

Note on a class of convex functions

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ABSTRACT. In this paper we define a general class of convex functions, denoted by $SL_{\beta}^c(q)$, with respect to a convex domain D ($q(z) \in \mathcal{H}_u(U)$, $q(0) = 1$, $q(U) = D$) contained in the right half plane by using the linear operator D_{λ}^{β} defined by

$$D_{\lambda}^{\beta} : A \rightarrow A,$$

$$D_{\lambda}^{\beta} f(z) = z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta} a_j z^j,$$

where $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$, $\lambda \geq 0$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$. This operator generalize the Sălăgean operator and the Al-Oboudi operator. Regarding the class $SL_{\beta}^c(q)$ we give a inclusion theorem, a preserving theorem (we use the Libera-Pascu integral operator) and many particular results.

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1 Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc U , $A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$, $\mathcal{H}_u(U) = \{f \in \mathcal{H}(U) : f \text{ is univalent in } U\}$ and $S = \{f \in A : f \text{ is univalent in } U\}$.

Let D^n be the Sălăgean differential operator (see [13]) defined as:

$$D^n : A \rightarrow A, \quad n \in \mathbb{N} \text{ and } D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z), \quad D^n f(z) = D(D^{n-1} f(z)).$$

Remark 1.1 If $f \in S$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, $z \in U$ then $D^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j$.

Let $n \in \mathbb{N}$ and $\lambda \geq 0$. Let denote with D_{λ}^n the Al-Oboudi operator (see [7]) defined by

$$D_{\lambda}^n : A \rightarrow A,$$

$$D_\lambda^0 f(z) = f(z), \quad D_\lambda^1 f(z) = (1 - \lambda)f(z) + \lambda z f'(z) = D_\lambda f(z),$$

$$D_\lambda^n f(z) = D_\lambda (D_\lambda^{n-1} f(z)).$$

We observe that D_λ^n is a linear operator and for $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ we have

$$D_\lambda^n f(z) = z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^n a_j z^j.$$

The aim of this paper is to define a general class of convex functions with respect to a convex domain D , contained in the right half plane, by using a operator which generalize the Sălăgean operator and the Al-Oboudi operator and to obtain some properties of this class.

2 Preliminary results

We recall here the definition of the well - known class of convex functions

$$S^c = CV = K = \left\{ f \in H(U); f(0) = f'(0) - 1 = 0, \operatorname{Re} \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} > 0, z \in U \right\}.$$

Remark 2.1 By using the subordination relation, we may define the class S^c thus

if $f(z) = z + a_2 z^2 + \dots$, $z \in U$, then $f \in S^c$ if and only if $\left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} \prec \frac{1+z}{1-z}$, $z \in U$, where by " $\prec$ " we denote the subordination relation.

Let consider the Libera-Pascu integral operator $L_a : A \rightarrow A$ defined as:

$$(1) \quad f(z) = L_a F(z) = \frac{1+a}{z^a} \int_0^z F(t) \cdot t^{a-1} dt, \quad a \in \mathbb{C}, \quad \operatorname{Re} a \geq 0.$$

In the case $a = 1$ this operator was introduced by R.J.Libera and it was studied by many authors in different general cases. In this general form ($a \in \mathbb{C}$, $\operatorname{Re} a \geq 0$) was used first time by N.N. Pascu in [12].

Definition 2.1 [6] Let $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$, $\lambda \geq 0$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$. We denote by D_λ^β the linear operator defined by

$$D_\lambda^\beta : A \rightarrow A,$$

$$D_\lambda^\beta f(z) = z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^\beta a_j z^j.$$

Remark 2.2 It is easy to observe that for $\beta = n \in \mathbb{N}$ we obtain the Al-Oboudi operator and for $\beta = n \in \mathbb{N}$, $\lambda = 1$ we obtain the Sălăgean operator.

The next theorem is result of the so called "admissible functions method" introduced by P.T. Mocanu and S.S. Miller (see [9], [10], [11]).

Theorem 2.1 Let h convex in U and $\operatorname{Re}[\beta h(z) + \gamma] > 0$, $z \in U$. If $p \in H(U)$ with $p(0) = h(0)$ and p satisfied the Briot-Bouquet differential subordination

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z), \quad \text{then } p(z) \prec h(z).$$

3 Main results

Definition 3.1 Let $q(z) \in \mathcal{H}_u(U)$, with $q(0) = 1$ and $q(U) = D$, where D is a convex domain contained in the right half plane, $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$ and $\lambda \geq 0$. We say that a function $f(z) \in A$ is in the class $SL_\beta^c(q)$ if $\frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta+1} f(z)} \prec q(z)$, $z \in U$.

Remark 3.1 Geometric interpretation: $f(z) \in SL_\beta^c(q)$ if and only if $\frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta+1} f(z)}$ take all values in the convex domain D contained in the right half-plane.

Remark 3.2 It is easy to observe that if we choose different function $q(z)$ we obtain variously classes of convex functions, such as (for example), for $\beta = n \in \mathbb{N}$ the class $SL_n^c(q)$ (see [3]), for $\lambda = 1$ and $\beta = 0$, the class of convex functions, the class of convex functions of order γ (see [8]), the class of convex functions with respect to a hyperbola (see [5]), and, for $\beta = n \in \mathbb{N}$ and $\lambda = 1$, the class of n -convex functions (see [2]), the class of n -convex functions with respect to a hyperbola (see [1]), the class of n -convex functions with respect to a convex domain contained in the right half-plane (see [2]), for $\beta \in \mathbb{R}$ and $\lambda = 1$, the class $S_\beta^c(q)$ of the β - q -convex functions (see [4]).

Remark 3.3 For $q_1(z) \prec q_2(z)$ we have $SL_\beta^c(q_1) \subset SL_\beta^c(q_2)$. From the above we obtain $SL_\beta^c(q) \subset SL_\beta^c\left(\frac{1+z}{1-z}\right)$.

Theorem 3.1 Let $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$ and $\lambda > 0$. We have

$$SL_{\beta+1}^c(q) \subset SL_\beta^c(q).$$

Proof. Let $f(z) \in SL_{\beta+1}^c(q)$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$.

With notation

$$p(z) = \frac{D_{\lambda}^{\beta+2} f(z)}{D_{\lambda}^{\beta+1} f(z)}, \quad p(0) = 1,$$

we obtain

$$(2) \quad \frac{D_{\lambda}^{\beta+3} f(z)}{D_{\lambda}^{\beta+2} f(z)} = \frac{D_{\lambda}^{\beta+3} f(z)}{D_{\lambda}^{\beta+1} f(z)} \cdot \frac{D_{\lambda}^{\beta+1} f(z)}{D_{\lambda}^{\beta+2} f(z)} = \frac{1}{p(z)} \cdot \frac{D_{\lambda}^{\beta+3} f(z)}{D_{\lambda}^{\beta+1} f(z)}$$

We have

$$\frac{D_{\lambda}^{\beta+3} f(z)}{D_{\lambda}^{\beta+1} f(z)} = \frac{z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta+3} a_j z^j}{z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta+1} a_j z^j}$$

and

$$\begin{aligned} zp'(z) &= \frac{z \left(D_{\lambda}^{\beta+2} f(z) \right)'}{D_{\lambda}^{\beta+1} f(z)} - \frac{D_{\lambda}^{\beta+2} f(z)}{D_{\lambda}^{\beta+1} f(z)} \cdot \frac{z \left(D_{\lambda}^{\beta+1} f(z) \right)'}{D_{\lambda}^{\beta+1} f(z)} \\ &= \frac{z \left(D_{\lambda}^{\beta+2} f(z) \right)'}{D_{\lambda}^{\beta+1} f(z)} - p(z) \cdot \frac{z \left(D_{\lambda}^{\beta+1} f(z) \right)'}{D_{\lambda}^{\beta+1} f(z)} \\ &= \frac{z \left(z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta+2} a_j z^j \right)'}{D_{\lambda}^{\beta+1} f(z)} - p(z) \cdot \frac{z \left(z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta+1} a_j z^j \right)'}{D_{\lambda}^{\beta+1} f(z)} \\ &= \frac{z \left(1 + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta+2} j a_j z^{j-1} \right)}{D_{\lambda}^{\beta+1} f(z)} - p(z) \cdot \frac{z \left(1 + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta+1} j a_j z^{j-1} \right)}{D_{\lambda}^{\beta+1} f(z)} \end{aligned}$$

or

$$(3) \quad zp'(z) = \frac{z + \sum_{j=2}^{\infty} j (1 + (j-1)\lambda)^{\beta+2} a_j z^j}{D_{\lambda}^{\beta+1} f(z)} - p(z) \cdot \frac{z + \sum_{j=2}^{\infty} j (1 + (j-1)\lambda)^{\beta+1} a_j z^j}{D_{\lambda}^{\beta+1} f(z)}.$$

Also, we have

$$\begin{aligned} z + \sum_{j=2}^{\infty} j (1 + (j-1)\lambda)^{\beta+1} a_j z^j &= z + \sum_{j=2}^{\infty} ((j-1) + 1) (1 + (j-1)\lambda)^{\beta+1} a_j z^j \\ &= z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta+1} a_j z^j + \sum_{j=2}^{\infty} (j-1) (1 + (j-1)\lambda)^{\beta+1} a_j z^j \end{aligned}$$

$$\begin{aligned}
&= D_{\lambda}^{\beta+1} f(z) + \sum_{j=2}^{\infty} (j-1) (1+(j-1)\lambda)^{\beta+1} a_j z^j \\
&= D_{\lambda}^{\beta+1} f(z) + \frac{1}{\lambda} \sum_{j=2}^{\infty} ((j-1)\lambda) (1+(j-1)\lambda)^{\beta+1} a_j z^j \\
&= D_{\lambda}^{\beta+1} f(z) + \frac{1}{\lambda} \sum_{j=2}^{\infty} (1+(j-1)\lambda-1) (1+(j-1)\lambda)^{\beta+1} a_j z^j \\
&= D_{\lambda}^{\beta+1} f(z) - \frac{1}{\lambda} \sum_{j=2}^{\infty} (1+(j-1)\lambda)^{\beta+1} a_j z^j + \frac{1}{\lambda} \sum_{j=2}^{\infty} (1+(j-1)\lambda)^{\beta+2} a_j z^j \\
&= D_{\lambda}^{\beta+1} f(z) - \frac{1}{\lambda} \left(D_{\lambda}^{\beta+1} f(z) - z \right) + \frac{1}{\lambda} \left(D_{\lambda}^{\beta+2} f(z) - z \right) \\
&= D_{\lambda}^{\beta+1} f(z) - \frac{1}{\lambda} D_{\lambda}^{\beta+1} f(z) + \frac{z}{\lambda} + \frac{1}{\lambda} D_{\lambda}^{\beta+2} f(z) - \frac{z}{\lambda} \\
&= \frac{\lambda-1}{\lambda} D_{\lambda}^{\beta+1} f(z) + \frac{1}{\lambda} D_{\lambda}^{\beta+2} f(z) \\
&= \frac{1}{\lambda} \left((\lambda-1) D_{\lambda}^{\beta+1} f(z) + D_{\lambda}^{\beta+2} f(z) \right).
\end{aligned}$$

Similarly we have

$$z + \sum_{j=2}^{\infty} j (1+(j-1)\lambda)^{\beta+2} a_j z^j = \frac{1}{\lambda} \left((\lambda-1) D_{\lambda}^{\beta+2} f(z) + D_{\lambda}^{\beta+3} f(z) \right).$$

From (3) we obtain

$$\begin{aligned}
zp'(z) &= \frac{1}{\lambda} \left(\frac{(\lambda-1) D_{\lambda}^{\beta+2} f(z) + D_{\lambda}^{\beta+3} f(z)}{D_{\lambda}^{\beta+1} f(z)} - p(z) \frac{(\lambda-1) D_{\lambda}^{\beta+1} f(z) + D_{\lambda}^{\beta+2} f(z)}{D_{\lambda}^{\beta+1} f(z)} \right) \\
&= \frac{1}{\lambda} \left((\lambda-1) p(z) + \frac{D_{\lambda}^{\beta+3} f(z)}{D_{\lambda}^{\beta+1} f(z)} - p(z) ((\lambda-1) + p(z)) \right) \\
&= \frac{1}{\lambda} \left(\frac{D_{\lambda}^{\beta+3} f(z)}{D_{\lambda}^{\beta+1} f(z)} - p(z)^2 \right)
\end{aligned}$$

Thus

$$\lambda zp'(z) = \frac{D_{\lambda}^{\beta+3} f(z)}{D_{\lambda}^{\beta+1} f(z)} - p(z)^2$$

or

$$\frac{D_{\lambda}^{\beta+3} f(z)}{D_{\lambda}^{\beta+1} f(z)} = p(z)^2 + \lambda zp'(z).$$

From (2) we obtain

$$\frac{D_{\lambda}^{\beta+3} f(z)}{D_{\lambda}^{\beta+2} f(z)} = \frac{1}{p(z)} (p(z)^2 + \lambda zp'(z)) = p(z) + \lambda \frac{zp'(z)}{p(z)},$$

where $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$, $\lambda > 0$.

From $f(z) \in SL_{\beta+1}^c(q)$ we have

$$p(z) + \lambda \frac{zp'(z)}{p(z)} \prec q(z),$$

with $p(0) = q(0) = 1$, $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$ and $\lambda > 0$. In this conditions from Theorem 2.1, we obtain

$$p(z) \prec q(z)$$

or

$$\frac{D_{\lambda}^{\beta+2}f(z)}{D_{\lambda}^{\beta+1}f(z)} \prec q(z).$$

This means $f(z) \in SL_{\beta}^c(q)$.

Corollary 3.1 For every $\beta \in \mathbb{N}^*$ we have $SL_{\beta}^c(q) \subset SL_0^c(q) \subset S^c$.

Theorem 3.2 Let $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$ and $\lambda \geq 1$. If $F(z) \in SL_{\beta}^c(q)$ then $f(z) = L_a F(z) \in SL_{\beta}^c(q)$, where L_a is the Libera-Pascu integral operator defined by (1).

Proof. From (1) we have

$$(1+a)F(z) = af(z) + zf'(z)$$

and, by using the linear operator $D_{\lambda}^{\beta+1}$, we obtain

$$\begin{aligned} (1+a)D_{\lambda}^{\beta+1}F(z) &= aD_{\lambda}^{\beta+1}f(z) + D_{\lambda}^{\beta+1}\left(z + \sum_{j=2}^{\infty} ja_j z^j\right) \\ &= aD_{\lambda}^{\beta+1}f(z) + z + \sum_{j=2}^{\infty} (1+(j-1)\lambda)^{\beta+1} ja_j z^j \end{aligned}$$

We have (see the proof of the above theorem)

$$z + \sum_{j=2}^{\infty} j(1+(j-1)\lambda)^{\beta+1} a_j z^j = \frac{1}{\lambda} \left((\lambda-1)D_{\lambda}^{\beta+1}f(z) + D_{\lambda}^{\beta+2}f(z) \right)$$

Thus

$$\begin{aligned} (1+a)D_{\lambda}^{\beta+1}F(z) &= aD_{\lambda}^{\beta+1}f(z) + \frac{1}{\lambda} \left((\lambda-1)D_{\lambda}^{\beta+1}f(z) + D_{\lambda}^{\beta+2}f(z) \right) \\ &= \left(a + \frac{\lambda-1}{\lambda} \right) D_{\lambda}^{\beta+1}f(z) + \frac{1}{\lambda} D_{\lambda}^{\beta+2}f(z) \end{aligned}$$

or

$$\lambda(1+a)D_{\lambda}^{\beta+1}F(z) = ((a+1)\lambda-1)D_{\lambda}^{\beta+1}f(z) + D_{\lambda}^{\beta+2}f(z).$$

Similarly, we obtain

$$\lambda(1+a)D_{\lambda}^{\beta+2}F(z) = ((a+1)\lambda - 1)D_{\lambda}^{\beta+2}f(z) + D_{\lambda}^{\beta+3}f(z).$$

Then

$$\frac{D_{\lambda}^{\beta+2}F(z)}{D_{\lambda}^{\beta+1}F(z)} = \frac{\frac{D_{\lambda}^{\beta+3}f(z)}{D_{\lambda}^{\beta+1}f(z)} + ((a+1)\lambda - 1) \cdot \frac{D_{\lambda}^{\beta+2}f(z)}{D_{\lambda}^{\beta+1}f(z)}}{\frac{D_{\lambda}^{\beta+2}f(z)}{D_{\lambda}^{\beta+1}f(z)} + ((a+1)\lambda - 1)}.$$

With notation

$$\frac{D_{\lambda}^{\beta+2}f(z)}{D_{\lambda}^{\beta+1}f(z)} = p(z), \quad p(0) = 1,$$

we obtain

$$(4) \quad \frac{D_{\lambda}^{\beta+2}F(z)}{D_{\lambda}^{\beta+1}F(z)} = \frac{\frac{D_{\lambda}^{\beta+3}f(z)}{D_{\lambda}^{\beta+1}f(z)} + ((a+1)\lambda - 1) \cdot p(z)}{p(z) + ((a+1)\lambda - 1)}$$

We have (see the proof of the above theorem)

$$\frac{D_{\lambda}^{\beta+3}f(z)}{D_{\lambda}^{\beta+1}f(z)} = p(z)^2 + \lambda zp'(z).$$

From (4), we obtain

$$\frac{D_{\lambda}^{\beta+2}F(z)}{D_{\lambda}^{\beta+1}F(z)} = \frac{p(z)^2 + \lambda zp'(z) + ((a+1)\lambda - 1)p(z)}{p(z) + ((a+1)\lambda - 1)} = p(z) + \lambda \frac{zp'(z)}{p(z) + ((a+1)\lambda - 1)},$$

where $a \in \mathbb{C}$, $\operatorname{Re} a \geq 0$, $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$ and $\lambda \geq 1$. From $F(z) \in SL_{\beta}^c(q)$ we have

$$p(z) + \frac{zp'(z)}{\frac{1}{\lambda}(p(z) + ((a+1)\lambda - 1))} \prec q(z),$$

where $a \in \mathbb{C}$, $\operatorname{Re} a \geq 0$, $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$, $\lambda \geq 1$, and from her construction, we have $\operatorname{Re} q(z) > 0$. In this conditions we have from Theorem 2.1 we obtain

$$p(z) \prec q(z)$$

or

$$\frac{D_{\lambda}^{\beta+2}f(z)}{D_{\lambda}^{\beta+1}f(z)} \prec q(z).$$

This means $f(z) = L_a F(z) \in SL_{\beta}^c(q)$.

For $\beta = n \in \mathbb{N}$ and $\lambda = 1$ we obtain

Corollary 3.2 *If $F(z) \in CV_n(q)$ then $f(z) = L_a F(z) \in CV_n(q)$, where L_a is the Libera-Pascu integral operator and by $CV_n(q)$ we denote the class of n -convex functions subordinate to the function $q(z)$ (see [2]).*

For $\beta = n \in \mathbb{N}$ we obtain

Corollary 3.3 [3] *Let $n \in \mathbb{N}$ and $\lambda \geq 1$. If $F(z) \in SL_n^c(q)$ then $f(z) = L_a F(z) \in SL_n^c(q)$, where L_a is the Libera-Pascu integral operator defined by (1).*

For $\beta \in \mathbb{R}$ and $\lambda = 1$ we obtain

Corollary 3.4 [4] *If $F(z) \in S_\beta^c(q)$ then $f(z) = L_a F(z) \in S_\beta^c(q)$, where L_a is the Libera-Pascu integral operator defined by (1).*

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