n))$. In particular, if $\xi, \eta \in \mathbb{R}_+^n$ and

$$\prod_{i=1}^k \xi_{[i]} \leq \prod_{i=1}^k \eta_{[i]} \quad \text{for } k = 1, 2, \dots, n,$$

then $\xi \prec_w \eta$. See [4] for further details.

2. Results

First we consider the box product $A \square B$.

Lemma 1. For any $A, B \in M_n$

$$\begin{bmatrix} \mathcal{E}(B^* B) & (A \square B)^* \\ A \square B & \mathcal{E}(AA^*) \end{bmatrix} \geq 0, \quad (7)$$

where $\mathcal{E} : M_n \rightarrow M_n$ denotes the pinching, i.e.

$$\mathcal{E}(X) = [\delta_{ij} X_{ij}]_{i,j=1}^N \quad \text{for } X = [X_{ij}]_{i,j=1}^N \in M_n.$$

Proof. Take any vectors $\xi = [\xi_j]_{j=1}^N, \eta = [\eta_j]_{j=1}^N \in \mathbb{C}^n$ with $\xi_j, \eta_j \in \mathbb{C}^p$. Then

$$\begin{aligned} | \langle (A \square B) \xi | \eta \rangle |^2 &= \left| \sum_{i,j=1}^N \langle A_{ij} B_{ij} \xi_j | \eta_i \rangle \right|^2 \\ &= \left| \sum_{i,j=1}^N \langle B_{ij} \xi_j | A_{ij}^* \eta_i \rangle \right|^2 \\ &\leq \left\{ \sum_{i,j=1}^N \|B_{ij} \xi_j\| \cdot \|A_{ij}^* \eta_i\| \right\}^2 \\ &\leq \left\{ \sum_{i,j=1}^N \|B_{ij} \xi_j\|^2 \right\} \cdot \left\{ \sum_{i,j=1}^N \|A_{ij}^* \eta_i\|^2 \right\} \\ &= \left\{ \sum_{j=1}^N \langle (\sum_{i=1}^N B_{ij}^* B_{ij}) \xi_j | \xi_j \rangle \right\} \cdot \left\{ \sum_{i=1}^N \langle (\sum_{j=1}^N A_{ij} A_{ij}^*) \eta_i | \eta_i \rangle \right\} \\ &= \langle \mathcal{E}(B^* B) \xi | \xi \rangle \cdot \langle \mathcal{E}(AA^*) \eta | \eta \rangle, \end{aligned}$$

which shows that (7) holds. ■

Using this lemma we have the following.

Theorem 2. For any $A, B \in M_n$

$$\sum_{j=1}^k \sigma_j(A \square B)^2 \leq \sum_{j=1}^k \sigma_j(A)^2 \sigma_j(B)^2 \quad \text{for } k = 1, 2, \dots, n. \quad (8)$$

Proof. By (7) there is $C \in M_n$ such that $\|C\|_\infty \leq 1$ and

$$A \square B = \mathcal{E}(AA^*)^{1/2} \cdot C \cdot \mathcal{E}(B^*B)^{1/2}.$$

By (3) this implies

$$\prod_{j=1}^k \sigma_j(A \square B)^2 \leq \prod_{j=1}^k \sigma_j(\mathcal{E}(AA^*)) \sigma_j(\mathcal{E}(B^*B)) \quad \text{for } k = 1, 2, \dots, n,$$

and consequently

$$\sum_{j=1}^k \sigma_j(A \square B)^2 \leq \sum_{j=1}^k \sigma_j(\mathcal{E}(AA^*)) \sigma_j(\mathcal{E}(B^*B)) \quad \text{for } k = 1, 2, \dots, n.$$

Let ω be a primitive N th root of 1, and define the unitary matrix $U = [\delta_{ij} \omega^j I_p]_{i,j=1}^N \in M_n$. Since the pinching $\mathcal{E}$ can be written in the form

$$\mathcal{E}(X) = \frac{1}{N} \sum_{k=1}^N U^{*k} X U^k \quad \text{for } X \in M_n, \quad (9)$$

we get by (6)

$$\sum_{j=1}^k \sigma_j(\mathcal{E}(AA^*)) \leq \sum_{j=1}^k \sigma_j(AA^*) = \sum_{j=1}^k \sigma_j(A)^2$$

and

$$\sum_{j=1}^k \sigma_j(\mathcal{E}(B^*B)) \leq \sum_{j=1}^k \sigma_j(B^*B) = \sum_{j=1}^k \sigma_j(B)^2.$$

Hence, by elementary calculation, we have (8). ■

As the consequence of the last theorem we have the norm inequalities.

Corollary 3. Whenever $p, q, r \geq 2$ satisfy $1/r = 1/p + 1/q$,

$$\|A \square B\|_r \leq \|A\|_p \|B\|_q. \quad (10)$$

In particular

$$\|A \square B\|_\infty \leq \|A\|_\infty \|B\|_\infty. \quad (11)$$

Note that Lemma 1 and norm inequality (11) remain valid in the C^* -algebra setting. In fact, we can obtain

$$\|[A_{ij} B_{ij}]_{i,j=1}^N\| \leq \|[A_{ij}]_{i,j=1}^N\| \cdot \|[B_{ij}]_{i,j=1}^N\|, \quad (12)$$

where $A = [A_{ij}]_{i,j=1}^N, B = [B_{ij}]_{i,j=1}^N \in M_n(\mathcal{A})$ with a C^* -algebra $\mathcal{A}$.

Next we consider the box product $A \blacksquare B$. Let $\{e_i\}_{i=1}^n$ be the canonical basis of $\mathbb{C}^n$, and define the unitary matrix $V \in M_n$ by

$$V e_{N(k-1)+j} = e_{p(j-1)+k} \quad \text{for } j = 1, 2, \dots, N, \quad k = 1, 2, \dots, p.$$

For $A, B \in M_n$, let $C = V^* A V, D = V^* B V$. Then we have

$$A \blacksquare B = V(C \boxtimes D)V^*, \quad (13)$$

where the block product $\boxtimes$ in the right hand side is the one with respect to the partition into p^2 blocks; $C = [C_{k\ell}]_{k,\ell=1}^p, D = [D_{k\ell}]_{k,\ell=1}^p$ with $C_{k\ell}, D_{k\ell} \in M_N$.

The next theorem follows from (13) and Theorem 2.

Theorem 4. For any $A, B \in M_n$

$$\sum_{j=1}^k \sigma_j(A \blacksquare B)^2 \leq \sum_{j=1}^k \sigma_j(A)^2 \sigma_j(B)^2 \quad \text{for } k = 1, 2, \dots, n. \quad (14)$$

The following is a consequence of this theorem.

Corollary 5. Whenever $p, q, r \geq 2$ satisfy $1/r = 1/p + 1/q$,

$$\|A \blacksquare B\|_r \leq \|A\|_p \|B\|_q. \quad (15)$$

In particular

$$\|A \blacksquare B\|_\infty \leq \|A\|_\infty \|B\|_\infty. \quad (16)$$

Finally we remark that there is another approach to the norm inequalities of the box products. The idea is the following: let $\Phi(\cdot, \cdot)$ be a bilinear map from $M_n \times M_n$ to M_n . If there are linear maps Φ_ℓ from M_n to $M_{n,m}$ and Φ_r from M_n to $M_{m,n}$ (for some m) satisfying

$$\begin{aligned} \Phi(A, B) &= \Phi_\ell(A) \Phi_r(B), \\ \|\Phi_\ell(A)\|_\infty &\leq \|A\|_\infty \quad \text{and} \quad \|\Phi_r(B)\|_\infty \leq \|B\|_\infty, \end{aligned} \quad (16)$$

for any $A, B \in M_n$, then

$$\|\Phi(A, B)\|_\infty \leq \|A\|_\infty \|B\|_\infty.$$

When we consider the bilinear map $\Phi(A, B) = A \square B$, we can find nice maps Φ_ℓ and Φ_r : for $A = [A_{ij}]_{i,j=1}^N$ and $B = [B_{ij}]_{i,j=1}^N$ define

$$\Phi_\ell(A) = [\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n], \quad \Phi_r(B) = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ \vdots \\ \hat{B}_n \end{bmatrix},$$

where

$$\tilde{A}_k = [\delta_{ij} A_{ik}]_{i,j=1}^N, \quad \hat{B}_k = [\delta_{kj} B_{ij}]_{i,j=1}^N \in M_n \quad \text{for } k = 1, 2, \dots, n.$$

Then we can check that Φ_ℓ and Φ_r satisfy (16). This nice idea was discovered by P. Nylen.

3. Counterexample

For the box products, desired inequalities are the following:

$$\sum_{j=1}^k \sigma_j(A \square B) \leq \sum_{j=1}^k \sigma_j(A) \sigma_j(B) \quad \text{for } k = 1, 2, \dots, n. \quad (17)$$

Though inequalities (8) hold, (17) or even the weaker inequalities

$$\sum_{j=1}^k \sigma_j(A \square B) \leq \left\{ \sum_{j=1}^k \sigma_j(A) \right\} \cdot \sigma_1(B) \quad \text{for } k = 1, 2, \dots, n \quad (18)$$

do not hold. A counterexample is the following: taking the 4×4 matrices

$$A = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}, \quad B = \begin{bmatrix} E_{11} & E_{21} \\ E_{12} & E_{22} \end{bmatrix},$$

where E_{ij} is 2×2 matrix whose (i, j) -entry is equal to 1 and all other entries are 0, we can easily compute the block product

$$A \square B = \begin{bmatrix} E_{11} & E_{11} \\ E_{22} & E_{22} \end{bmatrix}.$$

Hence we have

$$\begin{aligned} \sigma(A) &= \{2, 0, 0, 0\}, \\ \sigma(B) &= \{1, 1, 1, 1\}, \\ \sigma(A \square B) &= \{\sqrt{2}, \sqrt{2}, 0, 0\}, \end{aligned}$$

which do not satisfy (18). In view of (13) the inequalities

$$\sum_{j=1}^k \sigma_j(A \boxtimes B) \leq \sum_{j=1}^k \sigma_j(A) \sigma_j(B) \quad \text{for } k = 1, 2, \dots, n \quad (19)$$

or even the weaker inequalities

$$\sum_{j=1}^k \sigma_j(A \boxtimes B) \leq \left\{ \sum_{j=1}^k \sigma_j(A) \right\} \cdot \sigma_1(B) \quad \text{for } k = 1, 2, \dots, n \quad (20)$$

do not hold.

Finally the box products do not meet our request. Our purpose does not have been attained. But we do not have another candidate.

References

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