

NONSTANDARD UNIVERSE

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ABSTRACT. The nonstandard universes are frameworks of nonstandard analysis. We find sheaf representation for a nonstandard universe. in Theorem 3.7.

1. NONSTANDARD UNIVERSE

Definitions 1.1 (superstructure, base set). Given a set X , we define the iterated power set $V_n(X)$ over X recursively by

$$V_0(X) = X, \quad \text{and} \quad V_{n+1}(X) = V_n(X) \cup \mathcal{P}(V_n(X)).$$

The *superstructure* $V(X)$ is the union $\bigcup_{n < \omega} V_n(X)$. The set X is said to be a *base set* if $\emptyset \notin X$ and each element of X is disjoint from $V(X)$.

Definition 1.2 (nonstandard universe). A *nonstandard universe* is a triple $\langle V(X), V(Y), \star \rangle$ such that:

- (1) X and Y are infinite base sets.
- (2) (Transfer Principle) The symbol $\star$ is a map from $V(X)$ into $V(Y)$ such that

$$V(X) \models \varphi(a_1, \dots, a_n) \quad \text{if and only if} \quad V(Y) \models \varphi(\star a_1, \dots, \star a_n)$$

holds for any bounded formula $\varphi(x_1, \dots, x_n)$ and $a_1, \dots, a_n \in V(X)$.

- (3) $\star X = Y$.
- (4) For every infinite subset A of X , $\{\star a \mid a \in A\}$ is a proper subset of $\star A$.

Definitions 1.3 (standard, internal). For $a \in V(\star X)$, we call a *standard* if there is an $x \in V(X)$ such that $a = \star x$.

For $a \in V(\star X)$, we call a *internal* if there is an $x \in V(X)$ such that $a \in \star x$. We denote by $\star V(X)$ the set of all internal elements in $V(\star X)$.

From now on, we denote a nonstandard universe by single $\star V(X)$.

Definitions 1.4 (norm, radius). The *norm (of standardness)* of an internal element a is a cardinal defined by

$$\text{nos}(a) = \min \{|x| \mid a \in \star x\}.$$

The *radius* of $\star V(X)$ is a cardinal defined by

$$\text{rad}(\star V(X)) = \min \{\kappa \mid \forall y \in \star V(X) \text{ nos}(y) < \kappa\}.$$

Definition 1.5 (covering number). Let a be an internal element. The *local ultra-power* at a is defined by

$$V(X)[a] = \{(*w)(a) \mid w \in V(X) \text{ and } a \in *(\text{dom}(w))\}.$$

For a subset $E \subseteq *V(X)$, we denote

$$V(X)[E] = \bigcup \{V(X)[s] \mid s \text{ is a finite subset of } E\}.$$

The *covering number* of $*V(X)$ is defined by

$$\text{cov}(*V(X)) = \min \{|E| \mid E \subseteq *V(X) \text{ and } V(X)[E] = *V(X)\}.$$

2. LOCALLY ATOMIC COMPLETE ALGEBRA

Definition 2.1 (regular complete subalgebra). Let $\langle \mathcal{B}, \wedge, \vee, \neg, \mathbf{0}_{\mathcal{B}}, \mathbf{1}_{\mathcal{B}} \rangle$ be a Boolean algebra. A subset $C \subseteq \mathcal{B}$ is said to be a *regular complete subalgebra* of $\mathcal{B}$ if C is a complete subalgebra of $\mathcal{B}$ and the inclusion map is also complete.

Notation. Let $\mathcal{B}$ be a Boolean algebra. For a subset $\mathcal{S} \subseteq \mathcal{P}(\mathcal{B})$, we denote

$$\mathcal{S}^{\diamond} = \{C \in \mathcal{S} \mid C \text{ is a regular complete subalgebra of } \mathcal{B}\}.$$

Definition 2.2 (LCA). A *locally complete algebra (LCA)* is a set Λ of subsets of a Boolean algebra $\mathcal{B}$ satisfying the conditions below.

- (1) $\bigcup \Lambda = \mathcal{B}$.
- (2) If $S_1, S_2 \in \Lambda$ then $S_1 \cup S_2 \in \Lambda$.
- (3) If $S \in \Lambda$ and $T \subseteq S$ then $T \in \Lambda$.
- (4) For every $S \in \Lambda$, there is a $C \in \Lambda^{\diamond}$ containing S .

For an LCA Λ , we denote by $\mathcal{B}(\Lambda)$ the Boolean algebra $\bigcup \Lambda$. We call the Boolean algebra $\mathcal{B}(\Lambda)$ the *base Boolean algebra of Λ* .

Definition 2.3 (LACA). An LCA Λ is a *locally atomic complete algebra (LACA)* if every $C \in \Lambda^{\diamond}$ is atomic. We denote the set of atoms of $C \in \Lambda^{\diamond}$ by $\text{Atom}(C)$.

Definition 2.4 (homomorphism). We introduce notation $R^{\ulcorner} \mathcal{S} = \{R^{\ulcorner} S \mid S \in \mathcal{S}\}$. Let Λ and Ξ be LCAs. A Boolean homomorphism $f: \mathcal{B}(\Lambda) \rightarrow \mathcal{B}(\Xi)$ is a *pseudo-homomorphism* of LCAs if $f^{\ulcorner} \Lambda \subseteq \Xi$. We denote a pseudo-homomorphism by $f: \Lambda \rightarrow \Xi$. A pseudo-homomorphism $h: \Lambda \rightarrow \Xi$ of LCAs is a *(complete) homomorphism* if $\bigvee h^{\ulcorner} S = h(\bigvee S)$ for all $S \in \Lambda$. An *embedding* or *monomorphism* $j: \Lambda \rightarrow \Xi$ is an injective homomorphism.

Definition 2.5 (subLCA). A *subLCA* of an LCA Λ is a nonempty subset of Λ which is itself an LCA and the inclusion map is an embedding.

Definition 2.6 (generator). Let Λ be an LCA. A subset $\mathcal{G} \subseteq \Lambda^{\diamond}$ is a *generator* of Λ or $\mathcal{G}$ *generates* Λ if Λ is the only subLCA of Λ containing $\mathcal{G}$.

Definitions 2.7 (radius, covering number, diameter). The *radius* of an LACA Λ is a cardinal defined by

$$\text{rad}(\Lambda) = \min \{ \kappa \mid \forall C \in \Lambda^\diamond \mid |\text{Atom}(C)| < \kappa \}$$

The *covering number* of an LCA Λ is a cardinal defined by

$$\text{cov}(\Lambda) = \min \{ |\mathcal{G}| \mid \mathcal{G} \text{ is a generator of } \Lambda \}.$$

The *diameter* of an LACA Λ is a cardinal defined by

$$\text{diam}(\Lambda) = \min \left\{ \sum_{C \in \mathcal{G}} |\text{Atom}(C)| \mid \mathcal{G} \text{ is a generator of } \Lambda \right\}.$$

Definition 2.8 (direct product). Let I be an index set. The *direct product* $\Lambda^{[I]}$ of the LCA Λ is defined by:

$$\Lambda^{[I]} = \left\{ S \subseteq \mathcal{B}(\Lambda)^I \mid \bigcup_{g \in S} \text{rng } g \in \Lambda \right\}$$

with the pointwise Boolean operations on $\mathcal{B}(\Lambda^{[I]}) = \bigcup \Lambda^{[I]} \subseteq \mathcal{B}(\Lambda)^I$. Then $\Lambda^{[I]}$ is an LCA. The LCA Λ is embedded into $\Lambda^{[I]}$ by the *canonical embedding* $b \mapsto I \times \{b\}$.

Definitions 2.9 (embedding system, direct limit). The *embedding system* of LCAs is a family of embeddings

$$\mathcal{E} = \{ j_d^{d'} : \Lambda_d \rightarrow \Lambda_{d'} \}_{d \leq d', d, d' \in D}$$

satisfying $j_{d'}^{d''} \circ j_d^{d'} = j_d^{d''}$ for all $d \leq d' \leq d''$, where D is an upper direct set. The *direct limit* of $\mathcal{E}$ is $\bigcup \{ j_d : \Lambda_d \mid d \in D \}$ where $\{ j_d : \mathcal{B}(\Lambda_d) \rightarrow \mathcal{B} \}_{d \in D}$ is the direct limit of $\{ j_d^{d'} : \mathcal{B}(\Lambda_d) \rightarrow \mathcal{B}(\Lambda_{d'}) \}_{d \leq d', d, d' \in D}$ as Boolean algebras.

Definition 2.10 (ultrafilter). Let Λ be an LCA. A subset $\mathcal{U}$ of $\mathcal{B}(\Lambda)$ is an *ultrafilter* of an LCA Λ if it is an ultrafilter of the base Boolean algebra $\mathcal{B}(\Lambda)$.

3. ULTRALIMIT

Definition 3.1 (LACA-valued model). Let Λ be an LACA and let M be a model for a language $\mathcal{L}$. The $\mathcal{B}(\Lambda)$ -valued universe of M is defined by

$$M^{\langle \Lambda \rangle} = \left\{ u : M \rightarrow \mathcal{B}(\Lambda) \mid \begin{array}{l} u(x) \wedge u(y) = \mathbf{0} \text{ for } x \neq y, \\ \text{rng } u \in \Lambda, \bigvee \text{rng } u = \mathbf{1} \end{array} \right\}.$$

For $u \in M^{\langle \Lambda \rangle}$, the *support* of u is a subset of M defined by

$$\text{supp } u = \{ x \in M \mid u(x) \neq \mathbf{0} \}.$$

To each function F of $\mathcal{L}(M)$ and each $u_1, \dots, u_n \in M^{\langle \Lambda \rangle}$, we assign a $\check{F}(u_1, \dots, u_n) \in M^{\langle \Lambda \rangle}$ by:

$$\check{F}(u_1, \dots, u_n)(y) = \bigvee \left\{ \bigwedge_{i=1}^n u_i(x_i) \mid M \models y = F(x_1, \dots, x_n) \right\} \quad \text{for } y \in M.$$

We regard a constant of $\mathcal{L}(M)$ as a function without any variables. Note that $\bigwedge_{i=1}^n u_i(x_i) = 1$ if $n = 0$. To each sentence φ of $\mathcal{L}(M^{\langle\Lambda\rangle})$ we assign a truth value $\llbracket\varphi\rrbracket \in \overline{\mathcal{B}(\Lambda)}$ by following recursive rules:

$$\begin{aligned} \llbracket u = v \rrbracket &= \bigvee \{u(x) \wedge v(x) \mid x \in M\}, \\ \llbracket R(u_1, \dots, u_m) \rrbracket &= \bigvee \left\{ \bigwedge_{i=1}^m u_i(x_i) \mid \mathfrak{M} \models R(x_1, \dots, x_m) \right\}, \\ \llbracket \neg\varphi \rrbracket &= \neg\llbracket\varphi\rrbracket, \\ \llbracket \varphi_1 \vee \varphi_2 \rrbracket &= \llbracket\varphi_1\rrbracket \vee \llbracket\varphi_2\rrbracket, \\ \llbracket \exists x \varphi(x) \rrbracket &= \bigvee \{ \llbracket\varphi(u)\rrbracket \mid u \in M^{\langle\Lambda\rangle} \}, \end{aligned}$$

where R is any predicate in $\mathcal{L}$.

Definition 3.2 (LACA-valued superstructure). Let Λ be an LACA. The Λ -valued superstructure of $V(X)$ is defined by

$$\widehat{V}(X)^{\langle\Lambda\rangle} = \{u \in V(X)^{\langle\Lambda\rangle} \mid \text{supp } u \in V(X)\}.$$

While the truth values range over $\overline{\mathcal{B}(\Lambda)}$ on this definition, we shall see $\llbracket\varphi\rrbracket_\Lambda \in \mathcal{B}(\Lambda)$.

Theorem 3.1. Let $\varphi(x_1, \dots, x_r)$ be a formula of $\mathcal{L}$ with only $x_1, \dots, x_r$ free. For $u_1, \dots, u_r \in M^{\langle\Lambda\rangle}$,

$$(*) \quad \llbracket\varphi(u_1, \dots, u_r)\rrbracket_\Lambda = \bigvee \left\{ \bigwedge_{i=1}^r u_i(x_i) \mid \mathfrak{M} \models \varphi(x_1, \dots, x_r) \right\}.$$

Proof. For φ either “ $x_1 = x_2$ ” or R , $(*)$ holds by definition. If $(*)$ holds for an atomic formula $\varphi(x)$ then, by simple calculus of Boolean algebra, $(*)$ holds for $\varphi(\mathbf{F}(x_1, \dots, x_n))$. Thus, by induction, $(*)$ holds for φ atomic. Suppose $(*)$ holds for φ , φ_1 and φ_2 . Since there is an atomic $C \in \Lambda^\diamond$ containing all the ranges of $u_1, \dots, u_r$ and every range of u_i is a partition of unity except for $\mathbf{0}$,

$$\llbracket \neg\varphi \rrbracket_\Lambda = \bigvee \left\{ \bigwedge_{i=1}^r u_i(x_i) \mid \mathfrak{M} \models \neg\varphi(x_1, \dots, x_r) \right\}.$$

It is easy to see:

$$\llbracket \varphi_1 \vee \varphi_2 \rrbracket_\Lambda = \bigvee \left\{ \bigwedge_{i=1}^r u_i(x_i) \mid \mathfrak{M} \models \varphi_1(x_1, \dots, x_r) \vee \varphi_2(x_1, \dots, x_r) \right\}.$$

Since $\llbracket\varphi(u)\rrbracket_\Lambda = \bigvee_{x \in M} (u(x) \wedge \llbracket\varphi(\check{x})\rrbracket_\Lambda)$, we have $\llbracket\exists x \varphi(x)\rrbracket_\Lambda = \bigvee_{x \in M} \llbracket\varphi(\check{x})\rrbracket_\Lambda$. Therefore $(*)$ holds for $\exists x \varphi(x)$. $\square$

Similarly, we shall obtain the superstructure version.

Corollary 3.2. Let $\varphi(x_1, \dots, x_r)$ be a formula of $\mathcal{L}_\infty$ with only $x_1, \dots, x_r$ free. For $u_1, \dots, u_r \in \widehat{V}(X)^{\langle\Lambda\rangle}$,

$$\llbracket\varphi(u_1, \dots, u_r)\rrbracket_\Lambda = \bigvee \left\{ \bigwedge_{i=1}^r u_i(x_i) \mid V(X) \models \varphi(x_1, \dots, x_r) \right\}.$$

By the theorem and the corollary above, we have a fundamental property $\llbracket u = v \rrbracket \wedge \llbracket \varphi(u) \rrbracket \leq \llbracket \varphi(v) \rrbracket$. We have just introduced $\mathcal{B}(\Lambda)$ -valued model $\mathfrak{M}^{\langle \Lambda \rangle} = \langle M^{\langle \Lambda \rangle}, \check{R}, \check{F}, \check{c} \rangle$ and $\mathcal{B}(\Lambda)$ -valued superstructure $\widehat{V}(X)^{\langle \Lambda \rangle}$. We say that a sentence φ of $\mathcal{L}(M^{\langle \Lambda \rangle})$ holds in $\mathfrak{M}^{\langle \Lambda \rangle}$ if $\llbracket \varphi \rrbracket_{\Lambda} = 1$ and that a sentence ψ of $\mathcal{L}_{\in}(\widehat{V}(X)^{\langle \Lambda \rangle})$ holds in $\widehat{V}(X)^{\langle \Lambda \rangle}$ if $\llbracket \psi \rrbracket_{\Lambda} = 1$. Theorem 3.1 and Corollary 3.2 follow that we consider the values $u(x)$ only for $x \in \text{supp } u$. For $E \subseteq M$, we may regard $E^{\langle \Lambda \rangle}$ as a subset of $M^{\langle \Lambda \rangle}$ by extending the domain of $u \in E^{\langle \Lambda \rangle}$ to M . This means that we define for $u \in E^{\langle \Lambda \rangle}$

$$u(x) = \mathbf{0} \quad \text{if } x \notin E.$$

In the superstructure version, if E is a set relative to $V(X)$ then we may assume

$$E^{\langle \Lambda \rangle} = \{u \in \widehat{V}(X)^{\langle \Lambda \rangle} \mid u \in \check{E} \text{ holds in } \widehat{V}(X)^{\langle \Lambda \rangle}\}.$$

Theorem 3.3 (Maximum principle). *Let $\varphi(x)$ be a formula of $\mathcal{L}(M^{\langle \Lambda \rangle})$ with only x free. Then there is $u \in M^{\langle \Lambda \rangle}$ such that $\llbracket \varphi(u) \rrbracket_{\Lambda} = \llbracket \exists x \varphi(x) \rrbracket_{\Lambda}$.*

Proof. Let $\{a_{\zeta}\}_{\zeta < \alpha}$ be a well-ordering for M . By theorem 3.1, there is $C \in \Lambda^{\diamond}$ containing $\{\llbracket \varphi(\check{x}) \rrbracket \mid x \in M\}$. Putting $b_{\zeta} = \llbracket \varphi(\check{a}_{\zeta}) \rrbracket \wedge \neg \bigvee_{\xi < \zeta} \llbracket \varphi(\check{a}_{\xi}) \rrbracket$, we have $\{b_{\zeta}\}_{\zeta < \alpha} \subseteq C$. Since $\{b_{\zeta}\}_{\zeta < \alpha}$ is a pairwise disjoint family, we can pick $u \in M^{(C)}$ with $u(a_{\zeta}) \geq b_{\zeta}$. Then $\llbracket \varphi(u) \rrbracket \geq u(a_{\zeta}) \wedge \llbracket \varphi(\check{a}_{\zeta}) \rrbracket \geq b_{\zeta}$ for any $\zeta < \alpha$. Since $\llbracket \exists x \varphi(x) \rrbracket = \bigvee_{\zeta < \alpha} \varphi(\check{a}_{\zeta}) = \bigvee_{\zeta < \alpha} b_{\zeta}$, we have $\llbracket \varphi(u) \rrbracket \geq \llbracket \exists x \varphi(x) \rrbracket$. $\square$

Corollary 3.4. *Let $\varphi(x)$ be a formula of $\mathcal{L}_{\in}(\widehat{V}(X)^{\langle \Lambda \rangle})$ with only x free and let v be an element of $\widehat{V}(X)^{\langle \Lambda \rangle}$. Then there is $u \in \widehat{V}(X)^{\langle \Lambda \rangle}$ such that $\llbracket u \in v \wedge \varphi(u) \rrbracket_{\Lambda} = \llbracket \exists x \in v \varphi(x) \rrbracket_{\Lambda}$.*

Proof. Since there is n such that $\text{supp } v \subseteq V_{n+1}(X)$,

$$\llbracket \check{x} \in v \rrbracket = \bigvee \{v(y) \mid x \in y \in \text{supp } v\} = \mathbf{0} \quad \text{for } x \notin V_n(X).$$

Therefore we can choose u whose support is a subset of $V_n(X)$. $\square$

Definition 3.3 (ultralimit). We denote by $u/\mathcal{U}$ the equivalence class of $u \in M^{\langle \Lambda \rangle}$ by the equivalence relation

$$x \sim_{\mathcal{U}} y \quad \equiv \quad \llbracket x = y \rrbracket_{\Lambda} \in \mathcal{U}.$$

The ultralimit $\mathfrak{M}^{\langle \Lambda \rangle}/\mathcal{U}$ of $\mathfrak{M}$ modulo $\mathcal{U}$ of Λ is defined by:

$$\begin{aligned} M^{\langle \Lambda \rangle}/\mathcal{U} &= \{u/\mathcal{U} \mid u \in M^{\langle \Lambda \rangle}\}. \\ \check{F}/\mathcal{U}(u_1/\mathcal{U}, \dots, u_n/\mathcal{U}) &= (\check{F}(u_1, \dots, u_n))/\mathcal{U}. \\ \mathfrak{M}^{\langle \Lambda \rangle}/\mathcal{U} \models \mathbf{R}(u_1/\mathcal{U}, \dots, u_m/\mathcal{U}) &\text{ iff } \llbracket \mathbf{R}(u_1, \dots, u_m) \rrbracket \in \mathcal{U}. \end{aligned}$$

Definition 3.4 (bounded ultralimit). We denote by $u/\mathcal{U}$ the equivalence class of $u \in \widehat{V}(X)^{\langle \Lambda \rangle}$ by the equivalence relation

$$x \sim_{\mathcal{U}} y \quad \equiv \quad \llbracket x = y \rrbracket_{\Lambda} \in \mathcal{U}.$$

The bounded ultralimit $\widehat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U}$ of $V(X)$ modulo $\mathcal{U}$ of Λ is defined by:

$$\begin{aligned}\widehat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U} &= \{u/\mathcal{U} \mid u \in \widehat{V}(X)^{\langle\Lambda\rangle}\}, \\ \widehat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U} \models u/\mathcal{U} \in v/\mathcal{U} &\text{ iff } \llbracket u \in v \rrbracket \in \mathcal{U}.\end{aligned}$$

Theorem 3.5 (Łoś Principle of Ultralimits). *Let $\varphi(x_1, \dots, x_r)$ be a formula of $\mathcal{L}$ with only $x_1, \dots, x_r$ free. For $u_1, \dots, u_r \in M^{\langle\Lambda\rangle}$,*

$$\mathfrak{M}^{\langle\Lambda\rangle}/\mathcal{U} \models \varphi(u_1/\mathcal{U}, \dots, u_r/\mathcal{U}) \text{ iff } \llbracket \varphi(u_1, \dots, u_r) \rrbracket \in \mathcal{U}.$$

Proof. The proof proceeds by induction on the complexity of formulae. The only nontrivial step is the case where φ is of the form $\exists x \psi(x)$. Suppose $\llbracket \exists x \psi(x) \rrbracket \in \mathcal{U}$. By the maximal principle (Theorem 3.3), there is u satisfying $\llbracket \psi(u) \rrbracket = \llbracket \exists x \psi(x) \rrbracket$. Then $\mathfrak{M}^{\langle\Lambda\rangle} \models \psi(u/\mathcal{U})$ by the induction assumption. We have thus $\mathfrak{M}^{\langle\Lambda\rangle} \models \exists x \psi(x)$. Conversely, suppose $\mathfrak{M}^{\langle\Lambda\rangle} \models \exists x \psi(x)$. Then there is some u such that $\mathfrak{M}^{\langle\Lambda\rangle} \models \psi(u/\mathcal{U})$. By the induction assumption, $\llbracket \exists x \psi(x) \rrbracket \geq \llbracket \psi(u/\mathcal{U}) \rrbracket \in \mathcal{U}$. $\square$

Corollary 3.6 (Łoś-Mostowski Principle of Bounded Ultralimits).

Let $\varphi(x_1, \dots, x_r)$ be a Δ_0 -formula of $\mathcal{L}_\in$ with only $x_1, \dots, x_r$ free. For $u_1, \dots, u_r \in \widehat{V}(X)^{\langle\Lambda\rangle}$,

$$\widehat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U} \models \varphi(u_1/\mathcal{U}, \dots, u_r/\mathcal{U}) \text{ iff } \llbracket \varphi(u_1, \dots, u_r) \rrbracket \in \mathcal{U}.$$

Proof. The proof is similar to that of Theorem 3.5. The only different part is the if-part of the case where φ is of the form $\exists x \in y \psi(x)$. Suppose $\llbracket \exists x \in u_k \psi(x) \rrbracket \in \mathcal{U}$. It follows from Corollary 3.4 that there is $u \in \widehat{V}(X)^{\langle\Lambda\rangle}$ satisfying $\llbracket u \in u_k \wedge \psi(u) \rrbracket = \llbracket \exists x \in u_k \psi(x) \rrbracket$. $\square$

A bounded ultralimit is a pre-nonstandard universe: that satisfies (1),(2) and (3) of Definition 1.2 with Mostowski collapsing.

Definition 3.5 (atlas). An *atlas* is a pair $\langle \Lambda, \mathcal{U} \rangle$ of an LACA Λ and an ultrafilter of Λ such that $\text{rad}(\widehat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U}) = \text{rad}(\Lambda)$ and $\text{cov}(\widehat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U}) = \text{cov}(\Lambda)$.

Theorem 3.7 (Sheaf representation Theorem for Nonstandard Universes). *For any nonstandard universe ${}^*V(X)$, there is an atlas $\langle \Lambda, \mathcal{U} \rangle$ such that $\widehat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U}$ isomorphic to ${}^*V(X)$.*

We prove the theorem in the next section.

4. LOCAL ULTRALIMITS

We shall see that a homomorphism of LACAs induces an elementary embedding of ultralimits and a bounded elementary embedding of bounded ultralimits. Let $h: \Lambda \rightarrow \Xi$ be a homomorphism. The induced map $h_*: \mathfrak{M}^{\langle\Lambda\rangle} \rightarrow \mathfrak{M}^{\langle\Xi\rangle}$ is defined by $h_*(u) = h \circ u$. Then we have the lemma below.

Lemma 4.1. *Let $\varphi(x_1, \dots, x_r)$ be a formula of $\mathcal{L}$ with only $x_1, \dots, x_r$ free. For $u_1, \dots, u_r \in M^{\langle\Lambda\rangle}$*

$$\llbracket \varphi(h_*(u_1), \dots, h_*(u_r)) \rrbracket_\Xi = h(\llbracket \varphi(u_1, \dots, u_r) \rrbracket_\Lambda).$$

Proof. There is $C \in \Lambda^\diamond$ containing all the ranges of u_k . Since $h \upharpoonright C$ is complete, we have from Theorem 3.1

$$\bigvee \left\{ \bigwedge_{i=1}^r h(u_i(x_i)) \mid \mathfrak{M} \models \varphi(x_1, \dots, x_r) \right\} = h \left(\bigvee \left\{ \bigwedge_{i=1}^r u_i(x_i) \mid \mathfrak{M} \models \varphi(x_1, \dots, x_r) \right\} \right)$$

$$\llbracket \varphi(h_*(u_1), \dots, h_*(u_r)) \rrbracket_\Xi = h(\llbracket \varphi(u_1, \dots, u_r) \rrbracket_\Lambda).$$

We have thus proved the lemma. $\square$

For $u \in \widehat{V}(X)^{\langle\Lambda\rangle}$, since $\text{supp}(h \circ u) \subseteq \text{supp } u$, we can define the *induced map* $h_*: \widehat{V}(X)^{\langle\Lambda\rangle} \rightarrow \widehat{V}(X)^{\langle\Lambda\rangle}$ similarly.

Corollary 4.2. *Let $\varphi(x_1, \dots, x_r)$ be a formula of $\mathcal{L}_\in$ with only $x_1, \dots, x_r$ free. For $u_1, \dots, u_r \in \widehat{V}(X)^{\langle\Lambda\rangle}$*

$$\llbracket \varphi(h_*(u_1), \dots, h_*(u_r)) \rrbracket_\Xi = h(\llbracket \varphi(u_1, \dots, u_r) \rrbracket_\Lambda).$$

Proof. Using Corollary 3.2, we see the proof is similar to that of Lemma 4.1. $\square$

Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters of Λ and Ξ , respectively. Suppose $h^{-1*}\mathcal{V} = \mathcal{U}$. Then we have from Lemma 4.1 or from Corollary 4.2

$$\llbracket u = u' \rrbracket_\Lambda \in \mathcal{U} \quad \text{iff} \quad \llbracket h_*(u) = h_*(u') \rrbracket_\Xi \in \mathcal{V}.$$

Therefore we can define the injection $h_*: M^{\langle\Lambda\rangle}/\mathcal{U} \rightarrow M^{\langle\Xi\rangle}/\mathcal{V}$, denoted by same h_* , by $h_*(u/\mathcal{U}) = h_*(u)/\mathcal{V}$. Since $\text{supp } h_*(u) \subseteq \text{supp } u$, we can define the injection $h_*: \widehat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U} \rightarrow \widehat{V}(X)^{\langle\Xi\rangle}/\mathcal{V}$ similarly.

Lemma 4.3. *The injection h_* is an elementary embedding of $M^{\langle\Lambda\rangle}/\mathcal{U}$ into $M^{\langle\Xi\rangle}/\mathcal{V}$.*

Proof. Let $\varphi(x_1, \dots, x_r)$ be a formula of $\mathcal{L}$ with only $x_1, \dots, x_r$ free. From Theorem 3.5, we have for $u_1, \dots, u_r \in M^{\langle\Lambda\rangle}$

$$\begin{aligned} h(\llbracket \varphi(u_1, \dots, u_r) \rrbracket_\Lambda) \in \mathcal{V} & \quad \text{iff} \quad \llbracket \varphi(u_1, \dots, u_r) \rrbracket_\Lambda \in h^{-1*}\mathcal{V} \\ \llbracket \varphi(h_*(u_1), \dots, h_*(u_r)) \rrbracket_\Xi \in \mathcal{V} & \quad \text{iff} \quad \llbracket \varphi(u_1, \dots, u_r) \rrbracket_\Lambda \in \mathcal{U} \\ \mathfrak{M}^{\langle\Xi\rangle}/\mathcal{V} \models \varphi(h_*(u_1/\mathcal{U}), \dots, h_*(u_r/\mathcal{U})) & \quad \text{iff} \quad \mathfrak{M}^{\langle\Lambda\rangle}/\mathcal{U} \models \varphi(u_1, \dots, u_r). \end{aligned}$$

$\square$

Corollary 4.4. *The injection h_* is a bounded elementary embedding of $\widehat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U}$ into $\widehat{V}(X)^{\langle\Xi\rangle}/\mathcal{V}$.*

Proof. Using Corollary 3.6, we see the proof is similar to that of Lemma 4.3. $\square$

Let I be a set relative to $V(X)$. We shall find a one-to-one correspondence between $\mathcal{P}(I)^{\langle\Lambda\rangle}$ and $\mathcal{B}(\Lambda^{[I]})$. Note that $\mathcal{P}(I)^{\langle\Lambda\rangle}$ is the set of “the subsets of $\check{I}$ in $\widehat{V}(X)^{\langle\Lambda\rangle}$ ”. For $A \in \mathcal{P}(I)^{\langle\Lambda\rangle}$, there is $C \in \Lambda^\diamond$ such that $\text{rng } A \subseteq C$. Define $g: I \rightarrow \mathcal{B}(\Lambda)$ by $g(i) = \llbracket \check{i} \in A \rrbracket_\Lambda$. Then we have $\text{rng } g \subseteq C$ and $g \in \mathcal{B}(\Lambda^{[I]})$. Conversely, for $g \in \mathcal{B}(\Lambda^{[I]})$, there is $C \in \Lambda^\diamond$ such that $\text{rng } g \subseteq C$. Define $A: \mathcal{P}(I) \rightarrow C$ by

$$A(x) = \bigwedge_{i \in I} \text{sg}_x(i, g(i)), \quad \text{where } \text{sg}_x(i, b) = \begin{cases} b & \text{if } i \in x, \\ \neg b & \text{if } i \in I \setminus x. \end{cases}$$

Since C is completely distributive, we have $A \in \mathcal{P}(I)^{\langle\Lambda\rangle}$. Suppose $g(i) = \llbracket \check{i} \in A \rrbracket_\Lambda$ and $g'(i) = \llbracket \check{i} \in A' \rrbracket_\Lambda$. Then we see $(g \wedge g')(i) = \llbracket \check{i} \in A \cap A' \rrbracket_\Lambda$ and $(\neg g)(i) = \llbracket \check{i} \in \check{I} \setminus A \rrbracket_\Lambda$. In the context above, the relation $g(i) = \llbracket \check{i} \in A \rrbracket_\Lambda$ sets up a one-to-one correspondence between $\mathcal{P}(I)^{\langle\Lambda\rangle}$ and $\mathcal{B}(\Lambda^{[I]})$ as Boolean algebras. From now on, we identify $\mathcal{P}(I)^{\langle\Lambda\rangle}$ with $\mathcal{B}(\Lambda^{[I]})$.

We shall define the special element $\delta \in I^{\langle\Lambda^{[I]}\rangle} \subseteq \widehat{V}(X)^{\langle\Lambda^{[I]}\rangle}$ by

$$\delta(x)(i) = \begin{cases} 1 & \text{if } x = i, \\ 0 & \text{if } x \neq i. \end{cases}$$

We call the δ *diagonal element* of I on Λ . Let $j :: \Lambda \rightarrow \Lambda^{[I]}$ be the canonical embedding. Then j is also a Boolean monomorphism of $\mathcal{B}(\Lambda)$ into $\mathcal{P}(I)^{\langle\Lambda\rangle}$. The diagonal element δ has following properties.

Lemma 4.5. *The following statements hold.*

- (1) $\llbracket j(b) = \check{I} \rrbracket_\Lambda = b$ for every $b \in \mathcal{B}(\Lambda)$.
- (2) $\llbracket \delta \in j_*(g) \rrbracket_{\Lambda^{[I]}} = g$ for every $g \in \mathcal{P}(I)^{\langle\Lambda\rangle}$.

$$\begin{array}{ccccc} & & \mathcal{B}(\Lambda^{[I]}) & \xleftarrow{\llbracket \delta \in \bullet \rrbracket_{\Lambda^{[I]}}} & \mathcal{P}(I)^{\langle\Lambda^{[I]}\rangle} \\ & \nearrow j & \parallel & \nwarrow j_* & \\ \mathcal{B}(\Lambda) & \xleftarrow{\llbracket \bullet = \check{I} \rrbracket_\Lambda} & \mathcal{P}(I)^{\langle\Lambda\rangle} & & \end{array}$$

Proof. Since $\llbracket \check{i} \in j(b) \rrbracket_\Lambda = j(b)(i) = b$ for all $i \in I$, $\llbracket j(b) = \check{I} \rrbracket_\Lambda = \llbracket j(b) \supseteq \check{I} \rrbracket_\Lambda = \bigwedge_{i \in I} \llbracket \check{i} \in j(b) \rrbracket_\Lambda = b$. From the definition of δ , it is clear that $\llbracket \delta = \check{i} \rrbracket_{\Lambda^{[I]}}(i) = \delta(i)(i) = 1$. Then we have $\llbracket \delta \in j_*(g) \rrbracket_{\Lambda^{[I]}}(i) = \llbracket \check{i} \in j_*(g) \rrbracket_{\Lambda^{[I]}}(i) = \llbracket \check{i} \in g \rrbracket_\Lambda = g(i)$. $\square$

Theorem 4.6. *For any $v \in \widehat{V}(X)^{\langle\Lambda^{[I]}\rangle}$, there is a map $w: \check{I} \rightarrow (\text{supp } v)^\vee$ in $\widehat{V}(X)^{\langle\Lambda\rangle}$ such that $v = j_*(w)(\delta)$ holds in $\widehat{V}(X)^{\langle\Lambda\rangle}$.*

Proof. Since $\text{rng } v \in \Lambda^{[I]}$, $\bigcup_{g \in \text{rng } v} \text{rng } g \in \Lambda$. Therefore we can define $w: (\text{supp } v)^I \rightarrow \mathcal{B}(\Lambda)$ by

$$w(s) = \bigwedge_{i \in I} v(s(i))(i).$$

Then we get w as required. First, we show $w \in \widehat{V}(X)^{\langle\Lambda\rangle}$. If $s \neq s'$, then there is $i_0 \in I$ such that $s(i_0) \neq s'(i_0)$. Since $\text{rng } v$ is pairwise disjoint, we have

$$w(s) \wedge w(s') \leq v(s(i_0))(i_0) \wedge v(s'(i_0))(i_0) = 0.$$

There is $C \in \Lambda^\diamond$ such that $\bigcup_{g \in \text{rng } v} \text{rng } g \subseteq C$. Then we have $\text{rng } w \subseteq C$ and then

$$\bigvee_{s \in (\text{supp } v)^I} w(s) = \bigvee_{s \in (\text{supp } v)^I} \bigwedge_{i \in I} v(s(i))(i) = \bigwedge_{i \in I} \bigvee_{y \in \text{supp } v} v(y)(i) = 1.$$

We have thus shown $w \in \widehat{V}(X)^{\langle\Lambda\rangle}$. For each $i \in I$, since $w(s) \leq v(s(i))(i)$ holds for every $s \in (\text{supp } v)^I$, we have

$$\begin{aligned} \llbracket v = j_*(w)(\delta) \rrbracket_{\Lambda^{[I]}}(i) &= \llbracket v = j_*(w)(\check{i}) \rrbracket_{\Lambda^{[I]}}(i) \\ &= \left(\bigvee \{ v(y) \wedge j(w(s)) \wedge \check{i}(x) \mid y = s(x) \} \right)(i) \\ &= \bigvee_{s \in (\text{supp } v)^I} (v(s(i))(i) \wedge w(s)) \\ &= \bigvee_{s \in (\text{supp } v)^I} w(s) = 1. \end{aligned}$$

We have thus proved the theorem. $\square$

Let $\mathcal{U}$ be an ultrafilter of an LACA Λ . A *local ultralimit* $\rho: \widehat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U} \rightarrow {}^*V(X)$ is a bounded elementary embedding satisfying $\rho(\check{x}/\mathcal{U}) = {}^*x$ for every $x \in V(X)$.

Theorem 4.7 (Local Ultralimit Theorem). *Let $\rho: \widehat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U} \rightarrow {}^*V(X)$ be a local ultralimit and let p be an internal element of ${}^*V(X)$. Then there is a local ultralimit $\tau: \widehat{V}(X)^{\langle\Lambda^{[I]}\rangle}/\mathcal{V} \rightarrow {}^*V(X)$ such that the following conditions hold.*

- (i) *The index set I is a set relative to $V(X)$ and $|I| = \text{nos}(p)$.*
- (ii) *Let $j: \Lambda \rightarrow \Lambda^{[I]}$ be the canonical embedding. Then $\mathcal{U} = j^{-1}\mathcal{V}$ and $\rho = \tau \circ j_*$.*
- (iii) *The submodel $\text{rng } \tau$ of ${}^*V(X)$ is the minimal bounded elementary submodel of ${}^*V(X)$ that contains $\{p\} \cup \text{rng } \rho$.*

$$\begin{array}{ccc} & \widehat{V}(X)^{\langle\Lambda^{[I]}\rangle}/\mathcal{V} & \xrightarrow{\tau} {}^*V(X) \\ & \uparrow j_* & \nearrow \rho \\ V(X) & \longrightarrow \widehat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U} & \end{array} \quad , p \in \text{rng } \tau.$$

Proof. Let I be a set relative to $V(X)$ such that $p \in {}^*I$ and $|I| = \text{nos}(p)$. We have identified $\mathcal{B}(\Lambda^{[I]})$ with $\mathcal{P}(I)^{\langle\Lambda\rangle}$. Define $\mathcal{V} \subseteq \mathcal{B}(\Lambda^{[I]})$ by

$$g \in \mathcal{V} \quad \text{iff} \quad p \in \rho(g/\mathcal{U}).$$

Then $\mathcal{V}$ is an ultrafilter of $\Lambda^{[I]}$. Let b be an element of $\mathcal{U}$. From (1) of Lemma 4.5, $\rho(j(b)/\mathcal{U})$ coincides *I . Then we have $j(b) \in \mathcal{V}$ from the definition of $\mathcal{V}$. Since $\mathcal{U}$ and $\mathcal{V}$ are maximal filters, we obtain $j^{-1}\mathcal{V} = \mathcal{U}$. Let $\varphi(x_1, \dots, x_r)$ be a Δ_0 -formula of $\mathcal{L}_\in$ with only $x_1, \dots, x_r$ free. Let $v_1, \dots, v_r$ be elements of $\widehat{V}(X)^{\langle\Lambda^{[I]}\rangle}$. By Theorem 4.6, there are maps $w_1, \dots, w_r$ from $\check{I}$ in $\widehat{V}(X)^{\langle\Lambda\rangle}$ such that $v_k = j_*(w_k)(\delta)$ hold, where δ is the diagonal element of I on Λ . Putting $g_0 = \{i \in \check{I} \mid \varphi(w_1(i), \dots, w_r(i))\}$ in $\widehat{V}(X)^{\langle\Lambda\rangle}$, we have from (2) of Lemma 4.5

$$\begin{aligned} g_0 &= \llbracket \delta \in j_*(g_0) \rrbracket_{\Lambda^{[I]}} \\ &= \llbracket \delta \in j_*(\{i \in \check{I} \mid \varphi(w_1(i), \dots, w_r(i))\}) \rrbracket_{\Lambda^{[I]}} \\ &= \llbracket \delta \in \{i \in \check{I} \mid \varphi(j_*(w_1)(i), \dots, j_*(w_r)(i))\} \rrbracket_{\Lambda^{[I]}} \\ &= \llbracket \varphi(j_*(w_1)(\delta), \dots, j_*(w_r)(\delta)) \rrbracket_{\Lambda^{[I]}} \\ &= \llbracket \varphi(v_1, \dots, v_r) \rrbracket_{\Lambda^{[I]}} \end{aligned}$$

and we have

$$\begin{aligned}\widehat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U} \models g_0/\mathcal{U} &= \{i \in \check{I}/\mathcal{U} \mid \varphi((w_1/\mathcal{U})(i), \dots, (w_r/\mathcal{U})(i))\} \\ \rho(g_0/\mathcal{U}) &= \{i \in {}^*I \mid \varphi(\rho(w_1/\mathcal{U})(i), \dots, \rho(w_r/\mathcal{U})(i))\}.\end{aligned}$$

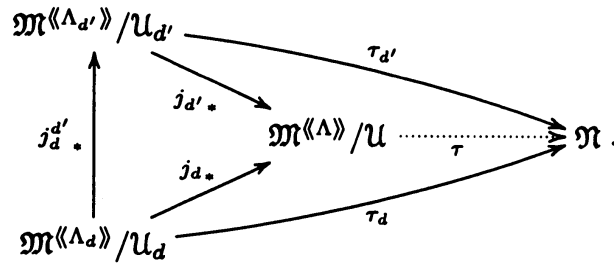
By the definition of $\mathcal{V}$, we obtain

$$\begin{aligned}g_0 \in \mathcal{V} &\text{ iff } p \in \rho(g_0/\mathcal{U}) \\ \llbracket \varphi(v_1, \dots, v_r) \rrbracket_{\Lambda^{[I]}} \in \mathcal{V} &\text{ iff } \varphi(\rho(w_1/\mathcal{U})(p), \dots, \rho(w_r/\mathcal{U})(p)).\end{aligned}$$

The case $\varphi(x_1, x_2) \equiv "x_1 = x_2"$ enables us to define the operation $v/\mathcal{V} \mapsto \rho(w/\mathcal{U})(p)$ where $v = j_*(w)(\delta)$ holds in $\widehat{V}(X)^{\langle\Lambda^{[I]}\rangle}$. Thus, defining $\tau: \widehat{V}(X)^{\langle\Lambda^{[I]}\rangle}/\mathcal{V} \rightarrow {}^*V(X)$ by $\tau(v/\mathcal{V}) = \rho(w/\mathcal{U})(p)$ where $v = j_*(w)(\delta)$ holds in $\widehat{V}(X)^{\langle\Lambda^{[I]}\rangle}$, we get τ as required. In fact, it is clear in the preceding context that τ is an bounded elementary embedding of $\widehat{V}(X)^{\langle\Lambda^{[I]}\rangle}/\mathcal{V}$ into ${}^*V(X)$. Let ι be the identity map on I , then we see $\tau(j_*(\check{I}/\mathcal{U})(\delta/\mathcal{V})) = \rho(\check{I}/\mathcal{U})(p) = \iota(p) = p$. For $u/\mathcal{U} \in \widehat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U}$, let $\tilde{u}$ be the constant map from $\check{I}$ onto $\{u\}$ in $\widehat{V}(X)^{\langle\Lambda\rangle}$, then we have $\tau(j_*(u/\mathcal{U})) = \rho(\tilde{u}/\mathcal{U})(p) = \rho(u/\mathcal{U})$. Suppose a bounded elementary submodel W of ${}^*V(X)$ contains $\{p\} \cup \text{rng } \rho$. From the definition of τ , $\tau(v/\mathcal{V}) = \rho(w/\mathcal{U})(p) \in W$ for some $w/\mathcal{U} \in \widehat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U}$. Therefore $\text{rng } \tau$ is the minimum. We have completed the proof of Theorem 4.7. $\square$

Let $\{j_d^{d'}: \Lambda_d \rightarrow \Lambda_{d'}\}_{d \leq d', d, d' \in D}$ be an embedding system of LACAs with direct limit $\{j_d: \Lambda_d \rightarrow \Lambda\}_{d \in D}$. Let $\mathcal{U}$ be an ultrafilter of Λ , then each $\mathcal{U}_d = j_d^{-1}\mathcal{U}$ is an ultrafilter of Λ_d .

Theorem 4.8 (Elementary Net Theorem of Ultralimits). *Let $\mathfrak{M}$ and $\mathfrak{N}$ be models for $\mathcal{L}$. Suppose there are elementary embeddings $\tau_d: \mathfrak{M}^{\langle\Lambda_d\rangle}/\mathcal{U}_d \rightarrow \mathfrak{N}$ satisfying the condition $\tau_d = \tau_{d'} \circ j_d^{d'}$ for $d \leq d'$. Then there is an elementary embedding $\tau: \mathfrak{M}^{\langle\Lambda\rangle}/\mathcal{U} \rightarrow \mathfrak{N}$ such that $\tau_d = \tau \circ j_d$ for $d \in D$.*

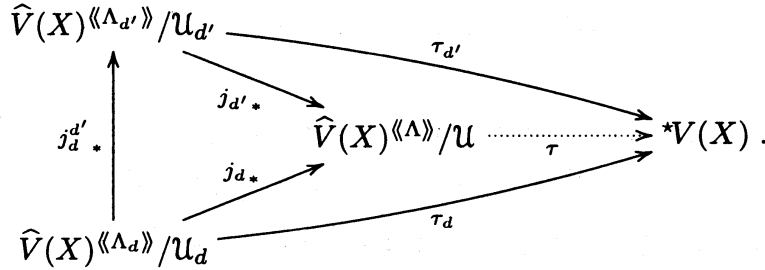


Proof. Let v be an element of $\mathfrak{M}^{\langle\Lambda\rangle}$. Since $\text{rng } v \in \Lambda = \{j_d S \mid d \in D \text{ and } S \in \Lambda_d\}$ from the definition of direct limits, there is $u \in \mathfrak{M}^{\langle\Lambda_d\rangle}$ such that $v = j_{d*}(u)$. Therefore defining $\tau(v/\mathcal{U}) = \tau_d(u/\mathcal{U}_d)$ where $v = j_{d*}(u)$, we get τ as required. Let $\varphi(x_1, \dots, x_r)$ be a formula of $\mathcal{L}$ with only $x_1, \dots, x_r$ free and let $v_1, \dots, v_r$ be elements $M^{\langle\Lambda\rangle}$. Then there are $d \in D$ and $u_1, \dots, u_r \in M^{\langle\Lambda_d\rangle}$ such that $v_k = j_{d*}(u_k)$. We conclude as below.

$$\begin{aligned}\mathfrak{M}^{\langle\Lambda_d\rangle}/\mathcal{U}_d \models \varphi(u_1/\mathcal{U}_d, \dots, u_r/\mathcal{U}_d) &\text{ iff } \mathfrak{N} \models \varphi(\tau_d(u_1/\mathcal{U}), \dots, \tau_d(u_r/\mathcal{U})) \\ \mathfrak{M}^{\langle\Lambda\rangle}/\mathcal{U} \models \varphi(v_1/\mathcal{U}, \dots, v_r/\mathcal{U}) &\text{ iff } \mathfrak{N} \models \varphi(\tau(v_1/\mathcal{U}), \dots, \tau(v_r/\mathcal{U})).\end{aligned}$$

Theorem 4.9 (Bounded Elementary Net Theorem of Bounded Ultralimits).

Suppose there are local ultralimits $\tau_d: \widehat{V}(X)^{\langle\Lambda_d\rangle}/\mathcal{U}_d \rightarrow {}^*V(X)$ satisfying the condition $\tau_d = \tau_{d'} \circ j_{d'}^{d*}$ for $d \leq d'$. Then there is a local ultralimit $\tau: \widehat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U} \rightarrow {}^*V(X)$ such that $\tau_d = \tau \circ j_{d*}$ for $d \in D$.



Proof. Similar to the proof of Theorem 4.8. □

We call the pair $\langle\Lambda, \mathcal{U}\rangle$ in Theorem 4.8 or Theorem 4.7 the *direct limit* of $\{\langle\Lambda_d, \mathcal{U}_d\rangle\}_{d \leq d', d, d' \in D}$.

Proof of Theorem 3.7 Let $\{p_\zeta\}_{\zeta < \kappa}$ be a sequence in ${}^*V(X)$ with $\kappa = \text{cov}({}^*V(X))$. We define local ultralimits $\{\rho_\zeta: \widehat{V}^{\langle\Lambda_\zeta\rangle}/\mathcal{U}_\zeta \rightarrow {}^*V(X)\}_{\zeta < \kappa}$ of ${}^*V(X)$ by:

$$\Lambda_0 = \mathcal{P}(\{0, 1\}), \mathcal{U}_0 = \{1\}.$$

$$\Lambda_{\zeta+1} = \Lambda^{[I_\zeta]}, \text{ where } |I_\zeta| = \text{nos}(p_\zeta), p_\zeta \in \text{rng } \rho_{\zeta+1} \text{ in Theorem 4.7.}$$

$$\langle\Lambda_\lambda, \mathcal{U}_\lambda\rangle \text{ is the direct limit of } \{\langle\Lambda_\zeta, \mathcal{U}_\zeta\rangle_{\zeta < \lambda}\} \text{ in Theorem 4.9.}$$

Then the direct limit $\langle\Lambda, \mathcal{U}\rangle$ of $\{\langle\Lambda_\zeta, \mathcal{U}_\zeta\rangle\}_{\zeta < \kappa}$ is an atlas of ${}^*V(X)$. □

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