

Characteristic quasi-polynomials of hyperplane arrangements and Ehrhart quasi-polynomials

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Abstract

This is a short summary of the author's talk "Characteristic quasi-polynomials of hyperplane arrangements and Ehrhart quasi-polynomials" in RIMS joint research "Research on finite groups, algebraic combinatorics, and vertex algebras", 5-8 December 2022.

1 Quasi-polynomial

Definition 1.1. A function $F : \mathbb{Z}$ (or $\mathbb{Z}_{\geq 0}$) $\longrightarrow \mathbb{C}$ is called a quasi-polynomial if there exist a positive integer $\rho > 0$ (called the period) and polynomials $f_1(t), f_2(t), \dots, f_\rho(t) \in \mathbb{C}[t]$ such that $F(q)$ is expressed as follows.

$$F(q) = \begin{cases} f_1(q), & \text{if } q \equiv 1 \pmod{\rho} \\ f_2(q), & \text{if } q \equiv 2 \pmod{\rho} \\ \vdots & \vdots \\ f_\rho(q), & \text{if } q \equiv \rho \pmod{\rho}. \end{cases} \quad (1)$$

Each polynomial $f_1(t), \dots, f_\rho(t)$ is called the constituent.

Example 1.2. $q_{10}(n) = \lfloor \frac{n}{10} \rfloor$ is a quasi-polynomial. Indeed,

$$q_{10}(n) = \begin{cases} \frac{n}{10}, & \text{if } n \equiv 0 \pmod{10} \\ \frac{n-1}{10}, & \text{if } n \equiv 1 \pmod{10} \\ \frac{n-2}{10}, & \text{if } n \equiv 2 \pmod{10} \\ \vdots & \vdots \\ \frac{n-9}{10}, & \text{if } n \equiv 9 \pmod{10} \end{cases}$$

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The quasi-polynomial is one of the well-known classes of counting functions that appear in enumerative problems. ([4, 11]). The definition of a quasi-polynomial is somewhat more complex than that of a polynomial. However, recent research has revealed that the very complexity of quasi-polynomials contain lots of subtle properties of the objects. The purpose of this note is to introduce recent research on the constituents of quasi-polynomials.

2 Characteristic quasi-polynomials of hyperplane arrangements

A subspace of codimension one in vector space, projective space, affine space, etc. is called a hyperplane. A (finite) collection of hyperplanes is called a **hyperplane arrangement**, and it appears in various fields of mathematics [10]. One of the most important invariants is called the **characteristic polynomial**. In this section we introduce the **characteristic quasi-polynomial** which is considered as a refinement of the characteristic polynomial. The characteristic quasi-polynomial is defined for hyperplane arrangement defined over integers, namely, defined by linear forms with integer coefficients.

Let $q \in \mathbb{Z}_{>0}$. For an integer vector $\mathbf{a} = (a_1, \dots, a_\ell) \in \mathbb{Z}^\ell$, consider the linear form

$$H_{\mathbf{a}} := \{(x_1, \dots, x_\ell) \mid a_1x_1 + \dots + a_\ell x_\ell = 0\}.$$

Denote “ mod q hyperplane” by

$$\overline{H}_{\mathbf{a}} := \{(x_1, \dots, x_\ell) \in (\mathbb{Z}/q\mathbb{Z})^\ell \mid a_1x_1 + \dots + a_\ell x_\ell \equiv 0 \pmod{q}\}$$

and “ mod q hyperplane” of $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\} \subset \mathbb{Z}^\ell$ by

$$M(\mathcal{A}, q) := (\mathbb{Z}/q\mathbb{Z})^\ell \setminus \bigcup_{i=1}^n \overline{H}_{\mathbf{a}_i}.$$

Then $M(\mathcal{A}, q)$ is a finite set and it is known that its cardinality is a quasi-polynomial in q .

Theorem 2.1. (*Kamiya-Takemura-Terao [7]*) *Under the above notation, $\#M(\mathcal{A}, q)$ is a quasi-polynomial in q (the characteristic quasi-polynomial). Furthermore, let $\rho > 0$ be the period and $f_1(t), f_2(t), \dots, f_\rho(t) \in \mathbb{Z}[t]$ be the constituents, The it has the following “**GCD-property**”.*

$$(i, \rho) = (j, \rho) \implies f_i(t) = f_j(t),$$

The above result shows that the constituents $f_i(t)$ depends only on the $\gcd(i, \rho)$. It means that the characteristic quasi-polynomials form a very special class of quasi-polynomials. Athanasiadis [2, 3] proved that the “prime constituent $f_1(t)$ ” is equal to the characteristic polynomial $\chi(\mathcal{A}, t) \in \mathbb{Z}[t]$ of $\mathcal{A}$. The characteristic polynomial captures lots of information on $\mathcal{A}$. For example, for a hyperplane arrangement in $\mathbb{C}^\ell$, the Poincaré polynomial of is expressed as

$$(-t)^\ell \cdot \chi(\mathcal{A}, -\frac{1}{t}). \quad (2)$$

Namely, the coefficients of the characteristic polynomial are Betti numbers of the complement (Orlik-Solomon).

Example 2.2. Let $\mathcal{A} = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\} \subset \mathbb{Z}^2$. Namely, consider three lines defined by $y = 0, x + y = 0, 3x + y = 0$. The characteristic quasi-polynomial has a period $\rho = 6$ and constituents are.

$$\#M(\mathcal{A}, q) = \begin{cases} q^2 - 3q + 2, & \text{if } q \equiv 1, 5 \pmod{6} \\ q^2 - 3q + 3, & \text{if } q \equiv 2, 4 \pmod{6} \\ q^2 - 3q + 4, & \text{if } q \equiv 3 \pmod{6} \\ q^2 - 3q + 5, & \text{if } q \equiv 0 \pmod{6}. \end{cases}$$

As we noticed, for i which is coprime to ρ , the i -th constituent is $f_i(t) = f_1(t)$, which is equal to the characteristic polynomial $\chi(\mathcal{A}, t)$. It is a natural question to ask what are the other constituents. Recently, in a joint work with Y. Liu, T. N. Tran, it has been shown that the constituents of the characteristic quasi-polynomial are closely related to the topology of toric arrangements. Here, “considering toric arrangement” means that taking tensor product with $\otimes \mathbb{C}^\times$. In other words, interpret the integer linear forms through $\otimes_{\mathbb{Z}} \mathbb{C}^\times$. For example, using Example 2.2, consider subtori of $(\mathbb{C}^\times)^2 \ni (t_1, t_2)$, defined by $t = 1, t_1 t_2 = 1, t_1^3 t_2 = 1$. Denote the complement of associated toric arrangement by $M(\mathcal{A}, \mathbb{C}^\times)$. Then on the contrary to the characteristic polynomial, the most degenerate constituent $f_\rho(t)$ is shown to be related to the Poincaré polynomial of $M(\mathcal{A}, \mathbb{C}^\times)$.

Theorem 2.3. ([8, 13])

1. The Poincaré polynomial of $M(\mathcal{A}, \mathbb{C}^\times)$ is equal to

$$(-t)^\ell \cdot f_\rho(-\frac{1+t}{t}). \quad (3)$$

2. The constituent $f_k(t)$ is equal to the characteristic polynomial of the poset of layers (connected components of subtori of intersections) which contain k -torsion points.

Remark 2.4. Observe that only the difference between formulae (2) and (3) is the numerator, one is 1 the other is $1 + t$. These are exactly equal to the Poincarè polynomial of $\mathbb{C}$ and $\mathbb{C}^\times$. This is not just by chance, but more generally, for any abelian Lie group G , one can define the G -characteristic polynomial, and when G is non-compact, we have a similar formula. Then $f_1(t)$, $f_\rho(t)$ are nothing but the G -characteristic polynomials for $G = \mathbb{C}, \mathbb{C}^\times$ respectively. See [8] for details.

3 Ehrhart quasi-polynomials of translated lattice polytopes

For a rational polytope $P \subset \mathbb{R}^n$, let

$$L_P(t) := \#(tP \cap \mathbb{Z}^n).$$

It is known that $L_P(t)$ is a quasi-polynomial in $t \in \mathbb{Z}_{\geq 0}$ (Ehrhart quasi-polynomial [4]). It is also known that the gcd of denominators of coordinates of vertices gives a period (not necessarily the minimum period). In particular, when P is a lattice polytope, $L_P(t)$ is a polynomial in t .

As was shown in the previous section, the characteristic quasi-polynomial of a hyperplane arrangement has GCD-property. On the other hand, the Ehrhart quasi-polynomials does not have GCD-property. However, there are some classes of polytopes which Ehrhart quasi-polynomials have GCD-property.

For example, Suter ([12]) computed Ehrhart quasi-polynomials for the fundamental alcoves of root systems, and observed that the Ehrhart quasi-polynomials have GCD-property¹.

To formulate the main result, we need the following.

Definition 3.1. A quasi-polynomial (1) is **symmetric** if the constituents satisfy $f_i(t) = f_{\rho-i}(t)$ ($i = 1, \dots, \rho - 1$).

Clearly, quasi-polynomial with GCD-property is symmetric.

Example 3.2. Let P_1, P_2, P_3 be

- $P_1 = \frac{1}{9} \cdot [0, 1]^3$ (the cube with edge length $\frac{1}{9}$).

¹At the moment of Suter's computation, the reason of GCD-property was not clear. However, later it was revealed that it comes from the GCD-property of the characteristic quasi-polynomial of arrangements [14].

- $P_2 = (\frac{5}{9}, \frac{5}{9}, \frac{2}{3})^t + \text{Conv}\{\pm e_i \mid i = 1, 2, 3\}$ (A rational translation of the regular octahedron $\text{Conv}\{\pm e_i \mid i = 1, 2, 3\}$).
- $P_3 = (\frac{1}{9}, \frac{2}{9}, \frac{1}{3})^t + [0, 1]^3$ (Rational translation of the unit cube).

All Ehrhart quasi-polynomials $L_{P_1}(t)$, $L_{P_2}(t)$, and $L_{P_3}(t)$ have the minimum period 9. The constituents are as follows.

$$L_{P_2}(t) = \begin{cases} \frac{4}{3}t^3 - \frac{4}{3}t, & (t \equiv 1, 8 \pmod{9}), \\ \frac{4}{3}t^3 + \frac{2}{3}t, & (t \equiv 2, 7 \pmod{9}), \\ \frac{4}{3}t^3 + t^2 + \frac{2}{3}t, & (t \equiv 3, 6 \pmod{9}), \\ \frac{4}{3}t^3 - \frac{1}{3}t, & (t \equiv 4, 5 \pmod{9}), \\ \frac{4}{3}t^3 + 2t^2 + \frac{8}{3}t + 1, & (t \equiv 9 \pmod{9}), \end{cases}$$

$$L_{P_3}(t) = \begin{cases} t^3 & (t \equiv 1, 2, 4, 5, 7, 8 \pmod{9}), \\ t^3 + t & (t \equiv 3, 6 \pmod{9}), \\ (t+1)^3 & (t \equiv 9 \pmod{9}). \end{cases}$$

Note that $L_{P_1}(t)$ has 9 mutually distinct constituents. $L_{P_2}(t)$ is symmetric, and $L_{P_3}(t)$ has GCD-property.

The purpose of this section is to formulate a relationship between these properties and the shape of polytopes.

Theorem 3.3. ([6]) Let $P \subset \mathbb{R}^d$ be a d -dimensional lattice polytopes. Then the following are equivalent.

- (1a) P is centrally symmetric.
- (1b) For any rational vector $v \in \mathbb{Q}^d$, $L_{v+P}(t)$ is symmetric quasi-polynomial.

Theorem 3.4. ([6]) Let $P \subset \mathbb{R}^d$ be a d -dimensional lattice polytopes. Then the following are equivalent.

- (2a) P is a zonotope.
- (2b) For any rational vector $v \in \mathbb{Q}^d$, $L_{v+P}(t)$ has GCD-property.

Simply speaking, symmetry or GCD-property of the Ehrhart quasi-polynomials are closely related to central symmetry and being zonotope.

Here we give a sketch of the proof. The implication (2a) $\implies$ (2b) is proved by using a recent formula Ardila-Beck-McWhirter [1], which is a generalization of Stanley's formula of the Ehrhart polynomial of a lattice zonotope.

(1a) $\implies$ (1b) needs a description of the constituents of rationally translated lattice polytope. We consider a variant of Ehrhart quasi-polynomial as follows. Let P be a lattice polytope, $\mathbf{v}$ be a rational vector. Define $L_{(P,\mathbf{v})}(t)$ to be

$$L_{(P,\mathbf{v})}(t) := \#(\mathbf{v} + tP) \cap \mathbb{Z}^n.$$

Then we have the following.

Lemma 3.5. ([6]) $L_{(P,\mathbf{v})}(t)$ is a polynomial in t .

Using this formula, we can describe constituents.

Lemma 3.6. ([6]) Let P be a lattice polytope, $\mathbf{v}$ be a rational vector. Then, the k -th constituent of the Ehrhart quasi-polynomial of $P + \mathbf{v}$ is equal to $L_{(P,k\mathbf{v})}(t)$.

Using this lemma, (1a) $\implies$ (1b) is easily obtained.

The converse implications (b) $\implies$ (a) for (1), (2) are not easy. It needs characterizations of centrally symmetric polytopes and zonotopes by Minkowski and McMullen. More explicitly, for non centrally symmetric (or non zonotope) polytope P , we need to find a rational vector $\mathbf{v}$ such that $L_{(P,\mathbf{v})}(t) \neq L_{(P,-\mathbf{v})}(t)$ ($L_{(P,\mathbf{v})}(t) \neq L_{(P,2\mathbf{v})}(t)$ (with the odd denominator $\mathbf{v}$)). See [6] for details.

The results may be summarized as in the table.

Polytopes	Zonotoes	$\subset$	Centrally symmetric	$\subset$	General
Quasi-polynomial	GCD-property	$\subset$	Symmetry	$\subset$	General

4 Future problems

So far, extensive research has been done on the Ehrhart (quasi-)polynomials of rational polytopes. One of the well studied subjects is the so-called the period collapse phenomenon. The present research may suggest a different direction, namely, “rational polytopes obtained by translating lattice polytopes with rational vectors” may be a noteworthy class.

It is also a natural to ask for what kind of rational polytopes, the Ehrhart quasi-polynomials have GCD-property. There are few known examples.

- The rational polytope P such that $2P$ is a lattice polytope (period is 2).
- The fundamental alcoves of root systems.

- A polytope obtained from a lattice polytope by translation with a rational vector (see below).

At this moment, the above are only known classes of rational polytopes whose Ehrhart quasi-polynomials have GCD-property.

Since GCD-property gives a very special type of quasi-polynomials. It would be also a natural question to ask the relationship between the Ehrhart quasi-polynomials of the translated lattice zonotopes and the characteristic quasi-polynomials of hyperplane arrangements.

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