

NAVIER-STOKES EQUATIONS WITH DISTRIBUTIONS AS INITIAL DATA

HIDEO KOZONO[†] AND MASAO YAMAZAKI[‡]
(小蘭 英雄) (山崎 昌男)

[†] Department of Applied Physics
Nagoya University

[‡] Department of Mathematics
Hitotsubashi University

§1 Introduction.

Let Ω be an exterior domain in $\mathbf{R}^n (n \geq 3)$, i.e., a domain having a compact complement $\mathbf{R}^n \setminus \Omega$, and assume that the boundary $\partial\Omega$ is of class $C^{2+\mu} (0 < \mu < 1)$. The motion of the incompressible fluid occupying Ω is governed by the Navier-Stokes equations:

$$(S) \quad \begin{cases} -\Delta w + w \cdot \nabla w + \nabla \pi = \operatorname{div} F & \text{in } \Omega, \\ \operatorname{div} w = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \quad w(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty, \end{cases}$$

where $w = w(x) = (w^1(x), \dots, w^n(x))$ and $\pi = \pi(x)$ denote the velocity vector and the pressure of the fluid at point $x \in \Omega$, respectively, while $F = F(x) = \{F_{ij}(x)\}_{i,j=1,\dots,n}$ is the given $n \times n$ matrices with $\operatorname{div} F$ the external force. In the previous paper [14], the first author and Ogawa showed the stability in L^n of solutions w in the class

$$(CL) \quad w \in L^n(\Omega) \quad \text{and} \quad \nabla w \in L^{n/2}(\Omega).$$

In case $n \geq 4$ we can show the existence and uniqueness for solutions w of (S) with (CL). In the three dimensional case, however, the solution in the class (CL) yields that the net force exerted to the body is equal to zero:

$$\int_{\partial\Omega} (T(w, \pi) + F) \cdot \nu dS = 0,$$

where $T(w, \pi) = \{\partial w^i / \partial x^j + \partial w^j / \partial x^i - \delta_{ij} \pi\}_{i,j=1,\dots,n}$ and ν denote the stress strain and the unit outer normal to $\partial\Omega$, respectively (see Kozono-Sohr [16]). Introducing another class

$$(CL') \quad \sup_{x \in \Omega} |x| |w(x)| + \sup_{x \in \Omega} |x|^2 |\nabla w(x)| \equiv C_w < \infty$$

Borchers-Miyakawa [3] constructed the solution with (CL') and showed that if C_w is small, then w is stable under the initial disturbance in weak- L^n space $L^{n,\infty}(\Omega)$.

The purpose of this note is to find a larger class of stable flows than (CL'). Indeed, we shall show that stationary flows in the class

$$(CL'') \quad w \in L^{n,\infty}(\Omega)$$

are stable under such perturbation as Borchers-Miyakawa's [3]. As a result, we shall obtain the *same class* of stable solutions and initial disturbances. More precisely, if w is perturbed by a , then the perturbed flow $v(x, t)$ is governed by the following *non-stationary* Navier-Stokes equations:

$$(N-S) \quad \begin{cases} \frac{\partial v}{\partial t} - \Delta v + v \cdot \nabla v + \nabla q = f & \text{in } \Omega, t > 0, \\ \operatorname{div} v = 0 & \text{in } \Omega, t > 0, \\ v = 0 & \text{on } \partial\Omega, t > 0, \quad v(x, t) \rightarrow 0 \text{ as } |x| \rightarrow \infty, \\ v(x, 0) = w(x) + a(x) & \text{for } x \in \Omega. \end{cases}$$

In this note we shall show: if the stationary flow w and the initial disturbance a are both small enough in $L^{n,\infty}(\Omega)$, then there is a unique *global strong solution* v of (N-S) such that the integrals

$$\int_{\Omega} |v(x, t) - w(x)|^r dx \quad \text{for } n < r < \infty$$

converges to zero with *definite decay rates* as $t \rightarrow \infty$. Let w and v be solutions of (S) and (N-S), respectively. Then the pair of functions $u \equiv v - w, p \equiv q - \pi$ satisfies

$$(N-S') \quad \begin{cases} \frac{\partial u}{\partial t} - \Delta u + w \cdot \nabla u + u \cdot \nabla w + u \cdot \nabla u + \nabla p = 0 & \text{in } \Omega, t > 0, \\ \operatorname{div} u = 0 & \text{in } \Omega, t > 0, \\ u = 0 & \text{on } \partial\Omega, t > 0, \quad u(x, t) \rightarrow 0 \text{ as } |x| \rightarrow \infty, \\ u|_{t=0} = a. \end{cases}$$

Hence our problem on the stability for (S) can now be reduced to investigation into asymptotic behaviour of the solution u of (N-S'). In a three-dimensional exterior domain, Heywood [10,11] and Masuda [18] considered inhomogeneous boundary condition at infinity like $w(x) \rightarrow w^\infty$ as $|x| \rightarrow \infty$, where w^∞ is a prescribed non-zero constant vector in $\mathbf{R}^3$. They showed the stability for such solutions in L^2 -spaces. On account of the parabolically wake region behind obstacles, their decay rates are slower than that of our solutions. To obtain sharper decay rates in L^r -spaces of the solutions of (N-S') with the initial data in weak- L^n space, we need to establish $L^{p,\infty} - L^r$ -estimates for the semigroup e^{-tL_r} , where L_r is the operator defined by

$$L_r u \equiv A_r u + P_r(w \cdot \nabla u + u \cdot \nabla w).$$

NAVIER-STOKES EQUATIONS WITH DISTRIBUTIONS AS INITIAL DATA

Here P_r is the projection operator from $L^r(\Omega)$ onto $L^r_\sigma(\Omega)$ and $A_r \equiv -P_r\Delta$ denotes the Stokes operator in $L^r_\sigma(\Omega)$.

In case $w \equiv 0$, we have $L_r = A_r$ and hence our problem coincides with obtaining a global strong solution and its decay properties of the Navier-Stokes equations in exterior domains. Since the pioneer work of Kato [13] and Ukai [23], many efforts have been made to get $L^p - L^r$ -estimates for the Stokes semigroup e^{-tA_r} in *exterior* domains and there are mainly two methods. One is to characterize the domain $D(A_r^\alpha)$ of fractional powers A_r^α ($0 < \alpha < 1$) due to Giga [7], Giga-Sohr [9] and Borchers-Miyakawa [2] and another is to obtain asymptotic expansion of the resolvent $(A_r + \lambda)^{-1}$ near $\lambda = 0$ due to Iwashita [12]. In our case, since L_r is the operator with *variable* coefficients, both of these methods seem to be difficult to get the same asymptotic behavior of e^{-tL_r} as that of e^{-tA_r} as $t \rightarrow \infty$. If we restrict ourselves to the case $n/(n-1) < r < \infty$, however, then L_r can be treated as a perturbation of A_r , and for such r , we can get satisfactory $L^{p,\infty} - L^r$ -estimates of e^{-tL_r} , which is enough to construct the global strong solution of (N-S'). Our proof needs neither estimates of the purely imaginary powers L_r^{is} ($s \in \mathbf{R}$) of L_r nor asymptotic expansion of $(L_r + \lambda)^{-1}$ near $\lambda = 0$; we need only such a standard resolvent estimate of elliptic differential operators as Agmon's [1].

On account of the restriction $n/(n-1) < r < \infty$, we cannot construct the strong solution directly in the same way as Giga-Miyakawa [8] and Kato [13]. Therefore, we need to first introduce a *mild solution* which is an intermediate between weak and strong solutions (see Definition below). This procedure is due to Kozono-Ogawa [14]. Then we shall show the existence and uniqueness of the *global* mild solution u of (N-S') in the class $C((0, \infty); L^{n,\infty}(\Omega))$ with decay property $\|u(t)\|_r = O(t^{-1/2+n/2r})$ as $t \rightarrow \infty$ for $n < r < \infty$. Using a similar uniqueness criterion to Serrin [21] and Masuda [19], we may identify the mild solution with the strong solution. As a result, it will be clarified that the restriction on r causes no obstruction for our purpose.

§2 Results.

Before stating our results, we introduce some notations and function spaces and then give our definition of mild solutions of (N-S'). Let $C_{0,\sigma}^\infty$ denote the set of all C^∞ -real vector functions $\phi = (\phi^1, \dots, \phi^n)$ with compact support in Ω , such that $\operatorname{div} \phi = 0$. L^r_σ is the closure of $C_{0,\sigma}^\infty$, with respect to the L^r -norm $\|\cdot\|_r$; $(\cdot, \cdot)$ denotes the L^2 -inner product and the duality pairing between L^r and $L^{r'}$, where $1/r + 1/r' = 1$. L^r stands for the usual (vector-valued) L^r -space over Ω , $1 < r < \infty$. $H_{0,\sigma}^{1,r}$ denotes the closure of $C_{0,\sigma}^\infty$ with respect to the norm

$$\|\phi\|_{H^{1,r}} = \|\phi\|_r + \|\nabla \phi\|_r,$$

where $\nabla \phi = (\partial \phi^i / \partial x_j; i, j = 1, \dots, n)$. When X is a Banach space, its norm is denoted by $\|\cdot\|_X$. Then $C^m((t_1, t_2); X)$ is a usual Banach space, where $m = 0, 1, 2, \dots$ and t_1 and t_2 are real numbers such that $t_1 < t_2$. $BC^m((t_1, t_2); X)$ is the set of all functions $u \in C^m((t_1, t_2); X)$ such that $\sup_{t_1 < t < t_2} \|\frac{d^m u(t)}{dt^m}\|_X < \infty$.

Let us recall the Helmholtz decomposition:

$$L^r = L_\sigma^r \oplus G^r \quad (\text{direct sum}), \quad 1 < r < \infty,$$

where $G^r = \{\nabla p \in L^r; p \in L_{loc}^r(\overline{\Omega})\}$. For the proof, see Fujiwara-Morimoto[6], Miyakawa[20] and Simader-Sohr[22]. P_r denotes the projection operator from L^r onto L_σ^r along G^r . The Stokes operator A_r on L_σ^r is then defined by $A_r = -P_r \Delta$ with domain $D(A_r) = \{u \in H^{2,r}(\Omega); u|_{\partial\Omega} = 0\} \cap L_\sigma^r$. It is known that

$$(L_\sigma^r)^*(\text{the dual space of } L_\sigma^r) = L_\sigma^{r'}, \quad A_r^*(\text{the adjoint operator of } A_r) = A_{r'},$$

where $1/r + 1/r' = 1$.

Furthermore, for $1 < r < \infty$ and $1 \leq q \leq \infty$, $L^{r,q}$ denotes the Lozentz space over Ω with norm $\|\cdot\|_{r,q}$. Then we define $L_\sigma^{r,q}$ as

$$L_\sigma^{r,q} \equiv \{u \in L^{r,q}; \operatorname{div} u = 0 \text{ in } \Omega, u \cdot \nu = 0 \text{ on } \partial\Omega\}.$$

Let us introduce the operator L_r in L_σ^r . To this end, we make the following assumption on w .

Assumption. w is a smooth solenoidal vector function on $\overline{\Omega}$ with $w|_{\partial\Omega} = 0$ in the class $w \in L_\sigma^{n,\infty}$

For the existence of such solutions w of (S), see Finn [4] and Fujita [5]. Under this assumption, we define the operator B_r on L_σ^r by

$$B_r u \equiv P_r(w \cdot \nabla u + u \cdot \nabla w) \quad \text{with domain } D(B_r) = H_{0,\sigma}^{1,r}.$$

L_r is now defined by

$$D(L_r) = D(A_r) \quad \text{and} \quad L_r \equiv A_r + B_r.$$

Since $\operatorname{div} w = 0$ in Ω , we can easily verify that the operator L' defined by

$$L'_r u = A_r u - P_r(w \cdot \nabla u + \sum_{j=1}^n w^j \nabla u^j), \quad D(L'_r) = D(A_r)$$

is the adjoint operator of L_r on $L_\sigma^{r'}$. It should be noted that the operator L' contains no derivative $\partial w / \partial x^j$ ($j = 1, \dots, n$) of w in its coefficients.

Our definition of mild solutions of (N-S') is as follows:

Definition. Let $a \in L_\sigma^{n,\infty}$ and let w satisfy the Assumption. Suppose that $n < r < \infty$. A measurable function u defined on $\Omega \times (0, \infty)$ is called a mild solution of (N-S') in L_σ^r if

- (1) $u \in BC((0, \infty); L_\sigma^{n,\infty})$ and $t^{(1-n/r)/2} u(\cdot) \in BC((0, \infty); L_\sigma^r)$;
- (2)

$$(u(t), \phi) = (e^{-tL} a, \phi) + \int_0^t (u(s) \cdot \nabla e^{-(t-s)L'} \phi, u(s)) ds$$

for all $\phi \in C_{0,\sigma}^\infty$ and all $0 < t < \infty$.

Our results now read:

NAVIER-STOKES EQUATIONS WITH DISTRIBUTIONS AS INITIAL DATA

Theorem 1. (1)(existence) Let $a \in L_{\sigma}^{n,\infty}$ and let w satisfy the Assumption. Then for every $n < r < \infty$, there is a positive number $\lambda = \lambda(n, r)$ such that if

$$\|a\|_{n,\infty} \leq \lambda, \quad \|w\|_{n,\infty} \leq \lambda,$$

there exists a mild solution u of (N-S') in L_{σ}^r such that

$$u(t) \rightarrow a \quad \text{weakly } * \text{ in } L_{\sigma}^{n,\infty} \quad \text{as } t \downarrow +0.$$

(2) (uniqueness) There is a constant $k = k(n, r)$ such that any mild solution u of (N-S') in L_{σ}^r with

$$\limsup_{t \rightarrow +0} t^{\frac{n}{2}(\frac{1}{n} - \frac{1}{r})} \|u(t)\|_r \leq k$$

is unique.

Concerning the regularity of the solution, we have

Theorem 2. The mild solution u given in Theorem 1 is actually a strong solution in the following sense:

- (1) $u \in C^1((0, \infty); L_{\sigma}^r)$;
- (2) $u(t) \in D(L_r)$ for $t \in (0, \infty)$ and $L_r u \in C((0, \infty); L_{\sigma}^r)$;
- (3) u satisfies

$$\frac{du}{dt} + L_r u + P_r(u \cdot \nabla u) = 0, \quad t > 0 \quad \text{in } L_{\sigma}^r$$

Remarks. (1) The above theorems show that the space $L_{\sigma}^{n,\infty}$ is the class of stable stationary flows and that it is the same class as that of initial disturbances. Borchers-Miyakawa [3] obtained, among others, similar results to ours including the uniform L^{∞} estimate in time. They make, however, such a stronger assumption as $\sup_{x \in \Omega} |x| |w(x)| + \sup_{x \in \Omega} |x|^2 |\nabla w(x)|$ is small enough. On the other hand, our theorems assert that the assumption on the spacial decay of $\nabla w(x)$ as $|x| \rightarrow \infty$ is not necessary. Moreover, the class of the space $L^{n,\infty}$ is larger than that of functions w such that $\sup_{x \in \Omega} |x| |w(x)| < \infty$.

(2) Since the semigroup $\{e^{-tL}\}_{t \geq 0}$ is not strongly continuous in $L_{\sigma}^{n,\infty}$, we cannot assure whether our solution u satisfies

$$\lim_{t \rightarrow +0} t^{\frac{n}{2}(\frac{1}{n} - \frac{1}{r})} \|u(t)\|_r = 0.$$

(3) When $\Omega = \mathbf{R}^n (n \geq 3)$, without assuming any regularity on the stationary flow w , Kozono-Yamazaki [17] obtained a similar strong solution with a uniform decay estimate.

REFERENCES

- [[1]] Agmon, S., *On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems*, Comm. Pure Appl. Math. **15** (1962), 119-147.
- [[2]] Borchers, W. and Miyakawa, T., *Algebraic L^2 decay for Navier-Stokes flows in exterior domains*, Acta Math. **165** (1990), 189-227.
- [[3]] Borchers, W. and Miyakawa, T., *On stability of exterior Navier-Stokes flows*, preprint.
- [[4]] Finn, R., *On exterior stationary problem for the Navier-Stokes equations and associated perturbation problems*, Arch. Rational Mech. Anal. **19** (1965), 363-406.
- [[5]] Fujita, H., *On the existence and regularity of steady state solutions of the*, J. Fac. Sci. Univ. Tokyo, Sec. IA Math. **9** (1961), 59-102.
- [[6]] Fujiwara, D. and Morimoto, H., *An L^r -theorem of the Helmholtz decomposition of vector fields*, J. Fac. Sci. Univ. Tokyo, Sec. IA Math. **24** (1977), 685-700.
- [[7]] Giga, Y., *Domains of fractional powers of the Stokes operator in L_r spaces*, Arch. Rational Mech. Anal. **89** (1985), 251-265.
- [[8]] Giga, Y. and Miyakawa, T., *Solution in L_r of the Navier-Stokes initial value problem*, Arch. Rational Mech. Anal. **89** (1985), 267-281.
- [[9]] Giga, Y. and Sohr, H., *On the Stokes operator in exterior domains*, J. Fac. Sci. Univ. Tokyo, Sec. I. **36** (1989), 103-130.
- [[10]] Heywood, J. G., *On stationary solutions of the Navier-Stokes equations as limits of nonstationary solutions*, Arch. Rational Mech. Anal. **37** (1970), 48-60.
- [[11]] Heywood, J. G., *The Navier-Stokes equations: On the existence, regularity and decay of solutions*, Indiana Univ. Math. J. **29** (1980), 639-681.
- [[12]] Iwashita, H., *$L_q - L_r$ estimates for solutions of the nonstationary Stokes equations in an exterior domain and the Navier-Stokes initial value problems in L_q spaces*, Math. Ann. **285** (1989), 265-288.
- [[13]] Kato, T., *Strong L^p -solution of the Navier-Stokes equation in $\mathbb{R}^m$, with applications to weak solutions*, Math. Z. **187** (1984), 471-480.
- [[14]] Kozono, H. and Ogawa, T., *On stability of the Navier-Stokes flows in exterior domains*, preprint.
- [[15]] Kozono, H. and Sohr, H., *On a new class of generalized solutions for the Stokes equations in exterior domains*, Ann. Scuola Norm. Sup. Pisa **19** (1992), 155-181.
- [[16]] Kozono, H. and Sohr, H., *On stationary Navier-Stokes equations in unbounded domains*, to appear, Ricerche Mat. Univ. Napoli.
- [[17]] Kozono, H. and Yamazaki, M., *Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data*, preprint.
- [[18]] Masuda, K., *On the stability of incompressible viscous fluid motions past object*, J. Math. Soc. Japan **27** (1975), 294-327.
- [[19]] Masuda, K., *Weak solutions of the Navier-Stokes equations*, Tohoku Math. J. **36** (1984), 623-646.
- [[20]] Miyakawa, T., *On nonstationary solutions of the Navier-Stokes equations in an exterior domain*, Hiroshima Math. J. **12** (1982), 115-140.
- [[21]] Serrin, J., *The initial value problem for the Navier-Stokes equations*, Nonlinear Problem, R. Longier ed., The University of Wisconsin Press, Madison, 1960, pp. 69-98.
- [[22]] Simader, C. G. and Sohr, H., *A new approach to the Helmholtz decomposition and the Neumann problem in L^q -spaces for bounded and exterior domains*, "Mathematical Problems Relating to the Navier-Stokes Equations", series on Advanced in Mathematics for Applied Sciences, G.P. Galdi ed., (1992), 1-35, World Scientific, Singapore-New Jersey-London-Hong Kong.
- [[23]] Ukai, S., *A solution formula for the Stokes equation in $\mathbb{R}_+^n$* , Comm. Pure Appl. Math. **40** (1987), 611-621.