

Coinvariant Algebras of Some Finite Groups

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0. Recently Y.Ito and I.Nakamura [IN2], [N2] studied the Hilbert scheme of G -orbits $\text{Hilb}^G(\mathbf{C}^2)$ for a finite group $G \subset SL(2, \mathbf{C})$ and showed a direct correspondence between the representation graph of G (McKay observation) and the singular fiber of the minimal resolution of $\mathbf{C}^2/G$ (Dynkin curve). In this article we report some attempts to extend the results to finite subgroups of $SL(3, \mathbf{C})$, which is being studied jointly with Iku Nakamura (Hokkaido Univ.) and Yasushi Gomi (Sophia Univ.). For simplicity we take the complex number field $\mathbf{C}$ as a ground field and representations considered are complex representations.

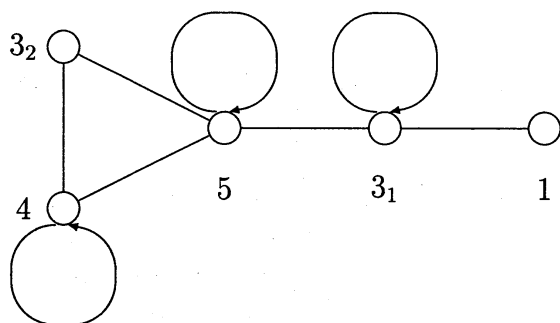
1. Let G be a finite group, $\text{Irr}(G) = \{\chi_1, \dots, \chi_s\}$ be the set of all irreducible characters of G and $\text{Irr}(G)^\# = \text{Irr}(G) - \{1_G\}$. Given a character χ of G , we can form the representation graph $\Gamma(G) = \Gamma_\chi(G)$ as follows: the set of vertices is $\text{Irr}(G)$ and the directed edge of weight m_{ij} from χ_i to χ_j is determined by the relation

$$\chi \cdot \chi_i = \sum_{j=1}^s m_{ij} \chi_j, \quad i = 1, \dots, s.$$

We use the convention that a pair of opposing directed edges of weight 1 is represented by a single edge and the weight m_{ij} is omitted if $m_{ij} = 1$.

Example 1. Let G be the quaternion group of order 8. Then $\text{Irr}(G)$ consists of 4 linear characters and the character χ of 2-dimensional representation. Then $\Gamma_\chi(G)$ is exactly the extended Dynkin diagram of type D_4 centered at χ .

Example 2. Let G be the alternating group of degree 5, A_5 . Then $\text{Irr}(G) = \{1, \chi = 3_1, 3_2, 4, 5\}$, (where the characters are expressed by the degrees of the corresponding representations), and $\Gamma_\chi(G)$ becomes as follows:



2. In [M] J. McKay stated the following which is now famous as McKay observation.

Proposition. Let G be a finite subgroup of $SL(2, \mathbf{C})$ and χ be the character of the inclusion representation. Then $\Gamma_\chi(G)$ is an extended Dynkin diagram of type A, D or E.

Conversely every such extended Dynkin diagram is obtained as a representation graph of a subgroup of $SL(2, \mathbb{C})$.

Thus McKay observation establishes a bijective correspondence between subgroups G of $SL(2, \mathbb{C})$ and the extended Dynkin diagram $\bar{X}_G$ of type A, D and E.

3. There is another famous correspondence between subgroups G of $SL(2, \mathbb{C})$ and the Dynkin diagram X_G of type A, D and E. (The extended Dynkin diagram of X_G is $\bar{X}_G$.) Let $S = \mathbb{C}^2/G$ and $p : \tilde{S} \rightarrow S$ be the minimal resolution of singularity. Then the singular fiber, $p^{-1}(0)$, is a union of projective lines, Dynkin curve of type X_G , having intersection matrix $-C$, where C is the Cartan matrix of type X_G . In particular the graph obtained by Dynkin curve as follows is the Dynkin diagram X_G : the set of vertices is that of projective lines appearing in Dynkin curve and two lines are joined iff they meet. For details, please see a survey article of R.Steinberg[St] or P.Slodowy[Sl].

These two correspondences were famous, but relations between them had not been clear. Recently an explanation of these correspondences was given by Y.Ito and I.Nakamura[IN1], [IN2] and I.Nakamura[N1], [N2], using Hilbert schemes.

4. Let $Hilb^n(\mathbb{C}^m)$ be the Hilbert scheme of $\mathbb{C}^m$ parametrizing all the 0-dimensional subschemes of length n and let $Symm^n(\mathbb{C}^m)$ be the n -th symmetric product of $\mathbb{C}^m$, that is, the quotient of n -copies of $\mathbb{C}^m$ by the natural action of the symmetric group of degree n . There is a canonical morphism π from $Hilb^n(\mathbb{C}^m)$ to $Symm^n(\mathbb{C}^m)$ associating to each 0-dimensional subscheme of $\mathbb{C}^m$ its support. Let G be a finite subgroup of $SL(m, \mathbb{C})$. The group G acts on $\mathbb{C}^m$ so that it acts naturally on both $Hilb^n(\mathbb{C}^m)$ and $Symm^n(\mathbb{C}^m)$. Since π is G -equivariant, π induces a morphism from the G -fixed point set $Hilb^n(\mathbb{C}^m)^G$ to the G -fixed point set $Symm^n(\mathbb{C}^m)^G$.

Now consider the special situation that n is the order of the group G and $m = 2$. Then $Symm^n(\mathbb{C}^2)^G$ is isomorphic to the quotient space $\mathbb{C}^2/G$ and there is a unique irreducible component of $Hilb^n(\mathbb{C}^2)^G$ dominating $Symm^n(\mathbb{C}^2)^G$, which we denote by $Hilb^G(\mathbb{C}^2)$ and call it the Hilbert scheme of G -orbits, following the notation and the definition by I.Nakamura. Notice that we have a morphism $p : Hilb^G(\mathbb{C}^2) \rightarrow \mathbb{C}^2/G$ induced by π . The following theorem is proved in a unified way.

Theorem. [IN2]. $Hilb^G(\mathbb{C}^2)$ is nonsingular and $p : Hilb^G(\mathbb{C}^2) \rightarrow \mathbb{C}^2/G$ is a minimal resolution of singularity.

5. Let $R = \mathbb{C}[x, y]$ be the ring of regular functions on $\mathbb{C}^2$ and M be the maximal ideal corresponding to the origin, that is $M = (x, y)$. For a finite group $G \subset SL(2, \mathbb{C})$ of order n , let R^G be the invariant algebra of G and N be the ideal of R generated by invariant homogeneous polynomials of positive degree which generate R^G . The ring $R_G = R/N$ is called the coinvariant algebra of G .

We identify a G -invariant 0-dimensional subscheme with its defining ideal of R . For $I \in Hilb^G(\mathbb{C}^2)$ with support origin, put $V(I) = I/(MI + N)$. Then $V(I)$ is a G -module and we denote its character by $\chi_{V(I)}$. Let E be the exceptional set of p and $Irr(E)$ be

the set of irreducible components of E . For $\chi \in \text{Irr}(G)^\sharp$, define

$$E(\chi) = \{I \in E \mid (\chi, \chi_{V(I)})_G \neq 0\}$$

where $(\cdot, \cdot)_G$ is the usual inner product on functions on G . Then by verifying every case the following theorem is obtained.

Theorem. [IN2],[N2].

$$E = \{ I \mid G\text{-invariant ideal of } R, N \subset I \subset M, R/I \simeq \mathbb{C}G \}$$

and the map $\chi \mapsto E(\chi)$ gives a bijective correspondence between $\text{Irr}(G)^\sharp$ and $\text{Irr}(E)$.

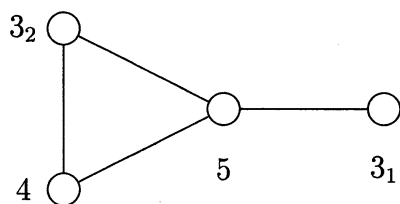
6. Let G be a subgroup of $SL(3, \mathbb{C})$. R, R^G, R_G, M and N are defined similarly for $\mathbb{C}^3$ and G as in 5. Now theorem 5 suggests the necessity to study

$$F_G := \{ I \mid G\text{-invariant ideal of } R, N \subset I \subset M, R/I \simeq \mathbb{C}G \},$$

which would be a fiber of the origin of the quotient space $\mathbb{C}^3/G$ in the Hilbert scheme of G -orbits. For that purpose we need detailed structures of the coinvariant algebras R_G . What we have mainly obtained so far are

- decomposition of R_G (or its overalgebra) into irreducible components, particularly for groups of orders $60(A_5)$, $168(PSL(2, 7))$, 108 , 180 , 216 , 504 , 648 , and 1080 ,
- explicit determination of basis for each irreducible component above for A_5 and $PSL(2, 7)$.

As an outcome of these calculations we can show that F_{A_5} is a union of projective lines whose graph is given by



and a graph for $PSL(2, 7)$ also can be given. Details will appear in [GNS].

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