

Some Remarks on Generalized Inverse $*$ -Semigroups

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1 Preliminaries

A semigroup S with a unary operation $*$: $S \rightarrow S$ is called a *regular $*$ -semigroup* if it satisfies the conditions:

- (1) $(a^*)^* = a$,
- (2) $(ab)^* = b^*a^*$
- (3) $aa^*a = a$

for any $a, b \in S$.

Let S be a regular $*$ -semigroup. An idempotent e in S is called a *projection* if $e^* = e$. For a subset A of S , denote the set of projections of A by $P(A)$.

A regular $*$ -semigroup S is called a *generalized inverse $*$ -semigroup* if $E(S)$, the set of idempotents of S , satisfies the identity:

$$x_1x_2x_3x_4 = x_1x_3x_2x_4 \quad (1.1)$$

Such a semigroup is orthodox in the usual sense that $E(S)E(S) \subseteq E(S)$.

Result 1.1 ([12]) *A regular $*$ -semigroup S is a generalized inverse $*$ -semigroup if, and only if, $P(S)$ satisfies the identity (1.1).*

Let S be a regular $*$ -semigroup. For $a, b \in S$, define a relation $\leq$ on S by

$$a \leq b \Leftrightarrow a = eb = bf \text{ for some } e, f \in P(S).$$

Result 1.2 ([5]) *Let a and b be elements of a regular $*$ -semigroup S . Then the following are equivalent:*

- (i) $a \leq b$.
- (ii) $aa^* = ba^*$ and $a^*a = b^*a$.
- (iii) $aa^* = ab^*$ and $a^*a = a^*b$.
- (iv) $a = aa^*b = ba^*a$.

Result 1.3 ([4]) *Let S be a regular $*$ -semigroup. Then*

- (i) $E(S) = P(S)^2$. In fact, for any $e \in E(S)$, there exist $f, g \in P(S)$ such that $f\mathcal{R}e\mathcal{L}g$ and $e = fg$.
- (ii) For any $a \in S$ and $e \in P(S)$, $a^*ea \in P(S)$.
- (iii) For $e, f \in P(S)$, $ef \in P(S)$ if, and only if, $ef = fe$.
- (iv) Each $\mathcal{L}$ -class and each $\mathcal{R}$ -class in S contains one and only one projection.

Result 1.4 ([8]) *Let S be an orthodox semigroup. Then*

$$\sigma = \{(a, b) \in S \times S : eae = ebe \text{ for some } e \in E(S)\}$$

is the minimum group congruence on S .

2 E -unitary generalized inverse $*$ -semigroups

PG^* -semigroups

Let (G, X, Y) be a McAlister triple, and let $\{P_\alpha : \alpha \in Y\}$ be a family of disjoint non-empty sets indexed by the elements of Y . Put $P = \bigcup_{\alpha \in Y} P_\alpha$. For each pair α, β of elements of Y where $\alpha \geq \beta$, let $\rho_{\alpha, \beta} : P_\alpha \rightarrow P_\beta$ be a mapping such that the following two axioms hold:

(PG*1) $\rho_{\alpha, \alpha}$ is the identity mapping on P_α .

(PG*2) If $\alpha \geq \beta \geq \gamma$ then $\rho_{\alpha, \beta} \rho_{\beta, \gamma} = \rho_{\alpha, \gamma}$.

We call such a quintet $(G, X, Y, P, \{\rho_{\alpha, \beta}\})$ a PG^* -quintet.

Proposition 2.1 *Let $(G, X, Y, P, \{\rho_{\alpha, \beta}\})$ be a PG^* -quintet. Then*

$$S = \{(\alpha, g, x_1, x_2) \in Y \times G \times P \times P : g^{-1}\alpha \in Y, x_1 \in P_\alpha, x_2 \in P_{g^{-1}\alpha}\},$$

with multiplication and a unary operation given by

$$\begin{aligned} (\alpha, g, x_1, x_2)(\beta, h, y_1, y_2) &= (\alpha \wedge g\beta, gh, x_1\rho_{\alpha, \alpha \wedge g\beta}, y_2\rho_{h^{-1}\beta, (gh)^{-1}(\alpha \wedge g\beta)}), \\ (\alpha, g, x_1, x_2)^* &= (g^{-1}\alpha, g^{-1}, x_2, x_1) \end{aligned}$$

is an E -unitary generalized inverse $*$ -semigroup.

We say that S is a PG^* -semigroup and denoted by $PG^*(G, X, Y, P, \{\rho_{\alpha, \beta}\})$, or simply by $PG^*(G, X, Y, P)$.

We now characterise the Green's relations $\mathcal{L}, \mathcal{R}$, the minimum group congruence σ and the natural order $\leq$ on $PG^*(G, X, Y, P, \{\rho_{\alpha, \beta}\})$.

Proposition 2.2 *Let $(\alpha, g, x_1, x_2), (\beta, h, y_1, y_2)$ be elements of $S = PG^*(G, X, Y, P, \{\rho_{\alpha, \beta}\})$.*

- (i) $(\alpha, g, x_1, x_2) \leq (\beta, h, y_1, y_2)$ if, and only if, $\alpha \leq \beta, g = h, y_1\rho_{\beta, \alpha} = x_1, y_2\rho_{h^{-1}\beta, g^{-1}\alpha} = x_2$.
- (ii) $(\alpha, g, x_1, x_2) \sigma (\beta, h, y_1, y_2)$ if, and only if, $g = h$.
- (iii) $(\alpha, g, x_1, x_2) \mathcal{L} (\beta, h, y_1, y_2)$ if, and only if, $g^{-1}\alpha = h^{-1}\beta$ and $x_2 = y_2$.
- (iv) $(\alpha, g, x_1, x_2) \mathcal{R} (\beta, h, y_1, y_2)$ if, and only if, $\alpha = \beta$ and $x_1 = y_1$.

Now we have the following theorem.

Theorem 2.3 *The semigroup $PG^*(G, X, Y, P, \{\rho_{\alpha, \beta}\})$ is an E -unitary generalized inverse $*$ -semigroup and maximum group homomorphic image isomorphic to G .*

Construction of E -unitary generalized inverse $*$ -semigroups

Let S be an E -unitary generalized inverse $*$ -semigroup. Put $G = S/\sigma$, and denote its identity by 1. Since $E(S)$ is a minimum group congruence class of S , $E(S)$ is the identity of G . Let $E(S) \sim \sum\{E_\alpha : \alpha \in Y\}$ be the structure decomposition of $E(S)$, that is $E(S)$ is a semilattice Y of rectangular bands E_α ($\alpha \in Y$). Put $\mathcal{E} = \{E_\alpha : \alpha \in Y\}$. We shall construct PG^* -quintet.

First, we define a relation ρ on $\mathcal{E} \times G$ by

$$(E_\alpha, g)\rho(E_\beta, h) \Leftrightarrow xx^* \in E_\alpha \text{ and } x^*x \in E_\beta \text{ for some } x \in g^{-1}h.$$

Lemma 2.4 *The relation ρ is an equivalence relation on $\mathcal{E} \times G$.*

We shall write $\mathcal{R}$ for $(\mathcal{E} \times G)/\rho$, and denote the ρ -class of $\mathcal{E} \times G$ which contains (E_α, g) by $(E_\alpha, g)\rho$. The following lemmas are immediate.

Lemma 2.5 For any element $x \in S$, $xx^* \in E_\alpha$, $x^*x \in E_\beta$ for some $\alpha, \beta \in Y$. Then

$$(E_\alpha, 1)\rho(E_\beta, x\sigma) \text{ and } (E_\beta, 1)\rho(E_\alpha, (x\sigma)^{-1}).$$

Lemma 2.6 Let $\alpha, \beta \in Y$ and $g \in G$ such that $(E_\alpha, g)\rho(E_\beta, g)$. Then $\alpha = \beta$.

Proposition 2.7 Let $E_\alpha, E_\beta, E_\gamma \in \mathcal{E}$ and $g, h \in G$. If $\alpha \leq \beta$ and $(E_\beta, g)\rho(E_\gamma, h)$, then there exists $\delta \in Y$ such that $\delta \leq \gamma$, $(E_\alpha, g)\rho(E_\delta, h)$.

We define a relation $\leq$ on $\mathcal{X}$ as follows:

$$A \leq B \Leftrightarrow \alpha \leq \beta, (E_\alpha, g) \in A, (E_\beta, g) \in B$$

for some $\alpha, \beta \in Y$ and $g \in G$. The proof of the following is straightforward from Proposition 2.7 and the definition of $\leq$.

Corollary 2.8 Let $A \leq B$, where $A, B \in \mathcal{X}$. If $(E_\gamma, h) \in B$, then there exists $\delta \in Y$ such that $\delta \leq \gamma$ and $(E_\delta, h) \in A$.

Lemma 2.9 The relation $\leq$ is a partial order on $\mathcal{X}$.

Let

$$\mathcal{Y} = \{(E_\alpha, 1)\rho : \alpha \in Y\}.$$

We define an action of G on $\mathcal{X}$ by order automorphisms. Suppose first that $(E_\alpha, g)\rho(E_\beta, h)$. This means that there exists $x \in g^{-1}h$ such that $xx^* \in E_\alpha$, $x^*x \in E_\beta$. Let $k \in G$. Then $x \in (kg)^{-1}(kh)$ and so $(E_\alpha, kg)\rho = (E_\beta, kh)\rho$. We can therefore define $\circ : G \times \mathcal{X} \rightarrow \mathcal{X}$ by

$$k \circ (E_\alpha, g)\rho = (E_\alpha, kg)\rho.$$

We shall show that the triple $(G, \mathcal{X}, \mathcal{Y})$ form a McAlister triple.

Lemma 2.10 The mapping $\varphi : Y \rightarrow \mathcal{Y}$ defined by $\alpha\varphi = (E_\alpha, 1)\rho$ is an order isomorphism.

Lemma 2.11 The mapping $\circ$ is an action of G on $\mathcal{X}$, on the left by order automorphisms.

Lemma 2.12 With the above notation:

- (i) $\mathcal{Y}$ is an order ideal of $\mathcal{X}$.
- (ii) $G \circ \mathcal{Y} = \mathcal{X}$.
- (iii) $g \circ \mathcal{Y} \cap \mathcal{Y} \neq \emptyset$ for all $g \in G$.

By the lemma above, we have that the triple $(G, \mathcal{X}, \mathcal{Y})$ is a McAlister triple. We shall construct PG^* -quintet by making use of McAlister triple $(G, \mathcal{X}, \mathcal{Y})$ and form the PG^* -semigroup $PG^*(G, \mathcal{X}, \mathcal{Y}, P)$. Put $P_\alpha = P(E_\alpha)$ for each $\alpha \in Y$ and let $P = \bigcup_{\alpha \in Y} P_\alpha$. For each pair α, β of elements of Y where $\alpha \geq \beta$, define the mapping

$$\rho_{\alpha, \beta} : P_\alpha \rightarrow P_\beta \text{ by } e\rho_{\alpha, \beta} = efe \text{ where } f \in P_\beta.$$

Lemma 2.13 With the definition above, $\rho_{\alpha, \beta}$ is a mapping satisfying the conditions (PG*1) and (PG*2).

Thus $PG^*(G, \mathcal{X}, \mathcal{Y}, P, \{\rho_{\alpha, \beta}\})$, constructed above, forms a PG^* -semigroup.

Lemma 2.14 For any $xx^* \in P_\alpha$ and $e \in P_\beta$, $xex^* \in P_{\alpha \wedge (x\sigma)\beta}$.

Lemma 2.15 The mapping $\theta : S \rightarrow PG^*(G, \mathcal{X}, \mathcal{Y}, P, \{\rho_{\alpha, \beta}\})$ defined by

$$x\theta = ((E_\alpha, 1)\rho, x\sigma, xx^*, x^*x),$$

where $xx^* \in E_\alpha$, is a $*$ -isomorphism.

Now we have the structure of generalized inverse $*$ -semigroups.

Proposition 2.16 A generalized inverse $*$ -semigroup is E -unitary if, and only if, it is $*$ -isomorphic to some PG^* -semigroup.

3 The compatibility relations

Let S be a regular $*$ -semigroup. For all $s, t \in S$, the *left compatibility relation* is defined by

$$s \sim_l t \Leftrightarrow st^* \in E(S),$$

the *right compatibility relation* is defined by

$$s \sim_r t \Leftrightarrow s^*t \in E(S),$$

and the *compatibility relation*, the intersection of the above two relations, is defined by

$$s \sim t \Leftrightarrow st^*, s^*t \in E(S).$$

It is clear that all three relations are reflexive and symmetric, but none of them need be transitive (see Theorem 3.2 for a characterisation of the generalized inverse $*$ -semigroups having a transitive compatibility relation). The next lemma describe some of the basic property of these relations.

Lemma 3.1 Let S be a generalized inverse $*$ -semigroup and ρ be any one of the three relations $\sim_l, \sim_r$, and $\sim$. Then the following two properties hold.

- (i) $s \rho t$ and $u \rho v$ imply that $su \rho tv$.
- (ii) $s \leq t, u \leq v$ and $t \rho v$ imply that $s \rho u$.

Theorem 3.2 Let S be a generalized inverse $*$ -semigroup. Then the compatibility relation is transitive if, and only if, S is E -unitary.

Proof Suppose that $\sim$ is transitive. Let $es \in E(S)$, where e is an idempotent. Then $s \sim es$ since elements $s(es)^*$ and s^*es are idempotents. Clearly $es \sim s^*s$, and so, by our assumption that the compatibility relation is transitive, we have that $s \sim s^*s$. But $s(s^*s)^* = s$, so that s is an idempotent.

Conversely, suppose that S is E -unitary and $s \sim t$ and $t \sim u$. Clearly $(s^*t)(t^*u)$ is an idempotent and

$$(s^*t)(t^*u) = s^*u(t^*u)^*(t^*u)$$

But S is E -unitary and so s^*u is an idempotent. Similarly, su^* is an idempotent. Hence $s \sim u$. ■

Proposition 3.3 Let S be a regular $*$ -semigroup. Then the following are equivalent:

- (i) The left and right compatibility relations are equal.
- (ii) For all $s, t \in S$, we have that $st \in E(S)$ if, and only if, $ts \in E(S)$.

A congruence ρ on an orthodox semigroup S is said to be *idempotent pure* if $a \in S, e \in E(S)$ and $(a, e) \in \rho$ then a is an idempotent.

Proposition 3.4 *Let S be an E -unitary regular $*$ -semigroup. Then a congruence ρ is idempotent pure if, and only if, $\rho \subseteq \sim$.*

Proof Let ρ be idempotent pure and let $(a, b) \in \rho$. Then $(ab^*, bb^*) \in \rho$. But ρ is idempotent pure and bb^* is an idempotent. Thus ab^* is an idempotent. Similarly, a^*b is an idempotent. Thus $a \sim b$.

Conversely, let ρ be a congruence contained in the compatibility relation. Let $(a, e) \in \rho$, where e is an idempotent. Then $a \sim e$. Thus $ae^* \in E(S)$. But e^* is an idempotent and so a is an idempotent, since S is E -unitary. $\blacksquare$

4 Enlargements

We proved in Section 2, that E -unitary generalized inverse $*$ -semigroups are essentially isomorphic to the generalized inverse $*$ -subsemigroups of PG^* -semigroups. The point is that if X is a meet semilattice, we can form the semigroup $PG^*(G, X, X, P, \{\rho_{\alpha, \beta}\})$, which contains $PG^*(G, X, Y, P, \{\rho_{\alpha, \beta}\})$ as a generalized inverse $*$ -subsemigroup. In the following proposition, we shall describe the *abstract* relationship between $PG^*(G, X, Y, P, \{\rho_{\alpha, \beta}\})$ and $PG^*(G, X, X, P, \{\rho_{\alpha, \beta}\})$.

Proposition 4.1 *Let $(G, X, Y, P, \{\rho_{\alpha, \beta}\})$ be a PG^* -quintet, where X is a meet semilattice.*

(i) *The idempotents of $PG^*(G, X, Y, P)$ form an order ideal of $PG^*(G, X, X, P)$.*

(ii) *If $(\alpha, g, x_1, x_2) \in PG^*(G, X, X, P)$ is such that*

$$(\alpha, g, x_1, x_2)^*(\alpha, g, x_1, x_2), (\alpha, g, x_1, x_2)(\alpha, g, x_1, x_2)^* \in PG^*(G, X, Y, P)$$

then $(\alpha, g, x_1, x_2) \in PG^*(G, X, Y, P)$.

(iii) *For each projection $(\alpha, 1, x, x) \in PG^*(G, X, X, P)$ there exists a projection $(\beta, 1, y, y) \in PG^*(G, X, Y, P)$ such that $(\alpha, 1, x, x) \mathcal{D} (\beta, 1, y, y)$.*

On the basis of the above proposition, we make the following definition. Let S be a generalized inverse $*$ -subsemigroup of a generalized inverse $*$ -semigroup T . We say that T is an *enlargement* of S if the following three axioms hold:

(E1) $E(S)$ is an order ideal of $E(T)$.

(E2) If $t \in T$ and $t^*t, tt^* \in S$ then $t \in S$.

(E3) For every projection $e \in T$ there exists a projection $f \in S$ such that $e \mathcal{D} f$.

The following is easy to prove.

Lemma 4.2 *Let S be a generalized inverse $*$ -subsemigroup of T . Then axiom (E1) holds if, and only if, S is an order ideal of T .*

We may find a PG^* -representation of an E -unitary generalized inverse $*$ -semigroup.

Theorem 4.3 *Let G be a group and X a semilattice, and let S be a generalized inverse $*$ -subsemigroup of the generalized inverse $*$ -semigroup $PG^*(G, X, X, P, \{\rho_{\alpha, \beta}\})$. Suppose that $PG^*(G, X, X, P, \{\rho_{\alpha, \beta}\})$ is an enlargement of S . Let*

$$Y = \{\alpha \in X : (\alpha, 1, x, y) \in E(S)\} \text{ and } Q = \{x \in P : (\alpha, 1, x, y) \in E(S)\}.$$

Then $(G, X, Y, Q, \{\rho_{\alpha, \beta}\})$ is a PG^* -quintet and $S = PG^*(G, X, Y, Q, \{\rho_{\alpha, \beta}\})$.

5 A Structure Theorem

We can now prove the uniqueness of the PG^* -representation of an E -unitary generalized inverse $*$ -semigroup.

Theorem 5.1 *Let $(G, X, Y, P, \{\rho_{\alpha, \beta}\})$ and $(G', X', Y', P', \{\rho'_{\alpha', \beta'}\})$ be two PG^* -quintets. Let $\theta : G \rightarrow G'$ be a group isomorphism and let $\psi : X \rightarrow X'$ be an order isomorphism such that $\psi|_Y$ is an isomorphism from the semilattice Y onto Y' ; now let $\xi : P \rightarrow P'$ be a bijection. Suppose also that, for all g in G , α in X and x in P_β .*

$$\begin{aligned}(g\alpha)\psi &= (g\theta)(\alpha\psi), \\ (x\rho_{\beta, \gamma})\xi &= (x\xi)\rho_{\beta\psi, \gamma\psi},\end{aligned}$$

where $\beta, \gamma \in Y$ such that $\beta \geq \gamma$. Then the mapping $\phi : PG^*(G, X, Y, P) \rightarrow PG^*(G', X', Y', P')$ defined by

$$(\alpha, g, x, y)\phi = (\alpha\psi, g\theta, x\xi, y\xi)$$

is a $*$ -isomorphism. Conversely, every $*$ -isomorphism from $PG^*(G, X, Y, P)$ onto $PG^*(G', X', Y', P')$ is of this type.

6 The minimum group congruence

In this subsection, we shall first give an alternative characterization of the minimum group congruence on a generalized inverse $*$ -semigroup.

Theorem 6.1 *If S is a generalized inverse $*$ -semigroup, then the relation*

$$\sigma = \{(a, b) \in S \times S : eaf = ebf \text{ for some } e, f \in P(S)\}$$

is the minimum group congruence on S .

Idempotent pure congruences, the minimum group congruence and E -unitary generalized inverse $*$ -semigroups are all linked by the following result.

Theorem 6.2 *Let S be a generalized inverse $*$ -semigroup. Then the following conditions are equivalent:*

- (i) S is E -unitary.
- (ii) $\sim = \sigma$.
- (iii) σ is idempotent pure.
- (iv) $\sigma(e) = E(S)$ for any idempotent e .

Proof (i) $\Leftrightarrow$ (iv). Immediate.

(i) $\Rightarrow$ (ii). Let $a \sim b$. Then $ab^*, a^*b \in E(S)$. Thus

$$\begin{aligned}(ab^*)(ab^*)^*a(a^*b)(a^*b)^* &= ab^*ba^*aa^*bb^*a \\ &= ab^*ba^*bb^*a \\ &= ab^*(ba^*)(ba^*)bb^*a \\ &= (ab^*)(ab^*)^*b(a^*b)(a^*b)^*.\end{aligned}$$

Hence $a \sigma b$.

Conversely, suppose $a \sigma b$. Then $ea f = eb f$ for some $e, f \in P(S)$ by Theorem 6.1. Thus we have

$$(ebf)(ebf)^* = eafb^*bb^*e = (eab^*)bfb^*e \in E(S).$$

But bfb^*e is an idempotent. Thus, by (i), $eab^* \in E(S)$. By using (i) again, we obtain $ab^* \in E(S)$ since $e \in E(S)$. Similarly, a^*b is an idempotent.

(ii) $\Rightarrow$ (iii). Let $(a, e) \in \sigma$, where e is an idempotent. Clearly, $e \sim a^*a$. But $\sim = \sigma$ and so $a \sim a^*a$. Hence a is an idempotent.

(iii) $\Rightarrow$ (i). Let $a \in S$ and $e \in E(S)$ such that $ea \in E(S)$. Then $eae = e(eae)e$. Thus, by Result 1.4, $(a, eae) \in \sigma$. But $eae = (ea)e \in E(S)$ and so a is an idempotent since σ is idempotent pure. ■

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