

On a generalization of Avhadiev and Aksentev's theorem

By

M. Nunokawa, T. Yaguchi, K. Takano and G. Salagean

1. Introduction

Let

$$F(z) = \sum_{n=0}^{\infty} a_n z^n$$

be analytic and univalent in $D = \{z \mid |z| < 1\}$ and suppose that $F(D) = E$. If $f(z)$ is analytic in D , $f(0) = F(0)$ and $f(D) \subset E$, then we call that $f(z)$ is subordinate to $F(z)$ in D , and we write $f(z) \prec F(z)$ in D . In 1943, Rogosinski [3] obtained the following theorem.

Theorem A. *If $f(z) \prec F(z)$ in D and if $0 < p$, then*

$$\int_0^{2\pi} |f(re^{i\theta})|^p d\theta \leq \int_0^{2\pi} |F(re^{i\theta})|^p d\theta$$

for $0 < r < 1$.

On the other hand, in 1973, Avhadiev and Aksentev [1] obtained a theorem which is congruently with Rogosinski's theorem.

Theorem B. *If $f(z)$ and $F(z)$, with $f(0) = F(0)$, are analytic in D and $f(z) \prec F(z)$ in D , then*

$$\int_0^{2\pi} |\operatorname{Re} f(re^{i\theta})| d\theta \leq \int_0^{2\pi} |\operatorname{Re} F(re^{i\theta})| d\theta$$

for $0 < r < 1$.

Applying Theorem A and B, Nunokawa, Fukui and Saitoh [2] obtained the following theorem.

Theorem C. If $f(z)$ and $F(z)$, with $f(0) = F(0)$, are analytic in D and $f(z) \prec F(z)$ in D , then

$$\int_0^{2\pi} |\operatorname{Re} f(re^{i\theta})|^2 d\theta \leq \int_0^{2\pi} |\operatorname{Re} F(re^{i\theta})|^2 d\theta$$

for $0 < r < 1$.

2. Main result

In this paper, we will prove the following theorem.

Theorem 1. If $f(z)$ and $F(z)$, with $f(0) = F(0)$, are analytic in D and

$$f(z) \prec F(z) \tag{1}$$

in D , then

$$\int_0^{2\pi} |\operatorname{Re} f(re^{i\theta})|^p d\theta \leq \int_0^{2\pi} |\operatorname{Re} F(re^{i\theta})|^p d\theta$$

where $1 < p$ and $0 < r < 1$.

Proof. From the hypothesis (1), we can write

$$f(z) = F(\phi(z))$$

where $\phi(z)$ is analytic in D , $|\phi(z)| < 1$ in D and $\phi(0) = 0$. Then, from the harmonic function theory, we can write

$$\begin{aligned} \operatorname{Re} f(z) &= \operatorname{Re} f(re^{i\theta}) \\ &= \operatorname{Re} F(re^{i\theta}) \\ &= \frac{1}{2\pi} \int_0^{2\pi} (\operatorname{Re} F(\rho e^{i\nu})) \operatorname{Re} \frac{\rho e^{i\nu} + \phi(re^{i\theta})}{\rho e^{i\nu} - \phi(re^{i\theta})} d\nu \end{aligned}$$

where

$$|\phi(re^{i\theta})| < |z| = |re^{i\theta}| = r < \rho < 1.$$

On the other hand, we will obtain the following easy calculation

$$\begin{aligned} & \left| \operatorname{Re} \frac{\rho e^{i\nu} + \phi(re^{i\theta})}{\rho e^{i\nu} - \phi(re^{i\theta})} \right| \\ &= \frac{\rho^2 - |\phi(re^{i\theta})|^2}{\rho^2 - 2\rho|\phi(re^{i\theta})|\cos(\nu - \arg \phi(re^{i\theta})) + |\phi(re^{i\theta})|^2} \\ &= \operatorname{Re} \frac{\rho e^{i\nu} + \phi(re^{i\theta})}{\rho e^{i\nu} - \phi(re^{i\theta})} > 0 \end{aligned} \tag{2}$$

and

$$\begin{aligned} \int_0^{2\pi} \frac{\rho e^{i\nu} + \phi(re^{i\theta})}{\rho e^{i\nu} - \phi(re^{i\theta})} d\theta &= \int_{|z|=r} \frac{\rho e^{i\nu} + \phi(re^{i\theta})}{\rho e^{i\nu} - \phi(re^{i\theta})} \frac{dz}{iz} \\ &= \frac{1}{i} \int_{|z|=r} \frac{\rho e^{i\nu} + \phi(z)}{z(\rho e^{i\nu} - \phi(z))} dz = 2\pi, \end{aligned}$$

and therefore we have

$$\int_0^{2\pi} \operatorname{Re} \frac{\rho e^{i\nu} + \phi(re^{i\theta})}{\rho e^{i\nu} - \phi(re^{i\theta})} d\theta = 2\pi \quad (3)$$

where $z = re^{i\theta}$ and $0 < r < \rho < 1$. Let us put

$$R(\rho, r, \phi, \theta) = \operatorname{Re} \frac{\rho e^{i\nu} + \phi(re^{i\theta})}{\rho e^{i\nu} - \phi(re^{i\theta})}.$$

Applying (2),(3) and Hölder's inequality, we have the following inequality

$$\begin{aligned} &\int_0^{2\pi} \left| \operatorname{Re} f(re^{i\theta}) \right|^p d\theta \\ &= \int_0^{2\pi} \left| \frac{1}{2\pi} \int_0^{2\pi} \left(\operatorname{Re} F(\rho e^{i\nu}) \right) R(\rho, r, \phi, \theta) d\nu \right|^p d\theta \\ &\leq \int_0^{2\pi} \frac{1}{(2\pi)^p} \left| \left\{ \int_0^{2\pi} |\operatorname{Re} F(\rho e^{i\nu})|^p \left(R(\rho, r, \phi, \theta) \right)^{\frac{1}{p}p} d\nu \right\}^{\frac{1}{p}} \right. \\ &\quad \cdot \left. \left\{ \int_0^{2\pi} \left(R(\rho, r, \phi, \theta) \right)^{\frac{1}{q}q} d\nu \right\}^{\frac{1}{q}} \right|^p d\theta \\ &= \frac{1}{(2\pi)^p} \int_0^{2\pi} \left| \left\{ \int_0^{2\pi} |\operatorname{Re} F(\rho e^{i\nu})|^p R(\rho, r, \phi, \theta) d\nu \right\}^{\frac{1}{p}} \right. \\ &\quad \cdot \left. \left\{ \int_0^{2\pi} R(\rho, r, \phi, \theta) d\nu \right\}^{\frac{1}{q}} \right|^p d\theta \\ &= (2\pi)^{-p+\frac{p}{q}} \int_0^{2\pi} \left(\int_0^{2\pi} |\operatorname{Re} F(\rho e^{i\nu})|^p R(\rho, r, \phi, \theta) d\nu \right) d\theta \\ &= (2\pi)^{-p+\frac{p}{q}} \int_0^{2\pi} |\operatorname{Re} F(\rho e^{i\nu})|^p \left(\int_0^{2\pi} R(\rho, r, \phi, \theta) d\theta \right) d\nu \\ &= (2\pi)^{-p+\frac{p}{q}+1} \int_0^{2\pi} |\operatorname{Re} F(\rho e^{i\nu})|^p d\nu \\ &= \int_0^{2\pi} |\operatorname{Re} F(\rho e^{i\nu})|^p d\nu \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p$ and $1 < q$. Putting $r \rightarrow \rho$, it completes the proof. $\square$

Remark 1. *In Theorem 1, we can not prove it for the case $0 < p < 1$. It is an open problem.*

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Mamoru Nunokawa
 Emeritus Professor of University of Gunma
 Hosikuki-cho 798-8, Chuou-ward, Chiba city 260-0808, Japan
 e-mail: mamoru-nuno@doctor.nifty.jp

Teruo Yaguchi
 Department of Computer Science and System Analysis,
 College of Humanities and Sciences, Nihon University
 3-25-40 Sakurajosui Setagaya Tokyo, 156-8550, Japan
 e-mail:yagichi@cssa.chs.nihon-u.ac.jp

Katsuo Takano
 Emeritus Professor of Ibaraki University
 Motoyosida 59-40, Mito city 310-0386, Japan
 e-mail: ktaka@mx.ibaraki.ac.jp

Grigore S. Salagean
 Faculty of Mathematics and Computer Science,
 Str. Kogalniceanu nr. 1, 400084 Cluj-Napoca, Romania
 e-mailsalagean@math.ubbcluj.ro