

Normal Forms of Vector Fields and Diffeomorphisms

By

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Abstract

We shall show simultaneous normal forms of a system of vector fields and diffeomorphisms under Brjuno condition. These results are proved by a new scheme of a rapidly convergent iteration with high loss of derivatives such that for some ε , $0 < \varepsilon < 1$, $\exp(\exp((\sigma - \sigma')^{-\varepsilon}))$, $0 < \sigma' < \sigma$.

We solve an overdetermined system of equations arising in the study of normal forms and diffeomorphisms by this method.

1 Normal forms of vector fields

Let us consider a system of analytic vector fields $X^\mu (\mu = 1, \dots, d)$ in some neighborhood of the origin of $x = (x_1, \dots, x_n) \in \mathbf{R}^n$,

$$X^\mu = \langle X^\mu, \partial_x \rangle = \sum_{j=1}^n X_j^\mu(x) \partial_{x_j}, \quad 1 \leq \mu \leq d, \quad (1.1)$$

with the convention that $\partial_x = (\partial_{x_1}, \dots, \partial_{x_n})$, $\partial_{x_j} = \partial/\partial x_j$. We assume

$$X^\mu \ (1 \leq \mu \leq d) \text{ are singular i.e. } X^\mu(0) = 0 \text{ for } 1 \leq \mu \leq d. \quad (1.2)$$

The linear parts of $X^\mu (1 \leq \mu \leq d)$ are semi-simple i.e.,

$$X^\mu(x) = (X_1^\mu(x), \dots, X_n^\mu(x)) = \Lambda^\mu x + R^\mu(x), \quad 1 \leq \mu \leq d, \quad (1.3)$$

where

$$\Lambda^\mu = \begin{pmatrix} \lambda_1^\mu & & 0 \\ & \ddots & \\ 0 & & \lambda_n^\mu \end{pmatrix}, \quad \lambda_j^\mu \in \mathbf{C}$$

and where $R^\mu(x)$ are analytic at the origin and satisfy

$$R^\mu(0) = \partial_x R^\mu(0) = 0, \quad 1 \leq \mu \leq d.$$

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$$X^\mu (1 \leq \mu \leq d) \text{ are pairwise commuting, i. e. } [X^\mu, X^\nu] = 0, \quad 1 \leq \nu, \mu \leq d. \quad (1.4)$$

Set $\lambda^\mu = (\lambda_1^\mu, \dots, \lambda_n^\mu)$, $(1 \leq \mu \leq d)$. We are interested in reduction of vector fields to normal forms. If $d = 1$ (single case), a normal form was obtained by Poincaré under the condition

$$(*) \quad |\lambda \alpha| \geq c_0 |\alpha| \quad \text{for } \alpha \in \mathbf{Z}_+^n, |\alpha| \gg 1$$

Roughly speaking, in order to find a change of variables which reduces a vector field to its normal form we must solve a nonlinear equation, a so-called homological equation. The condition $(*)$ implies the existence of the bounded inverse of the linearized operator. The solvability of certain nonlinear equations under Poincaré condition was proved by Kaplan for more general equation. ([6]).

The solvability of these nonlinear equations with unbounded inverse was proved by Siegel in case $d = 1$ ([12]) under a famous Siegel condition :

$$\exists c > 0, \exists \gamma > 0; |\lambda \alpha - \lambda_k| \geq c |\alpha|^{-\gamma} \quad \text{for } 1 \leq k \leq n, \alpha \in \mathbf{Z}_+^n. \quad (1.5)$$

Rüssman ([10]) generalized his idea and proved

Assume $d = 1$. Suppose (1.2), (1.3) and (1.5). Then the vector field (1.1) can be transformed to a normal form by a holomorphic change of variables.

By the studies of normal forms of mappings by Yoccoz ([13]) and M. Perez ([9]), it is natural to weaken the condition (1.5) to the following simultaneous Brjuno condition:

$\exists c > 0, \exists \gamma > 0$ such that

$$(Br) \quad \max_{1 \leq \mu \leq d} |\lambda^\mu \alpha - \lambda_k^\mu| \geq c \exp \left(- \frac{|\alpha|}{\log(2 + |\alpha|)^{1+\gamma}} \right) \quad \forall \alpha \in \mathbf{Z}_+^n, 1 \leq k \leq n.$$

We note that our condition is weaker because the bound from the below is exponentially small when $|\alpha| \rightarrow \infty$, and there is a maximum in μ in the left-hand side. Hence each vectors could be resonant and may not satisfy a Brjuno condition as a single equation, while they simultaneously satisfy (Br).

We note that (Br) implies that $\lambda_1^\mu, \dots, \lambda_n^\mu$ are non simultaneous resonant, namely

$$\max_{1 \leq \mu \leq d} |\lambda^\mu \cdot \alpha - \lambda_k^\mu| \neq 0, \quad \forall \alpha \in \mathbf{Z}_+^n, 1 \leq k \leq n. \quad (1.6)$$

Then we have

Theorem 1.1 *Let $X^1(x), \dots, X^d(x)$ be pairwise commuting holomorphic vector fields satisfying the conditions (1.2), (1.3) and (1.4). If $\lambda^1, \dots, \lambda^d$ verify the Brjuno condition (Br) we can find a neighborhood Ω of the origin and a holomorphic change of the variables $x = y + u(y)$, $y \in \Omega$ which transforms simultaneously $X^1(x), \dots, X^d(x)$ into $\lambda^1 y \partial_y, \dots, \lambda^d y \partial_y$, respectively. Moreover, u is a solution of the following equation*

$$\mathcal{L}_{\lambda^\mu} u - R^\mu(y + u) = 0, \quad 1 \leq \mu \leq d. \quad (1.7)$$

1.1 Approximate solution to a homological equation

First we need to introduce some Banach spaces of holomorphic functions. Let Ω be an open ball containing the origin in $\mathbb{C}^n$ and let $\mathcal{O}(\Omega)$ be the set of holomorphic functions on Ω . Following [4] we define for $0 < T < \text{diam}(\Omega)/2$

$$H(T) = \{u(x) = \sum_{\alpha \in \mathbb{Z}^n} u_\alpha x^\alpha \in \mathcal{O}(\Omega) : \|u\|_T = \sum_{\alpha \in \mathbb{Z}^n} |u_\alpha| T^{|\alpha|} < \infty\} \quad (1.8)$$

Theorem 1.2 *The following estimate is true*

$$\|D^\beta u\|_{T_1} \leq \frac{C}{(T - T_1)^{|\beta|}} \|u\|_T. \quad (1.9)$$

for all $0 < T_1 < T$.

We define

$$Mf = \sum_{\mu=1}^d \mathcal{L}_{\tilde{\lambda}^\mu} \mathcal{L}_{\lambda^\mu} f, \quad f \in (H(T))^n := H(T) \times \cdots \times H(T). \quad (1.10)$$

If we expand $f(x)$ into Taylor series $f(x) = \sum_\alpha f_\alpha x^\alpha$ and if we set $Mf = \sum M(\alpha) f_\alpha x^\alpha$ we can see that

$$M(\alpha) = \text{diag}(M_1(\alpha), \dots, M_n(\alpha)), \quad M_j(\alpha) = \sum_{\mu=1}^d |\lambda^\mu \cdot \alpha - \lambda_j^\mu|^2, \quad 1 \leq j \leq n. \quad (1.11)$$

Then we have

Lemma 1.3 *Let $T_0 > 0$ be given. Suppose that (Br) is satisfied. Then, for any $0 < T' < T < T_0$ there exists an inverse $M^{-1} : (H(T'))^n \rightarrow (H(T))^n$ and a constant $c_0 > 0$ such that the following estimate holds*

$$\|M^{-1}\|_{T' \rightarrow T} \leq \exp\left(2 \exp(c_0(T - T')^{-1/(1+\tau)})\right), \quad T_0 < \forall T' < \forall T < 2T_0. \quad (1.12)$$

Proof. Set $\delta = T'/T$. We have, for $f = \sum f_\alpha x^\alpha \in (H(T'))^n$

$$\begin{aligned} \|M^{-1}f\|_{T'} &= \sum_{\alpha} T'^{|\alpha|} |M^{-1}(\alpha) f_\alpha| \leq \\ &\leq \sum_{\alpha} \delta^{|\alpha|} \exp\left(2|\alpha|(\ln(|\alpha| + 2))^{-\tau-1}\right) |f_\alpha| T^{|\alpha|}. \end{aligned} \quad (1.13)$$

Since

$$\sup_{|\alpha| \geq 1} \left(\delta^{|\alpha|} \exp\left(2|\alpha|(\ln(|\alpha| + 2))^{-\tau-1}\right) \right) \leq \exp\left(\exp\left(c(\ln \delta^{-1})^{-1/(1+\tau)}\right)\right) \quad (1.14)$$

for some $c > 0$ and since $\ln(T/T') = \ln(1 + (T - T')/T')$ is bounded by the constant times of $T - T'$ from the above and the below we have (1.12). $\square$.

For the later use we define the approximate inverse to a homological operator as follows:

$$P(D)\vec{f} = \sum_{\mu=1}^d \mathcal{L}_{\bar{\lambda}^\mu} M^{-1} f_\mu, \quad \vec{f} = (f_1, \dots, f_d) \in (H(T))^{nd}. \quad (1.15)$$

We observe that

$$\mathcal{L}_{\lambda^\mu} P(D)\vec{f} = f_\mu + M^{-1} \sum_{\nu=1, \nu \neq \mu}^d \mathcal{L}_{\bar{\lambda}^\nu} (\mathcal{L}_{\lambda^\mu} f_\nu - \mathcal{L}_{\lambda^\nu} f_\mu) \quad 1 \leq \forall \mu \leq d. \quad (1.16)$$

1.2 Rapidly convergent iteration scheme

Now we will prove Theorem 2.1. We shall find $u(x)$ such that

$$\left(1 + \frac{\partial u}{\partial x}\right)^{-1} X^\mu(x + u(x)) = \Lambda^\mu x = {}^t(\lambda_1^\mu x_1, \dots, \lambda_n^\mu x_n) \quad (1.17)$$

for $1 \leq \mu \leq d$. The equation (1.17) is equivalent to solving the following overdetermined system of equations

$$\mathcal{L}_{\lambda^\mu} u(x) = R_\mu(x + u(x)), \quad 1 \leq \mu \leq d. \quad (1.18)$$

We set

$$v_0(x) = \sum_{\nu=1}^d \mathcal{L}_{\bar{\lambda}^\nu} M^{-1} R_\nu^0(x), \quad R_\nu^0(x) = R_\nu(x). \quad (1.19)$$

By a scale change of variables we may assume that $|v_0|_T \ll 1$. Then we consider the change of the variables $x + v_0(x)$ and obtain the new system of vector fields

$$X^{\mu,1}(x) = \left(1 + \frac{\partial v_0}{\partial x}\right)^{-1} X^\mu(x + v_0(x)) \equiv \Lambda^\mu x + R_\mu^1(x), \quad 1 \leq \mu \leq d. \quad (1.20)$$

Straightforward calculations show (multiplying by $(1 + \partial v_0/\partial x)$ from the left and recalling that $X^\mu(x + v_0(x)) = \Lambda^\mu x + \Lambda^\mu v_0(x) + R_\mu^0(x + v_0(x))$)

$$\Lambda^\mu x + \Lambda^\mu v_0(x) + R_\mu^0(x + v_0(x)) = \left(1 + \frac{\partial v_0}{\partial x}\right) \Lambda^\mu x + \left(1 + \frac{\partial v_0}{\partial x}\right) R_\mu^1(x),$$

i.e.,

$$\begin{aligned} \left(1 + \frac{\partial v_0}{\partial x}\right) R_\mu^1(x) &= -\frac{\partial v_0}{\partial x} \Lambda^\mu x + \Lambda^\mu v_0(x) + R_\mu^0(x + v_0(x)) \\ &= -\mathcal{L}_{\lambda^\mu} v_0 + R_\mu^0(x + v_0(x)) \end{aligned} \quad (1.21)$$

$$\begin{aligned}
&= -R_\mu^0(x) + \sum_{\nu=1}^d \mathcal{L}_{\bar{\lambda}^\nu} M^{-1} (\mathcal{L}_{\lambda^\nu} R_\mu^0 - \mathcal{L}_{\lambda^\mu} R_\nu^0) + R_\mu^0(x + v_0(x)) \\
&= v_0(x) \int_0^1 R_\mu^{0'}(x + tv_0(x)) dt + \sum_{\nu=1}^d \mathcal{L}_{\bar{\lambda}^\nu} M^{-1} (R_\mu^0 \partial_x R_\nu^0 - R_\nu^0 \partial_x R_\mu^0),
\end{aligned}$$

where we have used

$$\mathcal{L}_{\lambda^\mu} R^\nu(x) - \mathcal{L}_{\lambda^\nu} R^\mu(x) = \partial R^\mu(x) R^\nu(x) - \partial R^\nu(x) R^\mu(x), \quad 1 \leq \nu, \mu \leq d.$$

This is equivalent to $[X^\mu, X^\nu] = 0$. Therefore, R_μ^1 is estimated quadratically.

We continue this process. Suppose that we have constructed $v_0(x), \dots, v_{k-1}(x)$ such that after a change of variables

$$(1 + v_0) \circ (1 + v_1) \circ \dots \circ (1 + v_{k-1})(x)$$

we have obtained

$$X^{\mu,k}(x) = \Lambda^\mu x + R_\mu^k(x), \quad 1 \leq \mu \leq n. \quad (1.22)$$

Next we define

$$v_k(x) = \sum_{\nu=1}^d \mathcal{L}_{\bar{\lambda}^\nu} M^{-1} R_\nu^k(x), \quad (1.23)$$

and

$$\begin{aligned}
X^{\mu,k+1}(x) &= (1 + \partial_x v_k)^{-1} X^{\mu,k}(x + v_k(x)) \\
&= (1 + \partial_x v_k)^{-1} (1 + \partial_x v_{k-1})^{-1} \dots (1 + \partial_x v_0)^{-1} \\
&\quad \times X^\mu((1 + v_k) \circ (1 + v_{k-1}) \circ \dots \circ (1 + v_0)(x)) \\
&= \Lambda^\mu x + R_\mu^{k+1}(x).
\end{aligned} \quad (1.24)$$

As before we get

$$\begin{aligned}
&(1 + \partial_x v_k) R_\mu^{k+1}(x) = -\mathcal{L}_{\lambda^\mu} v_k + R_\mu^k(x + v_k(x)) \\
&= -R_\mu^k(x) + \sum_{\nu=1}^d \mathcal{L}_{\bar{\lambda}^\nu} M^{-1} (\mathcal{L}_{\lambda^\nu} R_\mu^k - \mathcal{L}_{\lambda^\mu} R_\nu^k) + R_\mu^k(x + v_k(x)) \\
&= v_k(x) \int_0^1 R_\mu^{k'}(x + tv_k(x)) dt + \sum_{\nu=1}^d \mathcal{L}_{\bar{\lambda}^\nu} M^{-1} (R_\mu^k \partial_x R_\nu^k - R_\nu^k \partial_x R_\mu^k). \quad (1.25)
\end{aligned}$$

Hence, there exist $c > 0$ and $c_1 > 0$ such that

$$\begin{aligned}
|R_\mu^{k+1}|_T &\leq c \left(|(1 + \partial_x v_k)^{-1}|_T |v_k|_T \frac{c_1}{T - T'} |DR_\mu^k|_{T'} + \right. \\
&\quad \left. + |(1 + \partial_x v_k)^{-1}|_T \exp(2 \exp(c(T - T')^{-1/(1+\tau)})) |R^k|_{T'} |DR_\mu^k|_{T'} \right). \quad (1.26)
\end{aligned}$$

By using the estimate for composition of maps we see that

$$(1 + v_0) \circ (1 + v_1) \circ \dots \circ (1 + v_k)(x) \longrightarrow 1 + u \quad \text{in } H(T) \text{ as } k \rightarrow \infty \quad (1.27)$$

and $|R^k|_T \rightarrow 0$, as $k \rightarrow \infty$. It follows that

$$(1 + \partial u_k)^{-1} X^\mu(x + u_k(x)) \longrightarrow (1 + \partial u)^{-1} X^\mu(x + u(x)) = \Lambda^\mu x. \quad (1.28)$$

1.3 Normal forms in the case of Jordan blocks

Now we shall remove the restriction (1.2). Namely,

$$X^\mu(x) = {}^t(X_1^\mu(x), \dots, X_n^\mu(x)) = J^\mu x + R^\mu(x), \quad 1 \leq \mu \leq d,$$

where the matrices J^μ are not necessarily semi-simple and we use the same notations as before. We define the homological operator $\mathcal{L}_{\lambda^\mu}$ by

$$\mathcal{L}_{\lambda^\mu} u = \partial u J^\mu x - J^\mu u, \quad (\mu = 1, \dots, d), \quad u \in (H(T))^n. \quad (1.29)$$

The commutativity of $X^\mu(x)$ imply that the matrices J^μ commute each other, namely $[J^\mu, J^\nu] = 0$ for every μ and ν . This determines J^μ up to their Jordan blocks if we fix one Jordan normal form of some J^μ . In the following, we may assume that all J^μ are diagonalized up to their Jordan blocks.

We assume the following simultaneous Poincaré condition; there exists $c > 0$ such that

$$\max_{1 \leq \mu \leq d} |\lambda^\mu \alpha - \lambda_k^\mu| \geq c|\alpha| \quad \forall \alpha \in \mathbf{Z}_+^n, 1 \leq k \leq n. \quad (1.30)$$

We note that this condition is stronger than the usual Poincaré condition if $d = 1$ because we have a nonresonance condition. Then we have the following

Theorem 1.4 *Let $X^1(x), \dots, X^d(x)$ be pairwise commuting holomorphic vector fields as above. If $\lambda^1, \dots, \lambda^d$ verify (1.30) we can find a neighborhood Ω of the origin and a holomorphic change of the variables $x = y + u(y)$, $y \in \Omega$ which transforms simultaneously $X^1(x), \dots, X^d(x)$ into their linear parts $J^1 y \partial_y, \dots, J^d y \partial_y$, respectively. Moreover, u is a solution of (1.7).*

Remark. The novelty of the theorem lies in the case $d \geq 2$ under natural extension of a usual Poincaré condition for a single equation.

In order to prove Theorem 2.4 we define the operator M by (1.10). Then we have

Lemma 1.5 *Suppose that (1.30) is satisfied. Then the followings holds;*

i) *There exists a scale change of variables $x_j = \rho_j y_j$ ($\rho_j > 0, j = 1, \dots, n$) and $T_0 > 0$ such that in the new coordinate y , the inverse $M^{-1} : (H(T))^n \rightarrow (H(T))^n$ exists as a continuous linear operator for any $0 < T < T_0$.*

ii) *We have*

$$[M^{-1}, \mathcal{L}_{\lambda^\mu}] = [M^{-1}, \mathcal{L}_{\bar{\lambda}^\mu}] = 0 \quad \text{for every } 1 \leq \mu \leq d. \quad (1.31)$$

Epecially, the operator $P(D)$, (1.15) satisfies (1.16).

Proof. We write $\mathcal{L}_{\lambda^\mu} = \mathcal{L}'_{\lambda^\mu} + \mathcal{L}''_{\lambda^\mu}$, where $\mathcal{L}'_{\lambda^\mu}$ and $\mathcal{L}''_{\lambda^\mu}$ correspond to semi-simple and nilpotent part of J^μ , respectively. Because the change of variables in the lemma transforms $x_{\nu+\ell} \partial_{x_\nu}$ into $\rho_{\nu+\ell} \rho_\nu^{-1} y_{\nu+\ell} \partial_{y_\nu}$, it follows that $(\sum_{\mu=1}^d \mathcal{L}'_{\bar{\lambda}^\mu} \mathcal{L}'_{\lambda^\mu})^{-1} \mathcal{L}''_{\lambda^\mu}$ can be made arbitrarily small by an appropriate choice of ρ_j . It follows that M has the

representation $M = \sum_{\mu=1}^d \mathcal{L}'_{\lambda^\mu} \mathcal{L}'_{\lambda^\mu} + \varepsilon$, where ε has small norm while the first term is invertible by (1.30). This proves i).

In order to prove ii) it is sufficient to show that $[M, \mathcal{L}_{\lambda^\mu}] = 0$ for every $1 \leq \mu \leq d$. In view of the definition of M we shall show the commutativity of $\mathcal{L}_{\lambda^\mu}$ and $\mathcal{L}_{\lambda^\nu}$. Noting that $[J^\mu, J^\nu] = 0$ and the symmetry of $\partial^2 u$ we have, for $u \in (H(T))^n$

$$\begin{aligned} [\mathcal{L}_{\lambda^\mu}, \mathcal{L}_{\lambda^\nu}]u &= \partial_x(\partial_x u J^\nu x - J^\nu u) J^\mu x - J^\mu(\partial_x u J^\nu x - J^\nu u) \\ - \partial_x(\partial_x u J^\mu x - J^\mu u) J^\nu x &+ J^\nu(\partial_x u J^\mu x - J^\mu u) = {}^t J^\mu x \partial^2 u J^\nu x - {}^t J^\nu x \partial^2 u J^\mu x = 0. \end{aligned}$$

This ends the proof.

The proof of Theorem 2.4 can be proved by the same argument as in Theorem 2.1.

2 Normal forms of commuting holomorphic diffeomorphisms

We consider d pairwise commuting local biholomorphic maps $\Phi_\mu : \mathbb{C}^n \rightarrow \mathbb{C}^n$, $\mu = 1, \dots, d$ in a neighbourhood Ω of the fixed point 0. Hence we can write

$$\Phi_\mu(x) = \Lambda_\mu x + \varphi_\mu(x), \quad \mu = 1, \dots, d \quad (2.1)$$

where $\Lambda_\mu \in GL(n : \mathbb{C})$ and $\varphi_\mu \in (\mathbb{C}_1\{x\})^n$, i.e.

$$\varphi_\mu(x) = O(|x|^2), \quad |x| \rightarrow 0, \quad \mu = 1, \dots, d \quad (2.2)$$

The commuting relation $\Phi_\mu \circ \Phi_\nu = \Phi_\nu \circ \Phi_\mu$ implies $\Lambda_\mu \circ \Lambda_\nu = \Lambda_\nu \circ \Lambda_\mu$ for all $\mu, \nu = 1, \dots, d$. Without loss of generality (after a linear change of the variables) we may assume that all matrices are in the Jordan normal forms with identical block structures. We set $\lambda_\mu = (\lambda_{\mu 1}, \dots, \lambda_{\mu n})$ to be the vector consisting of the eigenvalues of the matrix Λ_μ , $\mu = 1, \dots, d$.

Our result for (simultaneous) analytic equivalence of the maps to their linear parts will be proved under the additional requirement that all matrices are semisimple. Therefore we can set

$$\Lambda_\mu = \text{diag}\{\lambda_{\mu 1}, \dots, \lambda_{\mu n}\}, \quad \lambda_{\mu j} \in \mathbb{C}, \quad j = 1, \dots, n, \quad \mu = 1, \dots, d. \quad (2.3)$$

We suppose that the vectors λ_μ , $\mu = 1, \dots, d$ are nonresonant, namely

$$\lambda_\mu^\alpha = \lambda_{\mu 1}^{\alpha_1} \dots \lambda_{\mu n}^{\alpha_n} \neq \lambda_{\mu j}, \quad \alpha \in \mathbb{Z}_+^n, |\alpha| \geq 2, \quad j = 1, \dots, n, \quad \mu = 1, \dots, d. \quad (2.4)$$

In fact, we will impose a simultaneous Brjuno type condition: there exist two positive constants c_0 and τ such that

$$\min_{1 \leq j \leq n} \max_{1 \leq \mu \leq d} |\lambda_\mu^\alpha - \lambda_{\mu j}| \geq c_0 \exp \left(- \frac{|\alpha|}{(\ln |\alpha|)^{1+\tau}} \right), \quad \alpha \in \mathbb{Z}_+^n, |\alpha| \geq 2. \quad (2.5)$$

Furthermore, in the case of more than one diffeomorphism (i.e. $d \geq 2$) we impose the following restriction: there exist a constant $C_0 > 0$ such that

$$\frac{\sum_{\mu=1}^d |\lambda^{\mu\gamma}|}{1 + \sum_{\mu=1}^d |\lambda^{\mu\alpha}|} \leq C_0, \quad \forall \alpha, \gamma \in \mathbf{Z}_+^n, |\gamma| \geq 2, \gamma \leq \alpha. \quad (2.6)$$

We note that in the case of a single map ($d = 1$) the vector λ_1 belongs to the Poincaré domain if either $\min_{j=1,\dots,d} |\lambda_{1j}| > 1$ or $\max_{j=1,\dots,d} |\lambda_{1j}| < 1$ (cf. [1, p. 311]). In that case the condition (2.6) is always fulfilled. One checks easily that it is also true when the space dimension is one ($n = 1$) while d is arbitrary positive integer. Then we have

Theorem 2.1 *Let $\Phi_1, \dots, \Phi_d$ be pairwise commuting local biholomorphic maps preserving the origin and satisfying the conditions (2.5) and (2.6). Then we can find a neighborhood B of the origin and a holomorphic change of the variables $y \rightarrow x = u(y)$ which transforms simultaneously $\Phi_1, \dots, \Phi_d$ into their linear parts $\Lambda_1 x, \dots, \Lambda_d x$, respectively.*

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