

Complementary inequalities of the Furuta inequality

大阪教育大学 藤井正俊 (Masatoshi Fujii*)
前橋工科大学 亀井栄三郎 (Eizaburo Kamei**)
大阪府立東豊中高校 松本明美 (Akemi Matumoto***)

ABSTRACT. As a continuation of our preceding note, we discuss inequalities on the complementary domain of the Furuta inequality. For positive operators $A \geq B > 0$, it is shown that

$$A^t \natural_{\frac{\delta-t}{p-t}} B^p \geq A^\delta \geq B^\delta \geq B^t \natural_{\frac{\delta-t}{p-t}} A^p,$$

for $0 \leq \delta \leq t < p \leq 1$. This inequality is opposite to the inequality in [12].

1. Introduction. Throughout this note, a capital letter means a bounded linear operator on a Hilbert space H . An operator A is said to be positive (in symbol: $A \geq 0$) if $(Ax, x) \geq 0$ for all $x \in H$, and also an operator A is strictly positive (in symbol: $A > 0$) if A is positive and invertible. The Furuta inequality [5] established by Furuta himself in 1987 (cf.[6]) was given by the following form.

Furuta inequality: ([5], cf. [6]) If $A \geq B \geq 0$,
then for each $r \geq 0$,

$$(A^r A^p A^r)^{\frac{1}{q}} \geq (A^r B^p A^r)^{\frac{1}{q}}$$

and

$$(B^r A^p B^r)^{\frac{1}{q}} \geq (B^r B^p B^r)^{\frac{1}{q}}$$

holds for p and q such that $p \geq 0$ and $q \geq 1$ with
 $(1 + 2r)q \geq p + 2r$.

The best possibility of the conditions for p, q and r for the Furuta inequality is proved in [15]. In this inequality, if we take $r = 0$, then the following Löwner-Heinz inequality is obtained.

Löwner-Heinz inequality: If $A \geq B \geq 0$, then

$$A^\alpha \geq B^\alpha \text{ for } \alpha \in [0, 1].$$

We can review the Furuta inequality by using the operator mean theory established by Kubo-Ando[14]. Especially we use the α -power mean, $\sharp_\alpha$ which corresponds to the Löwner-Heinz inequality and is given by

$$A \sharp_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}, \text{ for } \alpha \in [0, 1].$$

Using it, we can reformulate the Furuta inequality as follows (cf.[1],[10],[13]):

$$A^t \sharp_{\frac{1-t}{p-t}} B^p \leq A \text{ and } B \leq B^t \sharp_{\frac{1-t}{p-t}} A^p \text{ for } p \geq 1 \text{ and } t \leq 0.$$

In our arguments of the Furuta inequality in [10], we obtained a chain of the following inequalities:

Satellite theorem of the Furuta inequality: If $A \geq B \geq 0$, then for $p \geq 1$ and $t \leq 0$,

$$A^t \sharp_{\frac{1-t}{p-t}} B^p \leq B \leq A \leq B^t \sharp_{\frac{1-t}{p-t}} A^p.$$

In our preceding notes [2],[3],[4] and [11], we discussed about the domain on which a similar formula to the Furuta inequality holds. In [12], we have given a unified form of the inequalities shown in [3], [4] and [11]; if $A \geq B > 0$ and $0 \leq t < p \leq 1$, then for each δ with $t \leq \delta \leq \min\{1, 2p\}$,

$$A^t \sharp_{\frac{\delta-t}{p-t}} B^p \leq A^\delta,$$

and

$$B^t \sharp_{\frac{\delta-t}{p-t}} A^p \geq B^\delta.$$

In particular, for each δ with $p \leq \delta \leq \min\{1, 2p\}$, we have another chain of inequalities:

$$A^t \sharp_{\frac{\delta-t}{p-t}} B^p \leq B^\delta \leq A^\delta \leq B^t \sharp_{\frac{\delta-t}{p-t}} A^p.$$

Recently, in [9], Furuta, Yamazaki and Yanagida have researched precisely the Furuta type inequalities on the complementary domain, $0 \leq t \leq 1$ and $0 \leq p \leq 1$, and investigate the relations.

In this note, we consider Furuta's type operator inequality in the case of $0 \leq \delta \leq t < p \leq 1$. Then we have the following inequality contrary to the above: If $A \geq B > 0$ and $\delta \leq t \leq \frac{1+\delta}{2}$, then

$$A^t \natural_{\frac{\delta-t}{p-t}} B^p \geq A^\delta \geq B^\delta \geq B^t \natural_{\frac{\delta-t}{p-t}} A^p.$$

In particular, if $0 \leq t \leq \frac{1}{2}$ and $0 \leq t < p \leq 1$, then

$$A^t \natural_{\frac{-t}{p-t}} B^p \geq 1 \geq B^t \natural_{\frac{-t}{p-t}} A^p.$$

2. Complementary inequalities.

The following lemmas shown in [11] are rewritten for the sake of convenience.

Lemma 1. *If $A \geq B > 0$, then the following inequalities hold;*

$$(i) \quad A^{-t} \sharp_s B^{-p} \geq A^{-(p-t)s-t}$$

for $0 \leq p \leq 1, 0 \leq s \leq 1$ and $t \in \mathbf{R}$,

$$(ii) \quad A^{-t} \natural_s B^{-p} \geq B^{-(p-t)s-t}$$

for $p \in \mathbf{R}, 1 \leq s \leq 2$ and $0 \leq t \leq 1$.

The following is proved from Lemma 1.

Lemma 2. *If $A \geq B > 0$, then the following inequalities hold;*

(i) *if $2n \leq s \leq 2n+1$ and $0 \leq p \leq 1$, then*

$$\begin{aligned} A^{-t} \sharp_s B^{-p} &= (B^{-p} A^t)^n (A^{-t} \sharp_{s-2n} B^{-p}) (A^t B^{-p})^n \\ &\geq (B^{-p} A^t)^n A^{-(p-t)(s-2n)-t} (A^t B^{-p})^n, \end{aligned}$$

(ii) *if $2n+1 \leq s \leq 2(n+1)$ and $0 \leq t \leq 1$, then*

$$\begin{aligned} A^{-t} \natural_s B^{-p} &= (B^{-p} A^t)^n (A^{-t} \natural_{s-2n} B^{-p}) (A^t B^{-p})^n \\ &\geq (B^{-p} A^t)^n B^{-(p-t)(s-2n)-t} (A^t B^{-p})^n. \end{aligned}$$

The next lemma is necessary to apply the Löwner Heinz inequality in the below.

Lemma 3. *Let $0 \leq \delta \leq t < p \leq 1$ and $(\frac{\delta}{2} \leq) t \leq \frac{1+\delta}{2}$. Then either*

(1) $2n \leq \frac{t-\delta}{p-t} \leq 2n+1$, *that is,* $\frac{t-\delta}{2n+1} \leq p-t \leq \frac{t-\delta}{2n}$

or

(2) $2n+1 \leq \frac{t-\delta}{p-t} \leq 2(n+1)$, *that is,* $\frac{t-\delta}{2(n+1)} \leq p-t \leq \frac{t-\delta}{2n+1}$

ensures

$$(a) \ 0 \leq 2(n-l)(p-t) + \delta \leq 1$$

and

$$(b) \ -1 \leq 2(n-l)p - 2(n-l+1)t + \delta \leq 0,$$

where $l = 0, 1, \dots, n-1$.

First of all, we discuss on the case of $\delta = 0$. Technically we will be along with our preceding argument in [2,3,4,11].

Theorem 1. Let $A \geq B > 0$, $0 \leq t < p \leq 1$ and $0 \leq t \leq \frac{1}{2}$. Then the following inequality holds;

$$A^t \natural_{\frac{-t}{p-t}} B^p \geq 1 \geq B^t \natural_{\frac{-t}{p-t}} A^p.$$

Consequently, if $0 \leq \delta \leq t \leq \frac{1}{2}$, then

$$A^t \natural_{\frac{\delta-t}{p-t}} B^p \geq A^\delta \geq B^\delta \geq B^t \natural_{\frac{\delta-t}{p-t}} A^p.$$

Proof. Under the assumption $A \geq B > 0$, $A^t \natural_{\frac{-t}{p-t}} B^p \geq 1$ is equivalent to $1 \geq B^t \natural_{\frac{-t}{p-t}} A^p$. We first consider the cases $\frac{t}{p-t} \in [0, 1]$ and $\frac{t}{p-t} \in [1, 2]$:

$$\begin{aligned} A^t \natural_{\frac{-t}{p-t}} B^p &= A^t (A^{-t} \sharp_{\frac{t}{p-t}} B^{-p}) A^t \\ &\geq A^t (A^{-t} \sharp_{\frac{t}{p-t}} A^{-p}) A^t = 1. \end{aligned}$$

If $1 \leq \frac{t}{p-t} \leq 2$, then

$$\begin{aligned} A^t \natural_{\frac{-t}{p-t}} B^p &= A^t (A^{-t} \natural_{\frac{t}{p-t}} B^{-p}) A^t \\ &\geq A^t (B^{-t} \natural_{\frac{t}{p-t}} B^{-p}) A^t \geq A^t B^{-2t} A^t \geq 1. \end{aligned}$$

In general, if $2n \leq \frac{t}{p-t} \leq 2n+1$, then

$$\begin{aligned} A^t \natural_{\frac{-t}{p-t}} B^p &= A^t (A^{-t} \natural_{\frac{t}{p-t}} B^{-p}) A^t \\ &= A^t (B^{-p} A^t)^n (A^{-t} \sharp_{\frac{t}{p-t}-2n} B^{-p}) (A^t B^{-p})^n A^t \\ &\geq A^t (B^{-p} A^t)^n A^{2np-2(n+1)t} (A^t B^{-p})^n A^t \quad \text{by Lemma 2 (i)} \\ &= A^t (B^{-p} A^t)^{n-1} B^{-p} A^{2n(p-t)} B^{-p} (A^t B^{-p})^{n-1} A^t \\ &\geq A^t (B^{-p} A^t)^{n-1} B^{2(n-1)p-2nt} (A^t B^{-p})^{n-1} A^t \quad \text{by Lemma 3 (a)} \\ &\geq A^t (B^{-p} A^t)^{n-1} A^{2(n-1)p-2nt} (A^t B^{-p})^{n-1} A^t \quad \text{by Lemma 3 (b)} \\ &\geq \dots \\ &\geq A^t (B^{-p} A^t)^{n-l} A^{2(n-l)p-2(n-l+1)t} (A^t B^{-p})^{n-l} A^t \quad \text{by Lemma 3 (b)} \\ &= A^t (B^{-p} A^t)^{n-l-1} B^{-p} A^{2(n-l)(p-t)} B^{-p} (A^t B^{-p})^{n-l-1} A^t \\ &\geq A^t (B^{-p} A^t)^{n-l-1} B^{2(n-l-1)p-2(n-l)t} (A^t B^{-p})^{n-l-1} A^t \quad \text{by Lemma 3 (a)} \end{aligned}$$

$$\begin{aligned}
&\geq \dots \\
&\geq A^t(B^{-p}A^t)A^{2p-4t}(A^tB^{-p})A^t \\
&= A^tB^{-p}A^{2p-2t}B^{-p}A^t \geq A^tB^{-2t}A^t \geq 1.
\end{aligned}$$

Moreover, if $2n+1 \leq \frac{t}{p-t} \leq 2(n+1)$, then

$$\begin{aligned}
A^t \natural_{\frac{t}{p-t}} B^p &= A^t(A^{-t} \natural_{\frac{t}{p-t}} B^{-p})A^t \\
&= A^t(B^{-p}A^t)^n(A^{-t} \natural_{\frac{t}{p-t}-2n} B^{-p})(A^tB^{-p})^nA^t \\
&\geq A^t(B^{-p}A^t)^nB^{2np-2(n+1)t}(A^tB^{-p})^nA^t \quad \text{by Lemma 2 (ii)} \\
&\geq A^t(B^{-p}A^t)^nA^{2np-2(n+1)t}(A^tB^{-p})^nA^t \quad \text{by Lemma 3 (b)} \\
&= A^t(B^{-p}A^t)^{n-1}B^{-p}A^{2n(p-t)}B^{-p}(A^tB^{-p})^{n-1}A^t \\
&\geq A^t(B^{-p}A^t)^{n-1}B^{2(n-1)p-2nt}(A^tB^{-p})^{n-1}A^t \quad \text{by Lemma 3 (a)} \\
&\geq \dots \\
&\geq A^t(B^{-p}A^t)^{n-l}B^{2(n-l)p-2(n-l+1)t}(A^tB^{-p})^{n-l}A^t \\
&\geq A^t(B^{-p}A^t)^{n-l}A^{2(n-l)p-2(n-l+1)t}(A^tB^{-p})^{n-l}A^t \quad \text{by Lemma 3 (b)} \\
&= A^t(B^{-p}A^t)^{n-l-1}B^{-p}A^{2(n-l)(p-t)}B^{-p}(A^tB^{-p})^{n-l-1}A^t \\
&\geq A^t(B^{-p}A^t)^{n-l-1}B^{2(n-l-1)p-2(n-l)t}(A^tB^{-p})^{n-l-1}A^t \\
&\geq \dots \\
&\geq A^t(B^{-p}A^t)A^{2p-4t}(A^tB^{-p})A^t = A^tB^{-p}A^{2p-2t}B^{-p}A^t \\
&\geq A^tB^{-2t}A^t \geq 1.
\end{aligned}$$

As a consequence, we obtain the second inequality as follows:

$$A^t \natural_{\frac{\delta-t}{p-t}} B^p = A^t \sharp_{\frac{\delta-t}{p-t}} (A^t \natural_{\frac{t}{p-t}} B^p) \geq A^t \sharp_{\frac{\delta-t}{p-t}} I = A^\delta.$$

Theorem 2. Let $A \geq B > 0$, $0 \leq \delta \leq t < p \leq 1$ and $t \leq \frac{1+\delta}{2}$. Then the following inequality holds;

$$A^t \natural_{\frac{\delta-t}{p-t}} B^p \geq A^\delta \geq B^\delta \geq B^t \natural_{\frac{\delta-t}{p-t}} A^p.$$

Proof. If $0 \leq \frac{t-\delta}{p-t} \leq 1$, then

$$A^t \natural_{\frac{\delta-t}{p-t}} B^p = A^t(A^{-t} \sharp_{\frac{t-\delta}{p-t}} B^{-p})A^t \geq A^t(A^{-t} \sharp_{\frac{t-\delta}{p-t}} A^{-p})A^t = A^\delta,$$

and if $1 \leq \frac{t-\delta}{p-t} \leq 2$, then

$$A^t \natural_{\frac{t-\delta}{p-t}} B^p = A^t(A^{-t} \natural_{\frac{t-\delta}{p-t}} B^{-p})A^t \geq A^tB^{\delta-2t}A^t \geq A^\delta.$$

In general, if $2n \leq \frac{t-\delta}{p-t} \leq 2n+1$, then

$$A^t \natural_{\frac{\delta-t}{p-t}} B^p = A^t(A^{-t} \natural_{\frac{t-\delta}{p-t}} B^{-p})A^t$$

$$\begin{aligned}
&= A^t(B^{-p}A^t)^n(A^{-t} \#_{\frac{t-\delta}{p-t}-2n} B^{-p})(A^tB^{-p})^nA^t \\
&\geq A^t(B^{-p}A^t)^nA^{2np-2(n+1)t+\delta}(A^tB^{-p})^nA^t \quad \text{by Lemma 2 (i)} \\
&= A^t(B^{-p}A^t)^{n-1}B^{-p}A^{2n(p-t)+\delta}B^{-p}(A^tB^{-p})^{n-1}A^t \\
&\geq A^t(B^{-p}A^t)^{n-1}B^{2(n-1)p-2nt+\delta}(A^tB^{-p})^{n-1}A^t \quad \text{by Lemma 3 (a)} \\
&\geq A^t(B^{-p}A^t)^{n-1}A^{2(n-1)p-2nt+\delta}(A^tB^{-p})^{n-1}A^t \quad \text{by Lemma 3 (b)} \\
&\geq \dots \\
&\geq A^t(B^{-p}A^t)^{n-l}A^{2(n-l)p-2(n-l+1)t+\delta}(A^tB^{-p})^{n-l}A^t \quad \text{by Lemma 3 (b)} \\
&= A^t(B^{-p}A^t)^{n-l-1}B^{-p}A^{2(n-l)(p-t)+\delta}B^{-p}(A^tB^{-p})^{n-l-1}A^t \\
&\geq A^t(B^{-p}A^t)^{n-l-1}B^{2(n-l-1)p-2(n-l)t+\delta}(A^tB^{-p})^{n-l-1}A^t \quad \text{by Lemma 3 (a)} \\
&\geq \dots \\
&\geq A^t(B^{-p}A^t)A^{2p-4t+\delta}(A^tB^{-p})A^t \\
&= A^tB^{-p}A^{2p-2t+\delta}B^{-p}A^t \geq A^tB^{-2t+\delta}A^t \geq A^\delta.
\end{aligned}$$

On the other hand, if $2n+1 \leq \frac{t-\delta}{p-t} \leq 2(n+1)$, then

$$\begin{aligned}
A^t \#_{\frac{\delta-t}{p-t}} B^p &= A^t(A^{-t} \#_{\frac{t-\delta}{p-t}} B^{-p})A^t \\
&= A^t(B^{-p}A^t)^n(A^{-t} \#_{\frac{t-\delta}{p-t}-2n} B^{-p})(A^tB^{-p})^nA^t \\
&\geq A^t(B^{-p}A^t)^nB^{2np-2(n+1)t+\delta}(A^tB^{-p})^nA^t \quad \text{by Lemma 2 (ii)} \\
&\geq A^t(B^{-p}A^t)^nA^{2np-2(n+1)t+\delta}(A^tB^{-p})^nA^t \quad \text{by Lemma 3 (b)} \\
&= A^t(B^{-p}A^t)^{n-1}B^{-p}A^{2n(p-t)+\delta}B^{-p}(A^tB^{-p})^{n-1}A^t \\
&\geq A^t(B^{-p}A^t)^{n-1}B^{2(n-1)p-2nt+\delta}(A^tB^{-p})^{n-1}A^t \quad \text{by Lemma 3 (a)} \\
&\geq \dots \\
&\geq A^t(B^{-p}A^t)^{n-l}B^{2(n-l)p-2(n-l+1)t+\delta}(A^tB^{-p})^{n-l}A^t \\
&\geq A^t(B^{-p}A^t)^{n-l}A^{2(n-l)p-2(n-l+1)t+\delta}(A^tB^{-p})^{n-l}A^t \\
&= A^t(B^{-p}A^t)^{n-l-1}B^{-p}A^{2(n-l)(p-t)+\delta}B^{-p}(A^tB^{-p})^{n-l-1}A^t \\
&\geq A^t(B^{-p}A^t)^{n-l-1}B^{2(n-l-1)p-2(n-l)t+\delta}(A^tB^{-p})^{n-l-1}A^t \\
&\geq \dots \\
&\geq A^t(B^{-p}A^t)A^{2p-4t+\delta}(A^tB^{-p})A^t \\
&= A^tB^{-p}A^{2p-2t+\delta}B^{-p}A^t \geq A^tB^{\delta-2t}A^t \geq A^\delta.
\end{aligned}$$

Remark. The assumption $t \leq \frac{1+\delta}{2}$ is needed to ensure the final inequality $A^tB^{\delta-2t}A^t \geq A^\delta$ in the proofs.

3. Brief proof of Theorem 2. Professor Furuta pointed out that the following known results [11] shorten proofs of Theorems above.

Theorem A. If $A \geq B > 0$, then
(i) in the case of $\frac{1}{2} \leq p \leq 1$ and $0 \leq t < p$,

$$A^t \natural_{\frac{1-t}{p-t}} B^p \leq B \leq A$$

and (ii) in the case of $0 \leq t < p \leq \frac{1}{2}$

$$A^t \natural_{\frac{2p-t}{p-t}} B^p \leq B^{2p} \leq A^{2p}.$$

Brief proof of Theorem 2. The assumption says $0 \leq \frac{t-\delta}{1-t} \leq 1$. If $\frac{1}{2} \leq p \leq 1$, then the above (i) implies as follows;

$$\begin{aligned} A^t \natural_{\frac{\delta-t}{p-t}} B^p &= A^t (A^{-t} \natural_{\frac{t-\delta}{p-t}} B^{-p}) A^{-t} \\ &= A^t (A^{-t} \sharp_{\frac{t-\delta}{1-t}} (A^{-t} \natural_{\frac{1-t}{p-t}} B^{-p}) A^{-t}) \\ &\geq A^t (A^{-t} \sharp_{\frac{t-\delta}{1-t}} A^{-1}) A^t = A^\delta. \end{aligned}$$

Suppose $0 \leq p \leq \frac{1}{2}$. Since $0 \leq \frac{t-\delta}{2p-t} \leq 1$, a similar calculation leads us the conclusion by the use of the result (ii) of Theorem A.

$$\begin{aligned} A^t \natural_{\frac{\delta-t}{p-t}} B^p &= A^t (A^{-t} \natural_{\frac{t-\delta}{p-t}} B^{-p}) A^{-t} \\ &= A^t (A^{-t} \sharp_{\frac{t-\delta}{2p-t}} (A^{-t} \natural_{\frac{2p-t}{p-t}} B^{-p}) A^{-t}) \\ &\geq A^t (A^{-t} \sharp_{\frac{t-\delta}{2p-t}} A^{-2p}) A^t = A^\delta. \end{aligned}$$

Acknowledgement. The authors would like to express their hearty thanks to Prof. T. Furuta for his kind instructions of the shorted proofs with respects and affection.

References

- [1] M. Fujii, Furuta's inequality and its mean theoretic approach, J. Operator Theory, 23(1990), 67-72.
- [2] M. Fujii, T. Furuta and E. Kamei, Complements to the Furuta inequality, Proc. Japan Acad., 70(1994), Ser. A, 239-242.
- [3] M. Fujii, T. Furuta and E. Kamei, Complements to the Furuta inequality, III, Math. Japon., 45(1997), 25-32.
- [4] M. Fujii, J. F. Jiang and E. Kamei, Complements to the Furuta inequality, IV, Math. Japon., 45(1997), 511-518.
- [5] T. Furuta, $A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p+2r$, Proc. Amer. Math. Soc., 101(1987), 85-88.
- [6] T. Furuta, Elementary proof of an order preserving inequality, Proc. Japan Acad., 65(1989), 126.
- [7] T. Furuta, Parametric operator function via Furuta inequality, Sci. Math., to appear.

- [8] T.Furuta, Operator functions involving order preserving inequalities, Math. Japon., to appear.
- [9] T.Furuta, T.Yamazaki and M.Yanagida, An extension of order preserving operator inequalities, preprint.
- [10] E.Kamei, A satellite to Furuta's inequality, Math. Japon., 33(1988), 883-886.
- [11] E.Kamei, Complements to the Furuta inequality, II, Math. Japon., 45(1997), 15-23.
- [12] E.Kamei, Monotonicity of the Furuta inequality on its complementary domain, Math. Japon., to appear.
- [13] E.Kamei, Parametrization of the Furuta inequality, Math. Japon., to appear.
- [14] F.Kubo and T.Ando, Means of positive linear operators, Math. Ann., 246(1980), 205-224.
- [15] K.Tanahashi, Best possibility of the Furuta inequality, Proc. Amer. Math. Soc., 124(1996), 141-146.

*) Department of Mathematics, Osaka kyoiku University,
Kashiwara, Osaka, 582-8582, Japan
e-mail: mfujii@cc.osaka-kyoiku.ac.jp

**) Maebashi Institute of Technology,
Kamisadori, Maebashi, Gunma, 371-0816, Japan
e-mail: kamei@maebashi-it.ac.jp

***) Higashi-Toyonaka Senior High School,
Shinsenriminami, Toyonaka, Osaka, 565-0084, Japan