

DISTRIBUTION OF RATIONAL POINTS ON FANO VARIETIES

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§0. Introduction

The Hardy–Littlewood circle method can be applied to certain problems of the following kind. Consider a system of homogeneous polynomial equations

$$V : F_i(x_0, \dots, x_n) = 0, \quad i = 1, \dots, r; \quad F_i \in \mathbb{Z}[x_0, \dots, x_n]; \quad \deg F_i = d_i.$$

We want to find an asymptotic for the number of primitive integral solutions of this system of bounded but growing size, e.g. fitting in the box $B_H = HB_1$ as $H \rightarrow \infty$.

The basic requirements for applicability are: a) $n + 1 - \sum_i^r d_i$ is positive; b) V has sufficiently many points locally over $\mathbb{R}$ and all $\mathbb{Q}_p$. It is easy to deduce formally what is expected to be the main term of the asymptotics. On the contrary, many technical problems crop up in estimating the remainder term. They are overcome by adopting more restrictions. Some of them must be rather arbitrary, other might be essential for the truth of the result, but nobody understands much about which are which. See [Bi] for some early work, and [S] for later development.

Whenever the technique works, it shows that the number of the primitive solutions of the system in question with, say, $\max(|x_i|) \leq H$ is $CH^{n+1-\sum_i^r d_i}(1 + o(1))$ where C is a product of local densities.

In this talk I review a number of notions, conjectures, and results that may be considered as an extension of this theorem in a wider context of Diophantine geometry. Roughly speaking, the generalization proceeds along the following lines.

a). The system of equations V is replaced by an arbitrary projective Fano manifold over a number field k . This means that the anticanonical class on it is ample. Notice that a smooth complete intersection as V above is Fano precisely when $n + 1 - \sum_i^r d_i > 0$.

b). "Size" of a solution is interpreted as the height h_L of a k -point $x \in V(k)$ with respect to a metrized line bundle L , and the dependence of the asymptotic (for the number of solutions of bounded size) on L and h_L is scrutinized.

c). The necessity to delete some accumulating subvarieties of V before starting counting is recognized, and the geometry of accumulation is discussed (cf. [BM]). Without this, no meaningful general picture emerges.

Below we start with describing an ideal situation. To simplify language, we present it in the form of a precise conjecture about the asymptotics. However, this conjecture should better be considered as a question, because not much evidence for its general validity exists. We then discuss partial results in the direction of this conjecture.

The program reviewed here is due to joint efforts of my friends and colleagues V. Batyrev, J. Franke, Yu. Tschinkel, A. Peyre, collaboration with whom was my

joy and privilege. Proofs and additional details can be found in the original works cited in References.

§1. Conjectural asymptotic formula

1.1. Notation. Let k be a number field of finite degree, v runs over places of k . The local norm $|\cdot|_v : k_v^* \rightarrow \mathbf{R}$ is defined by the condition $d\mu(ax) = |a|_v d\mu(x)$ where $d\mu$ is an additive Haar measure on k_v .

Let V be a projective manifold over k , L a metrized invertible sheaf on V . This means, in particular, that each geometric fiber $L(x)$, $x \in V(k)$, is endowed with v -adic norm $\|\cdot\|_v$ continuously depending on x and satisfying two conditions: i) $\|al\|_v = |a|_v \|l\|_v$ for all $l \in L(x)$; ii) $\|l\|_v = 1$ for $l \neq 0$ and almost all v .

The (exponential) height function $h_L : V(k) \rightarrow \mathbf{R}$ is then defined by the formula

$$h_L(x) = \prod_v \|s(x)\|_v^{-1}$$

where s is any local section of L nonvanishing at x . If L is ample, this is a counting function, i. e. there is only a finite number of rational points of bounded height.

For a subset $U \subset V(k)$ we put

$$N_U(h_L, H) = \sum_{x \in U, h_L(x) \leq H} 1,$$

$$Z_U(h_L, s) = \sum_{x \in U} h_L(x)^{-s}.$$

1.2. Main Conjecture. Let V be a Fano manifold over k such that $V(k)$ is Zariski dense (e.g. because V is k -unirational), U a sufficiently small dense Zariski open subset of $V(k)$. Then for $H \rightarrow \infty$ we have

$$N_U(h_L, H) = c(h_L) H^{a(L)} (\log H)^{t(L)-1} (1 + o(1)) \quad (1.1)$$

where $c(h_L)$, $a(L)$, $t(L)$ are defined below. (In 1.2.1 and 1.2.2 we identify L with its image in $\text{Pic } V \otimes \mathbf{R}$).

1.2.1. $a(\cdot)$ is the unique function on the open ample cone in $\text{Pic } V \otimes \mathbf{R}$ which is homogeneous of degree -1 and equals 1 on $P := -K_V + \text{the boundary of the effective cone}$.

In particular, $a(-K_V) = 1$. This is a strong form of the "Linear Growth Conjecture". For a weaker version which does not presuppose existence of an asymptotic formula and therefore is better suited for algebro-geometric study see [BM] and [M2].

1.2.2. $t(L)$ is the codimension in $\text{Pic } V \otimes \mathbf{R}$ of the smallest face of P to which a point proportional to L belongs.

This was suggested in [BM], together with a weaker, but unconditional version.

1.2.3. A formula for $c(h_L)$ was suggested by E. Peyre [P], but only for anti-canonical heights:

$$c(h_{-K_V}) = A.B.\tau \quad (1.2)$$

where A measures the size of the effective cone, B accounts for a possible weak approximation failure, and τ is a version of the Tamagawa number generalizing the classical product of the singular series by the singular integral. (Actually, Peyre did not introduce B explicitly).

To define A , consider the space N dual to $\text{Pic } V \otimes \mathbf{R}$ and the cone $N_+ \subset N$ dual to the effective cone. (N can be realized as the space of numerical equivalence classes of curves on V , but this is irrelevant here). Let D be the intersection of N_+ with the affine hyperplane $\langle -K_V, \cdot \rangle = 1$. Denote by dy the Haar measure in N for which the lattice dual to in $\text{Pic } V$ gets volume 1. Let $d_t y$ be the derivative measure on $\langle -K_V, \cdot \rangle = t$, in particular $d_1 y$ is a measure on D . We finally put:

$$A = \int_D d_1 y.$$

In the degenerate case of Picard group of rank 1, A equals to the inverse index of subgroup generated by $-K_V$.

Denote now by $\text{Br}(V)$ the Brauer group of V and put

$$B = |\text{Br}(V)/\text{Br}(k)| \quad (1.3)$$

(cf. comments 1.3c) below).

We now turn to the definition of the Tamagawa number. Start with recalling A. Weil's prescription for integrating a volume form over k_v -points of an algebraic (or analytic) manifold. For a relatively compact open subset $U \subset V(k_v)$ which belongs to a coordinate neighborhood $(x_1, \dots, x_n)$ in V we write $\omega = f dx_1 \wedge \dots \wedge x_n$ and then put

$$\int_U \omega = \int_U |f(x)|_v d\mu(x)$$

where $d\mu$ is the additive Haar measure on k_v^n giving volume $\delta_v^{-1/2}$ to $\mathcal{O}_v^n$, δ_v being the absolute local different.

Now if the sheaf of volume forms is v -adically metrized by $\|\cdot\|_v$, we define a measure μ_v on $V(k_v)$ by putting in any coordinate neighborhood $\mu_v = \omega / \|\omega\|_v$ where ω is any meromorphic volume form regular and non-vanishing in this neighborhood. So the measure varies with the change of local metrics.

In the global situation of 1.1, we might be tempted now to define the adelic Tamagawa measure as a product of μ_v , however in general the integral will diverge so that local corrections must be first inserted. To understand them, we notice that if $\|\cdot\|_v$ is induced by an integral structure on V with good reduction at v , then denoting the closed fiber by $\tilde{V}$ we have

$$\int_{V(k_v)} \mu_v = \frac{|\tilde{V}(F_v)|}{q_v^{\dim V}} = 1 + \frac{\text{Tr}(\Phi_v | \text{Pic } \tilde{V} \otimes \bar{F}_v)}{q_v} + O(q_v^{-3/2}) \quad (1.4)$$

where q_v is the cardinality of the residue field F_v and Φ_v is the local Frobenius.

The local factors of Artin's L -function (without archimedean ones)

$$L(s, \text{Pic}(V \otimes \bar{k})) := \prod_v \det(1 - q_v^{-1} \Phi_v | \text{Pic}(V \otimes \bar{k})^{I_v})^{-1} := \prod_v L_v(s)$$

can be used to cancel the divergence of the product of (1.4) because $L_v(1)$ differs from the first two summands of (1.4) by $O(q_v^{-2})$.

Following [P], we choose now a finite set S of places containing non-archimedean and bad reduction ones, put $L_S = \prod_{v \notin S} L_v$, $\lambda_v = L_v(1)$ for $v \notin S$, 1 for $v \in S$.

Finally, to get the number τ we integrate this measure over the closure of $V(k)$ in $V(A_k)$ in the direct product topology.

It might be helpful for the reader to look at the case $V = \mathbf{P}_k^n$ where the constant (for a natural $O(1)$ -height) was calculated by Schanuel. For the anticanonical height it takes form

$$\frac{h}{\zeta_k(n+1)} \left(\frac{2^{r_1} (2\pi)^{r_2}}{\sqrt{d}} \right)^{n+1} (n+1)^r \frac{R}{w}.$$

where w is the number of roots of unity in k , R regulator, d discriminant. On the other hand, $A = (n+1)^{-1}$, $B = 1$, and the rest is precisely the Tamagawa number.

1.3. Comments. a). As was mentioned, we expect a formula of the type (1.1) with predicted values of $a(L)$ only after removal of some varieties accumulating rational points. A standard example of such a variety is a k -rational exceptional curve on a del Pezzo surface (Fano variety of dimension 2, e.g. a cubic surface in $\mathbf{P}^3$, or intersection of two quadrics in $\mathbf{P}^4$). In fact, such a line contributes about CH^2 points of anticanonical height $\leq H$. This example teaches us something general about the geometry of accumulating subvarieties: if we expect the linear growth conjecture, then any subvariety having a "too small" restriction of the anticanonical class on it, must be $-K$ -accumulating.

Generally, accumulating subvarieties can be defined without any conjectures, and they form a sequence which hypothetically stabilizes for Fano manifolds, but may be infinite, say for K3 or Enriques surfaces: see a discussion in [BM] and [M2].

b). The formula for $t(L)$ implicitly assumes that the effective cone is finite polyhedral. This is known from Mori theory for the ample cone, and follows by duality for effective cone in dimension two and, with some effort, in dimension three (Batyrev). The problem in higher dimension seems open.

c). For a discussion of failure of weak approximation and its influence on the constant $c(h_{-K})$, see a controversy between [H-B] and [Sw-D]. To my opinion, Swinnerton-Dyer convincingly argues that a correction accounting for possible failure must be made, and that it can be done by defining the Tamagawa number via integration over the adelic closure of $V(k)$ instead of $V(A_k)$. the correction factor B . The tentative identification of the correction factor B with the order of the Brauer group is a flight of fancy, only partly supported by the philosophy of the Brauer-Manin obstruction: cf. [M1], [C-TS] and many other publications of

Colliot-Thélène and his collaborators. However, at least for compactifications of anisotropic tori this is justified: cf. [BT].

d). The height zeta-function introduced in 1.1 had never reappeared in the preceding discussion, but of course one hopes that the asymptotic formulas for the point counting function in fact reflect the analytic behavior of the zeta-function near the boundary of the convergence domain. This is actually the way some of the results cited in the next section were proved. The function itself is however very sensitive to the choice of the height, and e.g. for complete intersections its global properties are unknown.

§2. Results

We will refer to the asymptotic formula (1.1) as Conjecture, adding necessary qualifications about L, h_L etc when need arises.

2.1. Theorem. *Let V be a Fano smooth complete intersection of dimension ≥ 3 satisfying weak approximation. Then the main term of the circle method formula is equivalent to the asymptotics of the Conjecture for $L = -K_V$.*

This was stated in [FMT] without paying attention to the constant. A. Peyre [P] settled the value of the constant. Of course, the formula furnished by the circle method was anyway the principal source for the general Conjecture.

2.2. Theorem. *Let G be a semi-simple connected simply connected algebraic group over k , $P \subset G$ a parabolic k -subgroup, $V = P \backslash G$. Choose a standard family of maximal compact subgroups of $G(k_v)$. Then for any height function on V related to the metrics locally invariant with respect to these compact subgroups the Conjecture holds. For an anticanonical height of this type the constant is correct at least if G is quasi-split, i.e. contains a Borel subgroup defined over k .*

This was proved in [FMT] using the deep Langlands theory of Eisenstein series. A formula for the anticanonical constant was also given. A. Peyre [P] then established that it can be rewritten as in (1.2).

Note that unlike the case of complete intersections, a generalized flag space V can well have rank of the Picard group bigger than 1 so that powers of $\log H$ actually appear in the formula and the contribution A is not 1.

2.3. Theorem. *Let V be a smooth compactification of an anisotropic torus over k . Then the Conjecture holds for V and a certain family of heights defined (not uniquely) by the condition of local invariance with respect to the action of a compact subgroup of $T(A_k)$. If the height is anticanonical, it is a counting function even for non-Fano compactifications V , and the Conjecture holds for it with the correct constant.*

In this case treated in [BT] the non-trivial Brauer group appears for the first time. Note also that already in dimension three there exist non-rational anisotropic tori and their compactifications.

The authors use a generalization of the Tate method (adelic Poisson formula) in order to investigate the height zeta-function and its singularities.

2.4. Theorem. *If the Conjecture holds for (V_i, h_i) , $i = 1, 2$, with remainder terms $O(H(\log H)^{t_i-2})$, $t_i = \text{rk Pic}(V_i)$, then it holds for $(V_1 \times V_2, h_1 h_2)$. If h_i are anticanonical and the asymptotics have correct constants, the same is true for $h_1 h_2$.*

This was checked in [FMT] and [P] (anticanonical constants).

The last general result concerns the compatibility of the Conjecture with the change of height (or the box B_1 in the Introduction). Note that Theorems 2.2 and 2.3 were stated only for some particular classes of metrics. As we have already remarked, global properties of the height zetas are sensitive to the choice: compare e.g. zetas for two anticanonical heights on $\mathbf{P}^1(\mathbf{Q})$: $\max(x_0^2, x_1^2)$ and $x_0^2 + x_1^2$.

A. Peyre has shown that the compatibility of the anticanonical constants with different choices of the height functions is equivalent to a certain equidistribution property.

More precisely, call an open subset $W \subset V(A_k)$ *good* if a Tamagawa measure of its boundary is zero. Let $U \subset V(k)$ be a Zariski open dense subset. We will say that U *satisfies the equidistribution property* E_U if there exists an anticanonical height h such that for all good W , $N_{U \cap W}(h, H)/N_U(h, H)$ tends to the quotient of the Tamagawa volumes of $\overline{V(k) \cap W}$ and $\overline{V(k)}$ (with respect to the measure associated with h) as $H \rightarrow \infty$.

Then we have ([P]):

2.5. Theorem. *a). Whenever V is a complete intersection, and the circle method is applicable, E_V holds. The same is true for quasisplit flag varieties.*

b). If E_U holds and the Conjecture is true for U and some anticanonical height, it is true for all anticanonical heights. Moreover, the average of the Dirac distributions supported by $V(k)$ and ordered by the increasing height tends to the respective Tamagawa measure.

c). Conversely, if for the complement of the accumulating subvarieties U the Conjecture holds for all anticanonical heights, then E_U holds.

2.6. Examples with accumulating subvarieties. No accumulating subvarieties appear in the situation of Theorems 2.1–2.3 (the divisor at infinity of a toric compactification can be accumulating for non-anisotropic tori, but it carries no k -points in the anisotropic case). Therefore the following examples treated by A. Peyre directly present considerable interest showing that the Conjecture still holds for them (they are all special examples of toric compactifications for split tori): $\mathbf{P}^2$ over $\mathbf{Q}$ with ≤ 3 points blown up; $\mathbf{P}^n$ with a certain family of projective subspaces blown up.

2.7. Further comments. The existing methods do not allow us to treat even cubic surfaces which remain the first challenge. Nevertheless, numerical calculations and an averaging argument are compatible with the Conjecture: see [H–B] and [Sw–D].

Note that for cubic curves the asymptotics $c(\log H)^{r/2}$ is known. It can be deduced from the Mordell–Weil theorem and the existence of exactly quadratic Néron–Tate heights. Poincaré formulated the Mordell–Weil property as a conjecture about the finite generation of k -points of a plane cubic curve with respect to the

composition described by secants and tangents. This construction remains valid, but the question remains unresolved for cubic surfaces. In my recent note [M3] a different version of finite generation is proved and some numerical evidence for the initial problem is presented. However, no connection between the finite generation and asymptotic count was established.

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