

BOREL DEFINABLE SUBGROUPS AND ALMOST INTERNALITY IN ROSY DEPENDENT GROUPS

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ABSTRACT. Let G be a rosy dependent group with a countable language, weak canonical bases and elimination of hyperimaginaries. Let p be a global $\mathfrak{p}$ -generic type and Σ be an $\emptyset$ -invariant family of partial types. Under a certain assumption on independence relations of global types we show that if $p|_{\emptyset}$ is not foreign to Σ , then there exists a Borel definable subgroup H such that G/H is almost Σ -internal.

1. PRELIMINARIES

We review some definitions and facts. We only consider theories having elimination of hyperimaginaries and we will work in the eq-structures.

ROSY THEORIES (See [A]): Let T be rosy and $\mathcal{M}$ be a big model. We work in $\mathcal{M}^{\text{eq}}$. We say that T has weak canonical bases if for any type p there exists the smallest algebraically closed subset $\text{wcb}_{\mathfrak{p}}(p)$ such that p does not $\mathfrak{p}$ -forks over $\text{wcb}_{\mathfrak{p}}(p)$. In [A] Adler showed that if T has weak canonical bases then $\text{wcb}_{\mathfrak{p}}(\bar{a}/B) = \text{aker}((a_i)_{i < \omega}) := \{d \in \text{acl}^{\text{eq}}((a_i)_{i < \omega}) : (a_i)_{i < \omega} \text{ is } d\text{-indiscernible}\}$, where $B = \text{acl}^{\text{eq}}(B)$ and $(a_i)_{i < \omega}$ is a Morley sequence of $\text{tp}(\bar{a}/B)$.

DEPENDENT THEORIES (See [HP]): Let T be dependent. In [HP], it is shown that a global type $p \in S(\mathcal{M})$ does not fork over an algebraically closed set $A \subseteq \mathcal{M}$ if and only if p is $\text{acl}^{\text{eq}}(A)$ -invariant if and only if p is strongly Borel definable over $\text{acl}^{\text{eq}}(A)$. (We say that p is strongly Borel definable over A if for any formula $\varphi(x, y)$, there exists a finite Boolean combination of partial types over A , say $D_{p, \varphi}(y)$, such that $\varphi(x, b) \in p$ if and only if $\models D_{p, \varphi}(b)$ for any b . It is also mentioned that a global type p is $A(\subset \mathcal{M})$ -invariant, then any Morley sequence of p over A has the same type over A .)

ROSY GROUPS (See [EKP]): Let $(G(x), \cdot)$ be a rosy group over $\emptyset$ and let $\perp^{\mathfrak{p}}$ be thorn-non-forking relation. There exists $\mathfrak{p}$ -generic type $p \in S(A)$ for G over A : $p(x) \vdash G(x)$ and if $a, b \in G$ with $a \models p$ and $a \perp_A^{\mathfrak{p}} b$, then $b \cdot a \perp_{\emptyset}^{\mathfrak{p}} A, b$ holds. And then $\text{tp}(b \cdot a/A)$ is also $\mathfrak{p}$ -generic.

FOREIGNNESS and ALMOST INTERNALITY in ROSY THEORIES: Let

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Σ be A -invariant family of partial types. $\bar{b} \models \Sigma|B$ denotes that $\text{tp}(\bar{b}/B) \supseteq \Sigma_0$ for some $\Sigma_0 \subseteq \Sigma$. We say that $p \in S(A)$ is foreign to Σ if for any $B(\supseteq A)$, $\bar{a}, \bar{b}$ such that $\bar{a} \downarrow_A^p B$ and $\bar{b} \models \Sigma|B$ we have $\bar{a} \downarrow_B^p \bar{b}$. We say that $p \in S(A)$ is Σ -internal if for any $\bar{a} \in p$ there exist $B \supseteq A$ and $\bar{b} \models \Sigma|B$ such that $\bar{a} \downarrow_A^p B$ and $\bar{a} \in \text{acl}^{\text{eq}}(B, \bar{b})$.

2. THE PROPOSITION

Definition 2.1. Let G be a rosy group and let H be a Borel definable subgroup over B . We say that G/H is almost Σ -internal if for each $gH \in G/H$ there exist C and $D \models \Sigma|B$ such that $g \downarrow_B^p C$ and gH is $\text{acl}^{\text{eq}}(B, C, D)$ -invariant.

Here, gH is an ultrimaginary element, we do not consider thorn-independence relation on ultrimaginaries, so we do not define $gH \downarrow_B^p C$.

Proposition 2.2. *Let G be a rosy dependent group having weak canonical bases and a countable language. Suppose that G is sufficiently saturated and any global type (i.e. over G) does not $\mathfrak{p}$ -fork over A if and only if it does not fork over A . Let p be a global $\mathfrak{p}$ -generic type of G . If $p|_{\emptyset}$ is not foreign to $\emptyset$ -invariant family of partial types Σ , then there exists a Borel definable subgroup H such that $|G/H| \geq \omega$ and G/H is almost Σ -internal.*

Proof. Note that p is $\text{acl}^{\text{eq}}(\emptyset)$ -invariance: By the definition of $\mathfrak{p}$ -generics, p does not $\mathfrak{p}$ -fork over $\emptyset$. By our assumption p does not fork over $\emptyset$, so we see $\text{acl}^{\text{eq}}(\emptyset)$ -invariance of p by dependence of G .

As $p|_{\emptyset}$ is not foreign to Σ , there exist $A \subset G$, $a \models p|_A$ and $\bar{b} \models \Sigma$ such that $a \not\downarrow_A^p \bar{b}$. We may assume $a, \bar{b} \downarrow_A^p G$. By our assumption we have $a, \bar{b} \downarrow_A^f G$, where $\downarrow^f$ denotes non-forking relation. Let $B := \text{wcb}_p(a, \bar{b}/G) (= \text{wcb}_f(a, \bar{b}/G))$.

Claim. Put $H = \{g \in G : g \cdot a, \bar{b} \equiv_B a, \bar{b}\}$ is a Borel definable (over B) subgroup of G .

$H \leq G$: Let $g \in K$. As $g^{-1} \in G$, it holds that $a, \bar{b} = g^{-1} \cdot g \cdot a, \bar{b} \equiv_G g^{-1} \cdot a, \bar{b}$, we have $g^{-1} \in H$. Let $g, g' \in H$. As $g' \in G$, we have $g' \cdot g \cdot a, \bar{b} \equiv_G g' \cdot a, \bar{b} \equiv_G a, \bar{b}$, $g' \cdot g \in H$ follows.

The Borel definability of H over B : Let $\varphi(u \cdot x, \bar{y}, \bar{z}) \in L$ and put $q = \text{tp}(a, \bar{b}/G)$. There exists a $D_{q, \varphi}(u, \bar{z})$ which defines a strongly Borel definable set over B such that $\varphi(d \cdot x, \bar{y}, \bar{c}) \in q$ if and only if $\models D_{q, \varphi}(d, \bar{c})$ for any $d \in G, \bar{c} \subset_{\omega} G$. As $H = \bigcap_{\varphi \in L} \{g \in G : D_{q, \varphi}(d, \bar{c}) \leftrightarrow D_{q, \varphi}(d \cdot g^{-1}, \bar{c}) \text{ for any } d \in G, \bar{c} \subset_{\omega} G\}$, the Borel definability of H over B follows.

Claim. $|G/H| \geq \omega$

As p is $\text{acl}^{\text{eq}}(\emptyset)$ -invariant, take $(g_i)_{i < \omega} \subset G$ be the Morley sequence of $p|_B$. As $a, \bar{b} \downarrow_B^p (g_i)_{i < \omega}$, $g := g_j^{-1} \cdot g_i$ is $\mathfrak{p}$ -generic over $B, a, \bar{b}$. So we have $g \cdot a \downarrow_B^p \bar{b}$.

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As $a \not\perp_B^p \bar{b}$, we see that $g \notin H$ as desired.

Fix $g \in G$ and take a Morley sequence $(a_i)_{i < \omega} \subset G$ of $p|B, g$. Note that $g \perp_B^p (a_i)_{i < \omega}$.

Again take a Morley sequence $(g \cdot a'_i, \bar{b}_i)_{i < \omega}$ of $\text{tp}(a, \bar{b}/G)|B, g, (a_i)_{i < \omega}$. ($\text{tp}(a, \bar{b}/G)$ is B -invariant.) As $(a_i)_{i < \omega}$ and $(g \cdot a'_i)_{i < \omega}$ are Morley sequences of $p|B, g$, we have $(a_i)_{i < \omega} \equiv_{B, g} (g \cdot a'_i)_{i < \omega}$. Let $(b'_i)_{i < \omega} \subset G$ be such that $(a_i, b'_i)_{i < \omega}$ is a Morley sequence of $\text{tp}(a, \bar{b}/G)|B, h$. So $\text{wcb}_f(a, \bar{b}/G) = \text{wcb}_p(a, \bar{b}/G) \subseteq \text{acl}^{\text{eq}}((a_i, b'_i)_{i < \omega})$. Therefore $\text{tp}(a, \bar{b}/G)$ is $\text{acl}^{\text{eq}}((a_i, b'_i)_{i < \omega})$ -invariant and $b'_i \models \Sigma|B$.

Claim. *Let $\sigma \in \text{Aut}(G/\text{acl}^{\text{eq}}((a_i, b'_i)_{i < \omega}, B))$. Then $\sigma(gH) = gH$.*

As $\text{tp}(a, \bar{b}/G)$ is $\text{acl}^{\text{eq}}((a_i, b'_i)_{i < \omega})$ -invariant, we have $g \cdot a, \bar{b} \equiv_G \sigma(g) \cdot a', \bar{b}'$. As $\sigma(g) \in G$ and $\text{tp}(a, \bar{b}/G)$ is B -invariant, we see $g \cdot a, \bar{b} \equiv_G \sigma(g) \cdot a', \bar{b}' \equiv_G \sigma(g) \cdot a, \bar{b}$, as desired.

We get $g \perp_B^p (a_i)_{i < \omega}$, $b'_i \models \Sigma|B$ and gH is $\text{acl}^{\text{eq}}((a_i, b'_i)_{i < \omega}, B)$ -invariant. $\square$

- Question 2.3.** (1) *Let $\mathcal{M}$ be a sufficiently saturated rosy model and any global type (i.e. over $\mathcal{M}$) does not p -fork over A if and only if it does not fork over A . Then is $\mathcal{M}$ simple?*
- (2) *Can we find a Borel definable NORMAL subgroup H as in the Proposition?*
- (3) *Can we find a Σ -connected component G^Σ in a Borel definable way? (G^Σ want to be foreign to Σ , connected and invariant under any definable automorphism as in [W].)*
- (4) *Does any superrosy field has monomial U^p -rank? If so, Nubling's proof [N] that any supersimple field is n -ample for any $n < \omega$ works for any superrosy field.*

REFERENCES

- [A] H.Adler, Explanation of independence, Ph.D thesis, Freiburg, 2005, arXiv:math.LO/0511616 v1, 24 Nov 2005.
- [EKP] C.Ealy, K.Krupinski, A.Pillay, Superrosy dependent groups having finitely satisfiable generics, Annals of Pure and Applied Logic (151), 1-21, 2008.
- [HP] E.Hrushovski, A.Pillay, On NIP and invariant measures, 2007, arXiv:math.LO/0710.2330v1, 11 Oct 2007.
- [N] H.Nubling, Reducts and expansions of stable and simple theories, Ph.D thesis, the University of East Anglia, 2004.
- [W] F.Wagner, Stable groups, London Mathematical Society Lecture Note Series, 240, Cambridge University Press, Cambridge, 1997.

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