

SOME REMARKS ON $\mathcal{P}$ -CONGRUENCES ON $\mathcal{P}$ -REGULAR SEMIGROUPS I

- $\mathcal{P}$ -congruence pairs -

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Yamada and Sen introduced the new concept of $\mathcal{P}$ -regularity in the class of regular semigroups which is a generalization of both the concepts of "orthodox" and "(special) involution" (see [8],[9]). The purpose of this abstract is to characterize congruences on a $\mathcal{P}$ -regular semigroups by using " $\mathcal{P}$ -congruence pairs", which is a generalization of Petrich [7] for inverse semigroup and one of the authors [4] for regular $*$ -semigroups. Also, for a given congruence ρ on a $\mathcal{P}$ -regular semigroup S , the maximum and the minimum congruences on S whose traces coincide with the trace of ρ ($= \rho \cap E(S) \times E(S)$) are determined.

1. Introduction. Let S be a regular semigroup and E the set of idempotents of S . Let $P \subset E$. If S satisfies the following, it is called a $\mathcal{P}$ -regular semigroup:

- (1) $P^2 \subset E$,
- (2) $qPq \subset P$ for any $q \in P$,
- (3) for any $a \in S$, there exists $a^+ \in V(a)$ (the set of all inverses of a) such that $a^+P^1a \subset P$ and $aP^1a^+ \subset P$.

In such a case, S is denoted by $S(P)$ and P is called a C-set in S . Throughout this paper, let $S(P)$ be a $\mathcal{P}$ -regular semigroup

with a C-set P such that the set of idempotents of S is E . Let $a \in S(P)$ and $a^+ \in V(a)$. If a^+ satisfies that $a^+P^1a \subset P$ and $aP^1a^+ \subset P$, it is called a $\mathcal{P}$ -inverse of a , and the set of $\mathcal{P}$ -inverses of a is denoted by $V_P(a)$. An element of a C-set P in S is called a projection. The class of $\mathcal{P}$ -regular semigroups contains both the classes of orthodox semigroups and regular $*$ -semigroups. A good account of the concept of $\mathcal{P}$ -regularity can be seen in [8] and [9].

A congruence on S is sometimes called a $\mathcal{P}$ -congruence on $S(P)$. Let ρ be a $\mathcal{P}$ -congruence on $S(P)$, and put $\bar{x} = x\rho$ for any $x \in S$, $\bar{S} = \{\bar{x}: x \in S\}$ and $\bar{P} = \{\bar{q}: q \in P\}$. Then $\bar{S}(\bar{P})$ is also a $\mathcal{P}$ -regular semigroup with a C-set $\bar{P}$. So $\bar{S}(\bar{P})$ is called the factor $\mathcal{P}$ -regular semigroup of $S(P)$ mod. ρ , and it is denoted by $S(P)/(\rho)_{\mathcal{P}}$.

Let ρ be a $\mathcal{P}$ -congruence on $S(P)$. Then it is called an orthodox $\mathcal{P}$ -congruence on $S(P)$ if $S(P)/(\rho)_{\mathcal{P}}$ is an orthodox semigroup, and it is called a strong $\mathcal{P}$ -congruence on $S(P)$ if it satisfies that for $a \in S(P)$ and $e \in P$,

$$a \rho e \text{ implies } a^+ \rho e \text{ for all } a^+ \in V_P(a).$$

As was seen in [8], if ρ is a strong $\mathcal{P}$ -congruence on $S(P)$, then $S(P)/(\rho)_{\mathcal{P}}$ becomes a regular $*$ -semigroup with the set $\{e\rho: e \in P\}$ of projections if the $*$ -operation $\#$ on $S(P)/(\rho)_{\mathcal{P}}$ is defined by $(a\rho)^{\#} = a^+\rho$ ($a \in S(P)$, $a^+ \in V_P(a)$).

The set $\{a \in S(P): a \rho e \text{ for some } e \in E [e \in P]\}$ is called the $[\mathcal{P}]$ -kernel of ρ , and it denoted by $[\mathcal{P}]\text{-ker}\rho$. The restriction $\rho \cap (E \times E) [\rho \cap (P \times P)]$ of ρ is called the $[\mathcal{P}]$ -trace of ρ , and it is denoted by $[\mathcal{P}]\text{-tr}\rho$.

For any subset A of $S(P)$, define the terminology as follows:

A is [$\mathcal{P}$ -]full if $E \subset A$ [$P \subset A$],

A is a $\mathcal{P}$ -subset if $V_P(a) \subset A$ for any $a \in A$,

A is a $\mathcal{P}$ -self-conjugate

if $x^+Ax \subset A$ for any $x \in S(P)$ and $x^+ \in V_P(x)$,

A is weakly closed if $a^2 \in A$ for any $a \in A$.

The following results are fundamental and are used frequently in this abstract.

Result 1.1 (due to [8] and [9]). Let $a, b \in S(P)$, $e \in E$ and $q \in P$. Then

- (i) $V_P(b)V_P(a) \subset V_P(ab)$,
- (ii) if $a^+ \in V_P(a)$, then $a \in V_P(a^+)$,
- (iii) $V_P(e) \subset E$, (iv) $q \in V_P(q)$.

Result 1.2 (due to [2]). Let ρ be a $\mathcal{P}$ -congruence on $S(P)$ and $a, b \in S(P)$. Then $a \rho b$ if and only if

$$ba' \in \ker \rho, aa' \rho bb'aa', b'b \rho b'ba'a$$

for some $a' \in V(a)$ and $b' \in V(b)$.

In section 2, for a given $\mathcal{P}$ -congruence ρ on a $\mathcal{P}$ -regular semigroup $S(P)$, the maximum and the minimum $\mathcal{P}$ -congruences on $S(P)$ whose traces coincide with $\text{tr} \rho$ are determined, and the properties for those $\mathcal{P}$ -congruences are given.

The concept introduced in section 3 is " $\mathcal{P}$ -congruence pairs". This concept is a characterization of the pair $(\text{tr} \rho, \ker \rho)$ associated with a given $\mathcal{P}$ -congruence ρ on $S(P)$, and the

pair uniquely determines the $\mathcal{P}$ -congruence κ such that $\text{tr } \kappa = \text{tr } \rho$ and $\ker \kappa = \ker \rho$.

We use the notation and terminology of [3] and [9] unless otherwise stated.

2. $\mathcal{P}$ -congruences with the same trace. For any $\mathcal{P}$ -congruence ρ on $S(P)$, define a relation $\rho_{\max}$ on $S(P)$ as follows:

$$\rho_{\max} = \{(a,b): \text{there exist } a^+ \in V_P(a) \text{ and } b^+ \in V_P(b) \text{ such that } aea^+ \rho beb^+aea^+, beb^+ \rho aea^+beb^+, a^+ea \rho a^+eab^+eb \text{ and } b^+eb \rho b^+eba^+ea \text{ for all } e \in P\}.$$

Then we can easily see that

$$\rho_{\max} = \{(a,b): aea^+ \rho beb^+aea^+, beb^+ \rho aea^+beb^+, a^+ea \rho a^+eab^+eb \text{ and } b^+eb \rho b^+eba^+ea \text{ for all } a^+ \in V_P(a), b^+ \in V_P(b) \text{ and } e \in P\}$$

Lemma 2.1. Let ρ be a $\mathcal{P}$ -congruence on $S(P)$ and $a, b \in S(P)$. If $a \rho_{\max} b$, then

$aa^+ \rho bb^+aa^+$, $bb^+ \rho aa^+bb^+$, $a^+a \rho a^+ab^+b$, $b^+b \rho b^+ba^+a$ for any $a^+ \in V_P(a)$ and $b^+ \in V_P(b)$.

Theorem 2.2. For any $\mathcal{P}$ -congruence ρ on a $\mathcal{P}$ -regular semigroup $S(P)$, $\rho_{\max}$ is the greatest $\mathcal{P}$ -congruence on $S(P)$ whose trace coincides with $\text{tr } \rho$.

Theorem 2.3. For any orthodox $\mathcal{P}$ -congruence ρ on $S(P)$,

$\rho_{\max}$ is the greatest orthodox $\mathcal{P}$ -congruence on $S(P)$ whose trace coincides with $\text{tr } \rho$.

From now on, denote the maximum idempotent-separating congruence on a semigroup T by μ_T .

Corollary 2.4 (compare with [8, Proposition 4.1]). The maximum idempotent-separating $\mathcal{P}$ -congruence $\mu_{S(P)}$ on $S(P)$ is given as follows:

$$\begin{aligned} \mu_{S(P)} &= \{(a,b): \text{there exist } a^+ \in V_P(a) \text{ and } b^+ \in V_P(b) \text{ such} \\ &\quad \text{that } aea^+ = beb^+aea^+, beb^+ = aea^+beb^+, a^+ea = \\ &\quad a^+eab^+eb \text{ and } b^+eb = b^+eba^+ea \text{ for all } e \in P\}. \\ &= \{(a,b): aea^+ = beb^+aea^+, beb^+ = aea^+beb^+, a^+ea \\ &\quad = a^+eab^+eb \text{ and } b^+eb = b^+eba^+ea \text{ for all } a^+ \in \\ &\quad V_P(a), b^+ \in V_P(b) \text{ and } e \in P\} \end{aligned}$$

Let S be an orthodox semigroup and E the band of idempotents of S . Then it is easy to check that $S(E)$ is a $\mathcal{P}$ -regular semigroup with a C -set E in S . So we have immediately

Corollary 2.5 ([1, Theorem 4.2]). Let ρ be a congruence on an orthodox semigroup S with the band E of idempotents of S . Then

$$\begin{aligned} \rho_{\max} &= \{(a,b): \text{there exist } a' \in V(a) \text{ and } b' \in V(b) \text{ such} \\ &\quad \text{that } aea' \rho beb'aea', beb' \rho aea'beb', a'ea \rho \\ &\quad a'eab'eb, b'eb \rho b'eba'ea \text{ for any } e \in E\} \\ &= \{(a,b): aea' \rho beb'aea', beb' \rho aea'beb', a'ea \rho \end{aligned}$$

$a'eab'eb, b'eb \rho b'eba'ea$ for any $a' \in V(a), b' \in V(b)$ and $e \in E$

is the greatest congruence on S whose trace coincides with $\text{tr} \rho$.

On the other hand, the minimum $\mathcal{P}$ -congruence on $S(P)$ with the same trace is given as follows:

Theorem 2.6. For any $\mathcal{P}$ -congruence ρ on a $\mathcal{P}$ -regular semigroup $S(P)$, define a relation ρ_0 on $S(P)$ by

$$\rho_0 = \{(a,b): \text{there exist } x, y \in S(P)^1 \text{ and } e, f \in E \text{ such that } a = xey, b = xfy \text{ and } e \rho f\}$$

Then $\rho_{\min} = \rho_0^t$, the transitive closure of ρ_0 , is the least $\mathcal{P}$ -congruence on $S(P)$ whose trace coincides with $\text{tr} \rho$. In other words, the least $\mathcal{P}$ -congruence on $S(P)$ with $\text{tr} \rho$ as its trace is the $\mathcal{P}$ -congruence on $S(P)$ generated by $\text{tr} \rho$.

The following corollary gives us the characterization which is different from both [1, Theorem 4.1] and [7, Theorem 3.3], of the least congruence on an orthodox semigroup with the same trace.

Corollary 2.7. For any congruence ρ on an orthodox semigroup S , the congruence generated by $\text{tr} \rho$ is the least congruence on S whose trace coincides with $\text{tr} \rho$.

Proposition 2.9. For any $\mathcal{P}$ -congruence ρ on $S(P)$, $\rho = \rho_{\max}$ if and only if $S(P)/(\rho)_{\mathcal{P}}$ is a fundamental $\mathcal{P}$ -regular

semigroup.

For any $\mathcal{P}$ -congruences ρ and σ on $S(P)$ such that $\rho \subset \sigma$, define a relation σ/ρ on $S(P)/(\rho)_{\mathcal{P}}$ by

$$\sigma/\rho = \{(a\rho, b\rho) : (a, b) \in \sigma\}$$

Proposition 2.10. For any $\mathcal{P}$ -congruence ρ on $S(P)$, $\rho_{\max}/\rho$ is the maximum idempotent-separating $\mathcal{P}$ -congruence on $S(P)/(\rho)_{\mathcal{P}}$.

Let Λ be the lattice of all $\mathcal{P}$ -congruences on $S(P)$. Define a relation Θ on Λ as follows: for any $\rho, \sigma \in \Lambda$,

$$\rho \Theta \sigma \quad \text{if and only if} \quad \text{tr} \rho = \text{tr} \sigma.$$

It immediately follows from Theorems 2.2 and 2.6 that $\rho \Theta$, the Θ -class containing $\rho \in \Lambda$, is the interval $[\rho_{\min}, \rho_{\max}]$ of Λ .

Proposition 2.11 ([6, Theorem 5.1]). If $\mathcal{P}$ -congruences ρ and σ on $S(P)$ are Θ -equivalent, then $\rho\sigma = \sigma\rho$. Therefore, for any $\rho \in \Lambda$, $\rho \Theta$ is a complete modular subsemilattice of Λ .

Proposition 2.12. Let $\xi \in \Lambda$, and let Γ be the lattice of all idempotent-separating $\mathcal{P}$ -congruences on $S(P)/(\xi_{\min})_{\mathcal{P}}$. Then the mapping $\rho \rightarrow \rho/\xi_{\min}$ is a complete isomorphism of $\xi \Theta$ onto Γ .

3. $\mathcal{P}$ -congruence pairs. Let ξ be an equivalence on E .

Then ξ is called a normal equivalence on E if it satisfies the following conditions: for any $a \in S(P)$ and $e, f, g, h, i, j, k \in$

E,

- (a) if $e \xi f$ and $aea^+, afa^+ \in E$ for some $a^+ \in V_P(a)$,
then $aea^+ \xi afa^+$,
- (b) if $e \xi f$, $g \xi h$ and $eg, fh \in E$, then $eg \xi fh$,
- (c) if $\square \neq (e\xi)(f\xi) \cap E \subset h\xi$, $\square \neq (f\xi)(g\xi) \cap E \subset i\xi$ and
 $\square \neq (e\xi)(i\xi) \cap E \subset j\xi$ [$\square \neq (h\xi)(g\xi) \cap E \subset k\xi$],
then $\square \neq (h\xi)(g\xi) \cap E$ [$\square \neq (e\xi)(i\xi) \cap E$] and $j \xi k$.

Let ξ be a normal equivalence on E. Define a partial binary operation $\bullet$ on E/ξ as follows: for any $e, f, g \in E$,

$$e\xi \bullet f\xi = g\xi, \text{ where } \square \neq (e\xi)(f\xi) \cap E \subset g\xi.$$

It is easy to verify that the partial binary operation $\bullet$ is well-defined. The partial groupoid E/ξ satisfies the following:

- (w) if $e\xi \bullet f\xi$, $f\xi \bullet g\xi$ and $e\xi \bullet (f\xi \bullet g\xi)$ [$(e\xi \bullet f\xi) \bullet g\xi$] are defined in E/ξ , then $(e\xi \bullet f\xi) \bullet g\xi$ [$e\xi \bullet (f\xi \bullet g\xi)$] is defined in E/ξ and $(e\xi \bullet f\xi) \bullet g\xi = e\xi \bullet (f\xi \bullet g\xi)$.

Let K be a weakly closed full $\mathcal{P}$ -subset of $S(P)$ and ξ a normal equivalence on E. Then the pair (ξ, K) is called a $\mathcal{P}$ -congruence pair for $S(P)$ if it satisfies the following conditions: for any $a, b, c \in S(P)$, $c^+ \in V_P(c)$, $e, f, g \in E$ and $q \in P$,

- (C1) $a \in K$ implies $a^+a \xi a^+a^+aa$ for any $a^+ \in V_P(a)$,
- (C2) $aefb \in K$ and $e\xi \bullet f\xi = (a^+a)\xi$ for some $a^+ \in V_P(a)$
imply $ab \in K$,
- (C3) $ab^+ \in K$ and $aa^+ \xi bb^+aa^+$, $b^+b \xi b^+ba^+a$ for some
 $a^+ \in V_P(a)$ and $b^+ \in V_P(b)$ imply $aqb^+ \in K$ and
 $aqa^+ \xi bqb^+aq a^+$, $b^+qb \xi b^+qba^+qa$,

(C4) $a, b \in K$, $aa^+ \xi ee^+aa^+$, $ee^+ \xi aa^+ee^+$, $a^+a \xi a^+ae^+e$,
 $e^+e \xi e^+ea^+a$, $bb^+ \xi ff^+bb^+$, $ff^+ \xi bb^+ff^+$,
 $b^+b \xi b^+bf^+f$, $f^+f \xi f^+fb^+b$ and $e\xi \bullet f\xi = g\xi$ for some
 $a^+ \in V_P(a)$, $b^+ \in V_P(b)$, $e^+ \in V_P(e)$ and $f^+ \in V_P(f)$
 imply $ab \in K$,

(C5) $aq \in K$ and $aa^+ \xi qaa^+$, $q \xi qa^+a$ for some $a^+ \in V_P(a)$
 imply $cac^+ \in K$.

For any $\mathcal{P}$ -congruence pair (ξ, K) for $S(P)$, define a relation $\kappa_{(\xi, K)}$ on $S(P)$ as follows:

$$\kappa_{(\xi, K)} = \{(a, b) : ab^+ \in K \text{ and } aa^+ \xi bb^+aa^+, bb^+ \xi aa^+bb^+, a^+a \xi a^+ab^+b, b^+b \xi b^+ba^+a \text{ for some } [any] a^+ \in V_P(a) \text{ and } b^+ \in V_P(b)\}.$$

Now we can determine $\mathcal{P}$ -congruences on $S(P)$ by $\mathcal{P}$ -congruence pairs.

Theorem 3.1. For any $\mathcal{P}$ -congruence pair (ξ, K) for a $\mathcal{P}$ -regular semigroup $S(P)$, $\kappa_{(\xi, K)}$ is a $\mathcal{P}$ -congruence on $S(P)$ such that $\text{tr } \kappa_{(\xi, K)} = \xi$ and $\ker \kappa_{(\xi, K)} = K$. Conversely, for any $\mathcal{P}$ -congruence ρ on $S(P)$, $(\text{tr } \rho, \ker \rho)$ is a $\mathcal{P}$ -congruence pair for $S(P)$ and $\rho = \kappa_{(\text{tr } \rho, \ker \rho)}$.

Let $\mathcal{A}$ be the set of $\mathcal{P}$ -congruence pairs for $S(P)$. Define an order $\leq$ on $\mathcal{A}$ by

$$(\xi_1, K_1) \leq (\xi_2, K_2) \text{ if and only if } \xi_1 \subset \xi_2, K_1 \subset K_2.$$

Corollary 3.2. The mappings

$$(\xi, K) \rightarrow \kappa_{(\xi, K)}, \quad \rho \rightarrow (\text{tr } \rho, \ker \rho)$$

are mutually inverse order-preserving mappings of $\mathcal{A}$ onto Λ and of Λ onto $\mathcal{A}$, respectively. Therefore, $\mathcal{A}$ forms a complete lattice.

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