

フラスコの中の統計法則

Fundamental Properties of Homogeneous Multifractals¹

Iwao Hosokawa

University of Electro-Communications

It is proved from the requirement of scale-similarity of multifractals that the probability of spatial distribution of a certain measure supported by a multifractal, which may be called intrinsic probability, is uniquely determined for scale ratio tending to zero if the f - α spectrum of the multifractal is given. As a corollary, it is proved that there exists no nonlinear transformation of multifractals. Also, it is derived that intrinsic probabilities of many multifractals including multi-nomial generalized Cantor sets can be determined by the knowledge of intermittency exponents $\mu(q)$ (and then generalized dimensions $D(q)$) limited for $q =$ nonnegative integers only.

¹The content was spoken in the Symp. on "Generation and Statistical Law of Turbulence" at Res. Inst. Math. Sci., Kyoto Univ. on Jan. 21-23, 1992, except for the 1st corollary.

In the previous paper¹⁾, it was clarified that there are generalized dimensions $D(q)$, f - α spectrum $f(\alpha)$, intermittency exponents $\mu(q)$, and intrinsic probability $p(y; r/l)$ (for an arbitrary scale ratio r/l) associated with, and to characterize, every isotropic homogeneous multifractal; these quantities are equivalent to each other; and also $D(q)$ and $\mu(q)$ are continuous and differentiable in q , if $D(q)$ is defined in the sense of Hentschel and Procaccia²⁾.

Here we resume the proof of the uniqueness for a given $f(\alpha)$ of $p(y; r/l)$ that is the probability density of spatial distribution of a certain measure supported by the multifractal for a scale ratio r/l tending to zero, by a new argument using scale-similarity of a multifractal.

Returning to Hentschel and Procaccia's formula²⁾, we have

$$\sum_i (p_i^{(r/L)})^q = C_q (r/L)^{(q-1)D(q)} \quad (1)$$

as $r/L \rightarrow 0$. L is a main scale, C_q is proper proportionial constants, and $p_i^{(r/L)}$ is the normalized measure of the i th subbox of scale r in the box of scale L . The sum $\sum_i$ denotes summation over all subboxes of scale r , except for the ones with $p_i^{(r/L)} = 0$. Here we consider much finer subboxes of scale s . Then, of course, we should have a similar formula,

$$\sum_k (p_k^{(s/L)})^q = C_q (s/L)^{(q-1)D(q)} \quad (2)$$

Here we can find $(r/s)^d$ subboxes of scale s in each subbox of scale r . The measure of the j th subbox of scale s in the i th subbox of scale r may be expressed as $p_{ji}^{(s/r)} p_i^{(r/L)}$. Of course, we have $\sum_j p_{ji}^{(s/r)} = 1$. Then, if s/r goes to zero, we should have

$$\sum_j (p_{ji}^{(s/r)})^q = C_q (s/r)^{(q-1)D(q)} \quad (3)$$

in each subbox of scale r (irrespectively of i), so long as the multifractal nature of the total domain is the same as that of a partial domain, as is

illustrated in Fig. 1. Consider the measure of the k th subbox of scale s is that of the j th subbox of scale s in the i th subbox of scale r , so that

$$p_k^{(s/L)} = p_{ji}^{(s/r)} p_i^{(r/L)}. \quad (4)$$

Thus, we have

$$\sum_k (p_k^{(s/L)})^q = \sum_{i,j} [p_{ji}^{(s/r)} p_i^{(r/L)}]^q = \sum_i (p_i^{(r/L)})^q \sum_j (p_{ji}^{(s/r)})^q. \quad (5)$$

Hence, we have from (1), (2) and (3) that

$$C_q = C_q^2. \quad (6)$$

Therefore, $C_q = 1$ is the only meaningful case that we can have for all q . The above argument may be a little more relaxed for the most general case where C_q depends weakly on r/L , such as including the $\ln(r/L)$ factor. However, such a case should be prohibited that C_q is a power function of r/L , because it violates the definition of D_q . If we start from the premise of $C_q(r/L)$, we have

$$C_q(s/L) = C_q(s/r) C_q(r/L) \quad (7)$$

in place of (6). Then, the only possible solution is

$$C_q(r/L) = (r/L)^{v_q}, \quad v_q : \text{const.} \quad (8)$$

But this is the prohibited case unless $v_q = 0$.

Thus, if the left-hand side of (1) is replaced by the often-used heuristic expression in terms of $f(\alpha)$, we should accept the exact equality:

$$\int \rho(\alpha) (r/L)^{-f(\alpha)+q\alpha} d\alpha = (r/L)^{(q-1)D(q)}; \quad (9)$$

$\rho(\alpha)$ denotes a weight in the integration. Now we prove that $\rho(\alpha)$ can take the only one form. The steepest descent method allows us to write (9) as

$$\rho(\alpha_1) (r/L)^{q\alpha_1 - f(\alpha_1)} \int (r/L)^{-f''(\alpha)(\alpha-\alpha_1)^2/2} d\alpha = (r/L)^{(q-1)D_q} \quad (10)$$

under the condition of $q - f'(\alpha) = 0$ and $f''(\alpha) < 0$, and to give

$$\alpha_1 q - f(\alpha_1) = (q-1)D(q) \quad (11)$$

as well as

$$\rho(\alpha_1) = [f''(\alpha_1) \ln(r/L) / (2\pi)]^{1/2} \quad (12)$$

for every q in $(-\infty, \infty)$ and then for every α_1 in its whole range, because

α_1 must change continuously depending on q by (11), if $D(q)$ and $f(\alpha_1)$ are continuous functions. This concludes the proof. We note here that (12) is the lowest-order asymptotic formula to relate ρ to f , but it is easy to obtain higher-order formulas including the correction terms of $O[|\ln(r/L)|^n]$ ($n = 1, 2, \dots$). The special case with a one-point f - α spectrum cannot be treated by the above argument. The previous treatment¹⁾ contains exactly this case.

Thus, it is incorrect to presume $\rho(\alpha)$ in an arbitrary form, since it evidently destroys the equality of (10) and then (9) for all q or violates scale-similarity of multifractals. Physically, this means that the probability distribution of α in space for scale ratio $r/L \rightarrow 0$ should be decided by the f - α spectrum alone, and never interfered by an extra independent factor. The present proof is less axiomatic but more illustrative than the previous one.

As a corollary, we can find the transformation rule of multifractals. Suppose two multifractals with $f_i(\alpha_i)$, $\rho_i(\alpha_i)$, and $D_i(q)$ (for $i = 1, 2$). Then we have

$$\int \rho_1(\alpha_1) (r/L)^{-f(\alpha_1)+q\alpha_1} d\alpha_1 = (r/L)^{(q-1)D_1(q)}. \quad (13)$$

If α_2 is related to α_1 as $\alpha_2(\alpha_1)$, (13) for $i = 2$ is rewritten as

$$\int \rho_2(\alpha_2) \alpha_2'(\alpha_1) (r/L)^{-f(\alpha_2)+q\alpha_2} d\alpha_1 = (r/L)^{(q-1)D_2(q)}. \quad (14)$$

Since the right-hand side is the ensemble average of $(r/L)^{q\alpha_2-d}$ (d : spatial dimension), we must have just

$$\rho_2(\alpha_2) \alpha_2'(\alpha_1) = \rho_1(\alpha_1) \text{ and } f_2(\alpha_2(\alpha_1)) = f_1(\alpha_1). \quad (15)$$

As a result, we can produce a different multifractal with $f_1(\alpha_1)$ from a multifractal with $f_2(\alpha_2)$ and vice versa by way of (15), once a function $\alpha_2(\alpha_1)$ is given. How arbitrary is the functional form of $\alpha_2(\alpha_1)$?

According to (12), we have

$$\rho_1(\alpha_1) = [f_1''(\alpha_1) \ln(r/L)/(2\pi)]^{1/2}. \quad (16)$$

On the other hand, we have

$$\partial^2/\partial \alpha_1^2 f_2(\alpha_2(\alpha_1)) = f_2''(\alpha_2)(\alpha_2')^2 + f_2'(\alpha_2)\alpha_2'' \quad (17)$$

Thus, (15), (16) and (17) require

$$\alpha_2''(\alpha_1) = 0, \quad (18)$$

which means

$$\alpha_2(\alpha_1) = a \alpha_1 + b \quad (a, b : \text{const}). \quad (19)$$

Namely, any nonlinear transformation of α is prohibited. It is easy to obtain the transformation rule of $D(q)$ caused by (19) as

$$(q-1)D_2(q) = (aq-1)D_1(aq) + bq. \quad (20)$$

As another corollary, we can argue the moment problem of intrinsic probability. The intrinsic probability to characterize a multifractal may be written as¹⁾

$$(r/l)^{\mu(q)} = \int_0^{(l/r)^d} y^q p(y; r/l) dy. \quad (21)$$

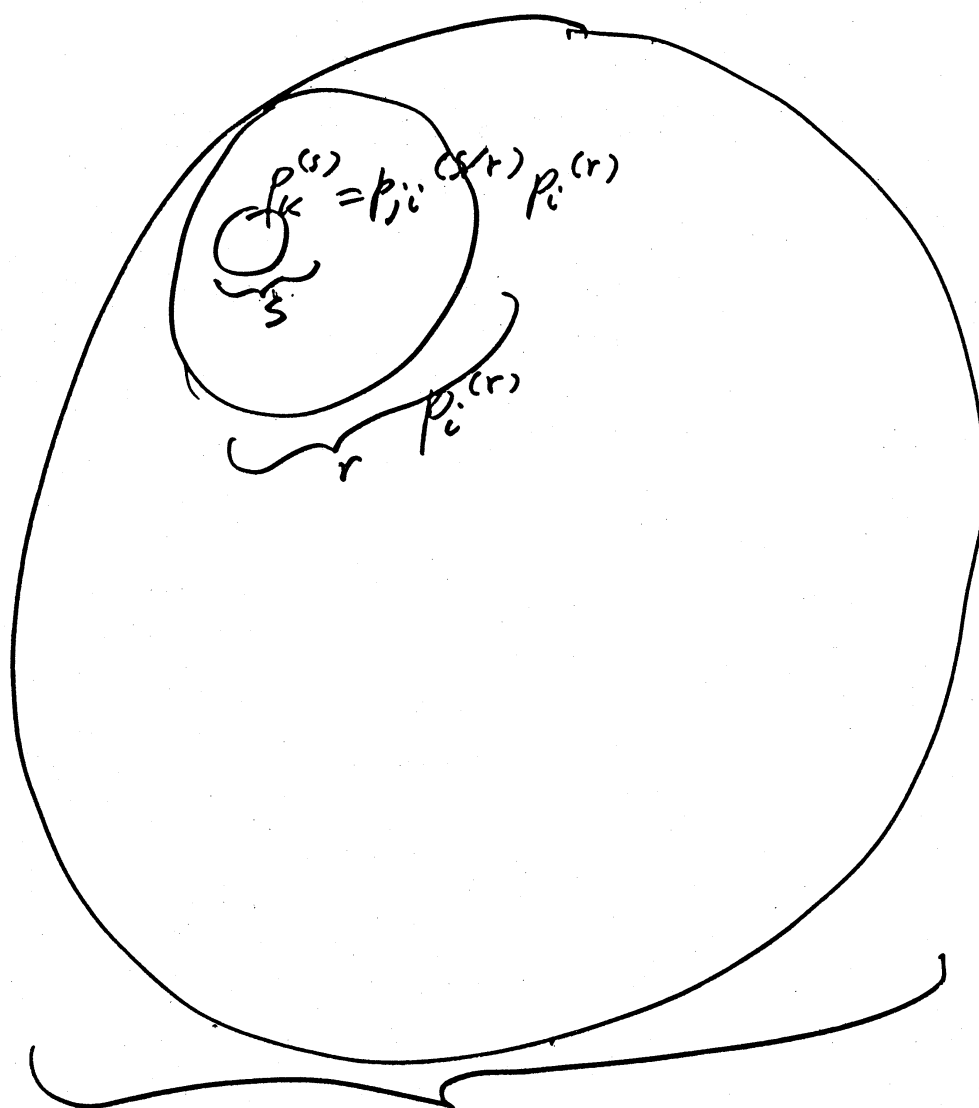
In this case, r/l does not necessarily tend to zero. Since (21) is a kind of the Mellin transform, we generally need the knowledge of $\mu(q)$ in the complex q -plane in order to determine the form of p . It is to be noted here that all formes for $\mu(q)$ do not necessarily give the intrinsic probability of a multifractal which is strongly conditioned to vanish towards $y = 0$ (rapidly) and for $y > (l/r)^d$; neither an exponential nor lognormal form as p is not exactly conditioned to do so. Many forms of p which can characterize multifractals were shown in Ref. 3, including binomial generalized Cantor sets; it is easy to extend the argument to multi-nomial Cantor sets. It is obvious that the characteristic functions $\phi(\theta; r/l)$ of these intrinsic probabilities have no essential singularity at $\theta = 0$; that is, ϕ can be expanded in a Taylor series. Since all the Taylor coefficients at the origin are given by all the nonnegative-integer-order moments of y , many intrinsic probabilities of multifractals can be determined by the limited knowledge of $\mu(q)$ only for $q = 0, 1, 2, \dots$. In these cases, all other values of $\mu(q)$ and $D(q)$ are redundant. Correspondingly, the right branch of $f(\alpha)$ is redundant because it is

decided by $D(q)$ for $q < 0$. Also, it is remarked that intrinsic probabilities of these multifractals are much less intermittent than the lognormal distribution that was mentioned by Orszag⁴⁾ as an example in which all the moments of nonnegative-integer-order cannot determine a unique probability. It is easy to see that the moments of y in generalized Cantor sets are within Carleman's criterion⁴⁾.

Finally, we note that the longitudinal velocity difference in isotropic turbulence is not supported by a multifractal in the present paradigm, because it is not exactly a measure in space. The statistical quality of it was discussed in Refs. 5 and 6.

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unit scale

の平均値

自己相似性

(全体と部分の構造が等しい)
(構造が同じ)

Fig. 1