

On uniformly convex functions

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Abstract

A.W.Goodman[1] introduced the geometrically defined class UCV of uniformly convex functions on the unit disk; he established some theorems for this class. Recently, some mathematicians showed one-variable characterization for function in UCV which are closely related to Goodman's characterizations, for example Ma and Minda[2], Ronning[3].

In this short paper, we give a examples and conjectured necessary and sufficient condition of the class UCV_p .

1 Introduction

Let A denote the class of functions of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. Let S denote the class of normalized analytic and univalent functions in U . We denote the subclass of S as follows :

$$(1.2) \quad K = \left\{ f(z) \in A : \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0, \quad z \in U \right\}.$$

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K is called the usual class of convex functions.

Definition 1. A function f is said to be uniformly convex in U if $f(z)$ is a normalized ($f(0) = f'(0) - 1 = 0$) convex function and has the property that for every circular arc γ contained in U , with center also in U , the image arc $f(\gamma)$ is a convex arc.

(1.3)

$$UCV = \left\{ f(z) \in K : \Re \left\{ 1 + \frac{(z - \zeta)f''(z)}{f'(z)} \right\} \geq 0, \quad ((z, \zeta) \in U \times U) \right\}$$

So far this two-variable characterization has not led to any sharp estimates for the class UCV . Ma and Minda introduced one-variable characterization for functions in UCV which are closely related to Goodman's characterizations.

Theorem A. Assume that $f(z)$ is holomorphic and locally univalent in U with $f(0) = f'(0) - 1 = 0$. Then the following are equivalent:

(i) $f(z) \in UCV$.

(ii) $\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \left| \frac{zf''(z)}{f'(z)} \right| \quad (z \in U).$

2 A example

The following example will be useful. Goodman proved that

Theorem B. The function

$$(2.1) \quad f(z) = \frac{z}{1 - Az} = z + \sum_{n=2}^{\infty} A^{n-1} z^n$$

is in UCV iff $|A| \leq \frac{1}{3}$.

It is clear from theorem B that the function $f(z) = \frac{z}{1 - z}$ is not in UCV , but it is in K . We have

Example 1. If $0 < r \leq \frac{2}{7} \approx 0.2857 < 0.29$, then the function

$$(2.2) \quad f(z) = \frac{z}{1-z} = \sum_{n=1}^{\infty} z^n$$

is in *UCV*.

proof. A simple computation shows that for this function

$$(2.3) \quad \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} - \left| \frac{zf''(z)}{f'(z)} \right| = \Re \left\{ 1 + \frac{2z}{1-z} \right\} - \left| \frac{2z}{1-z} \right|.$$

We set $z = re^{i\theta}$. For $0 < r \leq \frac{2}{7} \approx 0.2857 < 0.29$, we have

$$\frac{1 - r^2 - 2r\sqrt{2 - 2r\cos\theta}}{1 + r^2 - 2r\cos\theta} > 0$$

or

$$(2.4) \quad 1 - r^2 - 2r\sqrt{2 - 2r\cos\theta} > 0.$$

■

It is possible that in this example $\frac{2}{7}$ can be replaced by the larger constant.

Example 3. The function (2.2) is in *UCV* iff $0 < r \leq 1 - 2\rho, 0 < \rho \leq \frac{1}{2}$.

proof. A simple computation shows that for this function

$$(2.5) \quad \begin{aligned} Q(z, \zeta) &= 1 + \frac{(z - \zeta)f''(z)}{f'(z)} \\ &= \frac{1 + z - 2\zeta}{1 - z}. \end{aligned}$$

We set $z = re^{i\theta}$ and $\zeta = \rho e^{i\phi}$. Then $\Re Q(z, \zeta) \geq 0$ iff

$$\Re(1 + re^{i\theta} - 2\rho e^{i\phi})(1 - re^{-i\theta}) \geq 0$$

or

$$(2.6) \quad 1 - 2\rho\cos\phi - r^2 + 2r\rho\cos(\phi - \theta) \geq 0.$$

It is clear that the minimum of the expression on the left side of (2.6) occurs when $\phi = 0, \theta = \pi$. (Thus, $\zeta = \rho, z = -r$.) These values yield $1 - 2r\rho - 2\rho - r^2 \geq 0$, and this is true for $0 < r \leq 1 - 2\rho, 0 < \rho \leq \frac{1}{2}$. Thus, the condition is sufficient for (2.2) to be in UCV . By a limit argument, the condition is also necessary. ■

3 Conjecture

Let A_p denote the class of functions of the form

$$(3.1) \quad f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad (p \in N = 1, 2, 3, \dots)$$

which are analytic in $U = \{z : |z| < 1\}$.

A function $f(z) \in A_p$ is said to be p -valently convex iff

$$(3.2) \quad 1 + \Re \frac{zf''(z)}{f'(z)} > 0 \quad (z \in U).$$

We denote by K_p the subclass of A_p consisting of all p -valently convex functions in U .

Using the idea contained in Theorem A, we pose the following conjecture.

Conjecture. A function $f(z) \in A_p$ is said to be p -valent uniformly convex iff

$$(3.3) \quad \Re \left\{ \frac{zf''(z)}{f'(z)} - (p-2) \right\} - \left| \frac{zf''(z)}{f'(z)} - (p-1) \right| > 0, \quad (z \in U).$$

Also we denote by UCV_p the subclass of A_p consisting of all p -valent uniformly convex functions in U .

References

- [1] A.W.Goodman. On uniformly convex functions. Ann. Polon. Math. **56**(1991), 87-92.
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- [3] F.Rønning. Uniformly convex functions and a corresponding class of starlike functions. Proc. Amer. Math. Soc. **118**(1993), 189-196.