

Invariant Hilbert Scheme Resolution of Popov's $SL(2)$ -varieties

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The Invariant Hilbert Scheme

Question (general case).

Does the **Hilbert–Chow morphism**

$$\gamma : \text{Hilb}_h^G(X) \longrightarrow X//G \quad (\clubsuit)$$

give a **desingularization** of $X//G$?

Definition (Alexeev–Brion).

The **invariant Hilbert scheme** is defined as follows:

$$\text{Hilb}_h^G(X) = \left\{ Z \subset X \mid \begin{array}{l} Z \text{ is a closed } G\text{-subscheme of } X; \\ \mathbb{C}[Z] \cong \bigoplus_{M \in \text{Irr}(G)} M^{\oplus h(M)} \text{ as a } G\text{-module} \end{array} \right\}.$$

- G : a reductive algebraic group;
- X : an affine G -variety;
- $h : \text{Irr}(G) \rightarrow \mathbb{N}$: a **Hilbert function**.

Remark.

$\text{Hilb}_h^G(X)$ is a generalization of $G\text{-Hilb}(X)$ for a finite group G .

- $\pi : X \longrightarrow X//G$: the quotient morphism.
- $h :=$ Hilbert function of the **flat locus** $W \longrightarrow W//G$ of π .
- $\bullet \mathcal{H}^{main} := \gamma^{-1}(W)$: the **main component** of $\text{Hilb}_h^G(X)$.

$\leadsto \bullet \gamma|_{\mathcal{H}^{main}} : \mathcal{H}^{main} \longrightarrow X//G = \text{Spec } \mathbb{C}[X]^G$: **proj. birat.**

Popov's $SL(2)$ -varieties

Popov gave a complete **classification** of 3-dim. affine normal quasihomog. $SL(2)$ -varieties.

Theorem (Popov).

There is a one to one correspondence:

$$\left\{ \begin{array}{l} \text{3-dim. affine normal quasihomog.} \\ \text{\textit{SL}(2)-var. with a fixed point} \end{array} \right\} \longleftrightarrow \{ \mathbb{Q} \cap (0, 1) \} \times \mathbb{N}$$

$$E_{l,m} \longleftrightarrow (l, m)$$

$E_{l,m}$ contains three $SL(2)$ -orbits: $E_{l,m} = U \cup D \cup \{O\}$.

- U : the **dense open orbit**;
- D : a 2-dim. orbit;
- O : a unique $SL(2)$ -inv. **singular point**.

Popov's $SL(2)$ -varieties as a GIT Quotient

Batyrev and Haddad proved that Popov's variety has a **description as an affine quotient**.

Theorem (Batyrev–Haddad).

- $E_{l,m}$: 3-dim. affine normal quasihomog. $SL(2)$ -var.;
- $\mathbb{C}^5 \supset H_{q-p} := (X_0^{q-p} = X_1X_4 - X_2X_3)$.

Then,

$$E_{l,m} \cong H_{q-p} // (\mathbb{C}^* \times \mu_m).$$

Remark. Actions of $SL(2)$, $\mathbb{C}^*$, and μ_m on $\mathbb{C}^5$ are given as follows:

- $SL(2) \ni \forall g, \begin{pmatrix} a & c \\ b & d \end{pmatrix} \cdot \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} := \begin{pmatrix} X_0 \\ aX_1 + cX_2 \\ bX_1 + dX_2 \\ aX_3 + cX_4 \\ bX_3 + dX_4 \end{pmatrix};$
- $\mathbb{C}^* \ni \forall t, \quad t \cdot (X_0, X_1, X_2, X_3, X_4) := (tX_0, t^{-p}X_1, t^{-p}X_2, t^qX_3, t^qX_4);$
- $\mu_m \ni \forall \xi, \quad \xi \cdot (X_0, X_1, X_2, X_3, X_4) := (X_0, \xi^{-1}X_1, \xi^{-1}X_2, \xi X_3, \xi X_4).$

Question (our case). Apply $(\clubsuit)$ with

$$X = H_{q-p}, \quad G = \mathbb{C}^* \times \mu_m:$$

$$\gamma : \text{Hilb}_h^{\mathbb{C}^* \times \mu_m}(H_{q-p}) \longrightarrow E_{l,m}$$

Spherical Geometry

We use the **spherical geometry** of $E_{l,m}$ and $Bl_O^\omega(E_{l,m})$ to study

$$\gamma : \text{Hilb}_h^{\mathbb{C}^* \times \mu_m}(H_{q-p}) \longrightarrow E_{l,m}.$$

Theorem (Batyrev–Haddad).

$E_{l,m}$ and $Bl_O^\omega(E_{l,m})$ are **spherical** $SL(2) \times \mathbb{C}^*$ -varieties w.r.t. $B \times \mathbb{C}^*$.

- [Batyrev–Haddad] also computed the **colored fan** of $E_{l,m}$, $Bl_O^\omega(E_{l,m})$.
- [Brion–Pauer] Local structure th. for **toroidal spherical varieties**.

Theorem (Batyrev–Haddad).

- (i) $\exists!$ $C \cong \mathbb{P}^1$: a closed $SL(2)$ -orbit of $Bl_O^\omega(E_{l,m})$.
- (ii) Along C , $Bl_O^\omega(E_{l,m})$ is locally isomorphic to $\mathbb{C} \times \mathbb{C}^2/\mu_b$.
- (iii) $b = 1 \Leftrightarrow E_{l,m}$: **toric**.

Main Results

The Hilbert–Chow morphism decomposes as follows:

$$\begin{array}{ccc} \mathcal{H}^{main} & \xrightarrow{\exists \psi} & Bl_O^\omega(E_{l,m}) \\ & \searrow \gamma|_{\mathcal{H}^{main}} & \downarrow \\ & & E_{l,m} \end{array}$$

Main Theorem

(i) $E_{l,m}$: **toric**

$$\Rightarrow \begin{cases} \gamma : \mathcal{H}^{main} \longrightarrow E_{l,m} \text{ is a resol.} \\ \gamma^{-1}(O) \cong \mathbb{P}^1 \times \mathbb{P}^1. \end{cases}$$

$$(ii) \psi : \mathcal{H}^{main} \cong Bl_O^\omega(E_{l,m}) \iff E_{l,m} \text{ : toric.}$$

Outline of the Proof

Step 1. Determine the generators of ideals in $\gamma^{-1}(U)$.

Step 2. Construct ψ (use **Step 1.** + irred. decomp. of $\mathbb{C}[H_{q-p}]$):

$$\mathcal{H}^{main} \xrightarrow{\psi} E_{l,m} \times \mathbb{P}^1 \times \mathbb{P}^1.$$

Step 3. Use the spherical geometry of $E_{l,m}$ and $Bl_O^\omega(E_{l,m})$ to show

$$\psi(\mathcal{H}^{main}) \cong Bl_O^\omega(E_{l,m}) \hookrightarrow E_{l,m} \times \mathbb{P}^1 \times \mathbb{P}^1.$$

Step 4. $E_{l,m}$: toric $\Rightarrow \psi$ is a cl. imm. $[\cdot : \cdot]$ Description of generators for $\forall I_Z \in \mathcal{H}^{main}$.

Step 5. $E_{l,m}$: non-toric $\Rightarrow \mathcal{H}^{main} \not\cong Bl_O^\omega(E_{l,m})$ $[\cdot : \cdot]$ Calculation of flat limits of ideals.

Work in Progress

[Main Theorem] $E_{l,m}$: **non-toric** $\Rightarrow \mathcal{H}^{main} \not\cong Bl_O^\omega(E_{l,m})$.

$\leadsto$ We need a **further blow-up**.

- [Batyrev–Haddad] $E_{l,m}$: **non-toric** $\Rightarrow Bl_O^\omega(E_{l,m})$ is **sing.** along C .
- $\mathbb{C}^2/\mu_b$ has a **minimal resol.** described by **Hirzebruch–Jung continued fraction**.

Conjecture

$E_{l,m}$: **non-toric**

$$\Rightarrow \mathcal{H}^{main} \longrightarrow Bl_O^\omega(E_{l,m}) \text{ is a minimal resol.}$$