

MARKOV'S PRINCIPLE, CHURCH'S THESIS AND LINDELÖF'S THEOREM

石原 哉

北陸先端科学技術大学院大学
情報科学研究科

Abstract. The principal results of this paper are: in constructive mathematics (1) the theorem "Mappings from a complete metric space into a metric space are sequentially continuous" can be proved using a disjunctive form of Church's thesis only, and (2) the theorem "Every open cover of a complete separable metric space has an enumerable subcover" can be proved using the Extended Church's Thesis only; Markov's principle is not needed.

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In constructive mathematics which is formalized by, for example **EL** + AC_0 ¹, the extended Church's thesis, where A is almost negative:

$$\text{ECT}_0. \forall x(Ax \rightarrow \exists yBxy) \rightarrow \exists z\forall x(Ax \rightarrow \exists u(Tz xu \wedge B(x, Uu)))$$

yields, in combination with Markov's principle:

$$\text{MP}. \forall \alpha[\neg\neg\exists n(\alpha n \neq 0) \rightarrow \exists n(\alpha n \neq 0)],$$

interesting mathematical consequences, in particular the KLST-theorem ([3, Chapter 9], [6, 7.2.11])

Assuming $\text{ECT}_0 + \text{MP}$, all mappings from complete separable metric spaces into metric spaces are continuous

and a version of Lindelöf's theorem ([3, Theorem 1, 9.3], [6, 7.2.15])

Assuming $\text{ECT}_0 + \text{MP}$, every open cover of a complete separable metric space has an enumerable subcover.

Recently, the author investigated the interrelation between a weak version of Markov's principle:

$$\text{WMP}. \forall \alpha[\forall \beta(\neg\neg\exists n(\beta n \neq 0) \vee \neg\neg\exists n(\alpha n \neq 0 \wedge \beta n = 0)) \rightarrow \exists n(\alpha n \neq 0)]$$

and certain continuity principles [1, 2].

In this paper, we shall show that MP is equivalent to WMP and a disjunctive version of Markov's principle:

$$\text{MP}^\vee. \forall \alpha\beta[\neg\neg\exists n(\alpha n \neq 0 \vee \beta n \neq 0) \rightarrow \neg\neg\exists n(\alpha n \neq 0) \vee \neg\neg\exists n(\beta n \neq 0)],$$

and that the lesser limited principle of omniscience, here we call it SEP according to [5, 6]:

$$\text{SEP}. \forall \alpha\beta[\neg\exists n(\alpha n \neq 0 \vee \beta n \neq 0) \rightarrow \neg\exists n(\alpha n \neq 0) \vee \neg\exists n(\beta n \neq 0)]$$

¹For detailed exposition of **EL** and **AC**, see [5, 6]; also we follow notations and conventions in [5, 6]; for example $\alpha, \beta, \gamma, \delta$ range over $\mathbf{N} \rightarrow \mathbf{N}$.

implies $MP^\vee$. (WMP and $MP^\vee$ is equivalent to WLPE and LLPE in [4], respectively; see [2] and compare $MP^\vee$ and [4, Theorem 4.1 (j)]. So these results correspond to the results in [4].) Then we shall prove that WMP is derivable from Church's thesis for disjunctions:

$$CT_0^\vee. \forall x(Ax \vee Bx) \rightarrow \exists \alpha \in \text{TREC} \forall x((\alpha x = 0 \rightarrow Ax) \wedge (\alpha x \neq 0 \rightarrow Bx)),$$

So assuming $CT_0^\vee$, MP is equivalent to $MP^\vee$. Using these results we shall prove

Assuming $CT_0^\vee$, all mappings from complete metric spaces into metric spaces are sequentially continuous

and

Assuming ECT_0 , any open cover of a complete separable metric space has an enumerable subcover.

Now we turn to our first result [4].

Proposition 1

- (1) $MP \Leftrightarrow WMP + MP^\vee$;
- (2) $SEP \Rightarrow MP^\vee$.

Proof. (1). It is easy to see that $MP \Rightarrow WMP + MP^\vee$. To see the converse, let α be such that $\neg \neg \exists n(\alpha n \neq 0)$, and for arbitrary β define γ by

$$\gamma n := \begin{cases} 1 & \text{if } \alpha n \neq 0 \wedge \beta n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\neg \neg \exists n(\beta n \neq 0 \vee \gamma n \neq 0)$. Applying $MP^\vee$, we have

$$\neg \neg \exists n(\beta n \neq 0) \vee \neg \neg \exists n(\gamma n \neq 0)$$

or

$$\neg \neg \exists n(\beta n \neq 0) \vee \neg \neg \exists n(\alpha n \neq 0 \wedge \beta n = 0).$$

Since β is arbitrary, we have $\exists n(\alpha n \neq 0)$ by WMP.

(2). Let α and β be such that $\neg\neg\exists n(\alpha n \neq 0 \vee \beta n \neq 0)$, and define γ and δ by

$$\begin{aligned}\gamma n &:= \begin{cases} 1 & \text{if } \forall i < n(\alpha i = 0 \wedge \beta i = 0) \wedge \alpha n \neq 0 \wedge \beta n = 0, \\ 0 & \text{otherwise;} \end{cases} \\ \delta n &:= \begin{cases} 1 & \text{if } \forall i < n(\alpha i = 0 \wedge \beta i = 0) \wedge \alpha n = 0 \wedge \beta n \neq 0, \\ 0 & \text{otherwise,} \end{cases}\end{aligned}$$

Then $\neg(\exists n(\gamma n \neq 0) \wedge \exists n(\delta n \neq 0))$. Hence either $\neg\exists n(\gamma n \neq 0)$ or $\neg\exists n(\delta n \neq 0)$ by SEP. In the former case, if $\neg\exists n(\beta n \neq 0)$, then $\neg\exists n(\alpha n \neq 0)$, a contradiction; hence $\neg\neg\exists n(\beta n \neq 0)$. Similarly, in the latter case, we have $\neg\neg\exists n(\alpha n \neq 0)$. $\square$

Lemma 1 $\text{CT}_0^\forall$ proves

$$\forall x(\neg\neg\exists nTxnx \vee Ax) \rightarrow \exists y(\exists nTyyn \wedge Ay).$$

Proof. Assume $\forall x(\neg\neg\exists nTxnx \vee Ax)$ and apply $\text{CT}_0^\forall$. Then we find a total recursive γ such that

$$\forall x((\gamma x = 0 \rightarrow \neg\neg\exists nTxnx) \wedge (\gamma x \neq 0 \rightarrow Ax)).$$

Let $\{y\}$ be a partial recursive function such that

$$\gamma x \neq 0 \leftrightarrow \exists nTyxn,$$

and suppose that $\gamma y = 0$. Then $\neg\neg\exists nTyyn$ and $\neg\exists nTyyn$, a contradiction. Hence $\gamma y \neq 0$ and therefore $\exists nTyyn$ and Ay . $\square$

Proposition 2 Assuming $\text{CT}_0^\forall$, WMP holds.

Proof. Let the β in $\forall\beta(\neg\neg\exists n(\beta n \neq 0) \vee \neg\neg\exists n(\alpha n \neq 0 \wedge \beta n = 0))$ range over the characteristic function of $Txxn$ as predicate in n . Then

$$\forall x(\neg\neg\exists nTxnx \vee \neg\neg\exists n(\alpha n \neq 0 \wedge \neg Txn)).$$

Apply Lemma 1. Then we find y such that

$$\exists n Tyyn \wedge \neg \neg \exists n (\alpha n \neq 0 \wedge \neg Tyyn).$$

Choose n so that $Tyyn$ and suppose that $\forall k < n (\alpha k = 0)$. Then $\neg \exists n (\alpha n \neq 0 \wedge \neg Tyyn)$, a contradiction. Hence $\exists k < n (\alpha k \neq 0)$. $\square$

Theorem 1 *Assuming $CT_0^\vee$, MP is equivalent to $MP^\vee$.*

Proof. By Proposition 1 (1) and Proposition 2. $\square$

Theorem 2 *Assuming $CT_0^\vee$, all mappings from complete metric spaces into metric spaces are sequentially continuous.*

Proof. By $CT_0^\vee \Rightarrow \neg \forall\text{-PEM}$ ([5, Corollary 1, 4.3.4]), Proposition 2, and [2, Corollary 1]. $\square$

Let $M \equiv (X, \rho)$ be a complete separable metric space with basis $\langle p_n \rangle_n$. Let $\{x\}$ be a total recursive function such that

$$A_M(x) := \forall n (\rho(p_{\{x\}(n)}, p_{\{x\}(n+1)}) < 2^{-n-1}); \quad (*)$$

we write $[x]_M$ for $\lim \langle p_{\{x\}(n)} \rangle_n$.

If $M_0 := \{y \in \mathbf{R} : y > 0\}$ with metric inherited from $\mathbf{R}$, we let

$$S_n := U([j_1 n]_{M_0}, [j_2 n]_M);$$

here $U(r, c)$ denotes the open ball with radius r and center c , and j_1, j_2 are the projections for the pairing $j(x, y)$.

Here and in the sequel, the use of the notation $[x]_M$ is tacitly taken to imply that x satisfies (*).

A partial recursive α is said to be *effective covering of open sheres of M* if

$$\forall [x]_M (E(\alpha x) \wedge [x]_M \in S_{\alpha x}).$$

Proposition 3 *Assuming $\text{CT}_0^\vee$, every effective covering of open spheres of a complete separable metric space has an enumerable subcover.*

Proof. Let $\langle p_n \rangle_n$ be a basis for a complete separable metric space M . We put

$$\gamma(x, y, n) \simeq \gamma_1(x, y)(n) \simeq \begin{cases} \{x\}(n) & \text{if } \neg \exists k \leq nTyk, \\ \{x\}(\min_{k \leq n} Tyk) & \text{if } \exists k \leq nTyk. \end{cases}$$

Then

$$A_M(x) \rightarrow \forall y A_M(\gamma_1(x, y))$$

Let R be the r.e. predicate defined by

$$R(x, y) := \exists k [Tyk \wedge \forall i \leq k E\gamma(x, y, i) \wedge \forall i < k (\rho(p_{\gamma(x, y, i)}, p_{\gamma(x, y, i+1)}) < 2^{-i-1})].$$

So if $R(x, y)$, then $A_M(\gamma_1(x, y))$. Let δ_1 enumerate $\{j(x, y) : R(x, y)\}$, and put

$$\delta m := \gamma_1(j_1(\delta_1 m), j_2(\delta_1 m)).$$

Let α be an effective covering. Then

$$A_M(x) \rightarrow \forall y [E(\alpha(\gamma_1(x, y))) \wedge [\gamma_1(x, y)]_M \in S_{\alpha(\gamma_1(x, y))}].$$

We shall prove that $\{S_{\alpha(\delta m)} : m \in \mathbf{N}\}$ is a covering. To see this, let $rx := j_1(\alpha x)$ and $cx := j_2(\alpha x)$, and assume that $A_M(x)$. Then

$$\forall y [\rho([x]_M, [\gamma_1(x, y)]_M) > 0 \rightarrow \exists nTyyn]$$

and

$$\forall y [\rho([x]_M, [\gamma_1(x, y)]_M) < r(\gamma_1(x, y)) - \rho([\gamma_1(x, y)]_M, [c(\gamma_1(x, y))]_M) \rightarrow [x]_M \in S_{\alpha(\gamma_1(x, y))}];$$

hence

$$\forall y (\exists nTyyn \vee [x]_M \in S_{\alpha(\gamma_1(x, y))}).$$

Apply Lemma 1. Then we find z such that

$$\exists nTzzn \wedge [x]_M \in S_{\alpha(\gamma_1(x, z))}.$$

and therefore $R(x, z)$. Hence $\alpha\delta$ is an enumerable subcovering of α . $\square$

Theorem 3 *Assuming ECT_0 , any open cover of a complete separable metric space has an enumerable subcover.*

Proof. Similar to the proof of [6, Theorem 7.2.15] using Proposition 3. $\square$

References

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