

A SUBCLASS OF ANALYTIC FUNCTIONS WITH TWO FIXED POINTS

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ABSTRACT. Making use of operator of fractional calculus a subclass $\varphi(\alpha, \beta, \mu, \eta, \gamma, \delta, t; z_0)$ of univalent functions with fixed point in the unit disk E is introduced and obtained coefficient-estimates distortion theorem. Lastly we investigated Hadamard product property and linear combination function of $\varphi(\alpha, \beta, \mu, \eta, \gamma, \delta, t; z_0)$.

1. INTRODUCTION

Let A denote the class of functions of the form

$$(1.1) \quad f(z) = z - \sum_{n=2}^{\infty} a_n z^n (a_n \geq 0),$$

which are analytic in unit disc $E = \{z : |z| < 1\}$. Silvermann ([4]) studied the class of functions of the form

$$f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n (a_n \geq 0),$$

where either

$$(1.2) \quad f(z_0) = z_0 (-1 < z_0 < 1; z_0 \neq 0) \quad \text{or} \quad f'(z_0) = 1 (-1 < z_0 < 1).$$

Recently, Uralegadi and Somanatha([6]) studied the class of functions of the form

$$(1.3) \quad f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n (a_n \geq 0) \quad \text{with} \quad \frac{(1-t)f(z_0)}{z_0} + tf'(z_0) = 1,$$

where $-1 < z_0 < 1$, $0 \leq t \leq 1$. A function $f(z)$ is said to be convex of order α , if

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha (z \in E : 0 \leq \alpha < 1).$$

We denote by $C^*(\alpha)$ the class of convex functions of order α ($0 \leq \alpha < 1$).

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We now recall the following definition of a generalized fractional operator introduced by Srivastava et al([5]).

Definition 1 For real numbers $\eta(\eta > 0)$, γ and δ , the generalized fractional integral operator $I_{0,z}^{\eta,\gamma,\delta}$ of order η is defined for a function $f(z)$, by

$$I_{0,z}^{\eta,\gamma,\delta} f(z) = \frac{z^{-\eta-1}}{\Gamma(\eta)} \int_0^z (z-\xi)^{\eta-1} F(\eta+\gamma, -\delta; \eta; 1-\frac{\xi}{z}) f(\xi) d\xi,$$

where $f(z)$ is an analytic function in a simply-connected region of the z -plane containing the origin with the order

$$f(z) = O(|z|^\epsilon), (z \rightarrow 0), (\epsilon < \max\{0, \gamma, -\delta\} - 1),$$

$$(1.4) \quad F(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \quad (z \in D),$$

and $(v)_n$ being the pochhammer symbol defined by

$$(v)_n = \frac{\Gamma(v+n)}{\Gamma(v)},$$

if $n = 0$ then $(v)_n = 1$ and if $n \in N = \{1, 2, \dots\}$ then $v_n = v(v+1) \cdots (v+n-1)$, provided further that the multiplicity of $(z-\xi)^{\eta-1}$ is removed requiring $\log(z-\xi)$ to be real $(z-\xi) > 0$.

Definition 2 For real numbers $\eta(0 \leq \eta < 1)$, γ , and δ , the generalized fractional derivative operator $J_{0,z}^{\eta,\gamma,\delta}$ of order η is defined for a function $f(z)$, by

$$(1.5) \quad J_{0,z}^{\eta,\gamma,\delta} f(z) = \frac{1}{\Gamma(1-\eta)} \frac{d}{dz} \left\{ z^{\eta-\gamma} \int_0^z (z-\eta)^{-\eta} F\left(\gamma-\eta, -\delta; 1-\eta; 1-\frac{\xi}{z}\right) f(\xi) d\xi \right\},$$

where $f(z)$ is an analytic function in a simply-connected region of the z -plane containing the origin, and the multiplicity of $(z-\xi)^{-\eta}$ is removed as Definition 1 above.

It follows readily from Definition 2, $J_{0,z}^{\eta,\gamma,\delta} f(z) = D_z^\eta f(z)$ ($0 \leq \eta < 1$), where operator D_z^η is fractional derivatives operator which is defined by Owa([2]). Furthermore, in terms of Gamma functions, we have the following Lemma.

Lemma 3. ([5]) If $0 \leq \eta < 1$ and $n > \gamma - \delta - 2$, then

$$(1.6) \quad J_{0,z}^{\eta,\gamma,\delta} z^n = \frac{\Gamma(n+1) \Gamma(n-\gamma+\delta+2)}{\Gamma(n-\gamma+1) \Gamma(n-\eta+\delta+2)} z^{n-\gamma}.$$

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Lemma 4. If the form of a function $f(z)$ defined by (1.2) and satisfying (1.3), then

$$(1.7) \quad \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} z^{\gamma-1} J_{0,z}^{\eta,\gamma,\delta} f(z) = a_1 - \sum_{n=2}^{\infty} \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n z^{n-1},$$

where we denote $\Psi_n(\eta, \gamma, \delta) = \frac{\Gamma(n+1)\Gamma(n-\gamma+\delta+2)}{\Gamma(n-\gamma+1)\Gamma(n-\eta+\delta+2)}$.

proof. By Lemma3, we get a (1.7). $\square$

We will define the following definition.

Definition 5 A function $f(z)$ defined by (1.2) and satisfying (1.3) is said to be in the class $\varphi(\alpha, \beta, \mu, \eta, \gamma, \delta, t; z_0)$ if

$$(1.8) \quad \left| \frac{\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} z^{\gamma-1} J_{0,z}^{\eta,\gamma,\delta} f(z) - a_1}{\mu \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} z^{\gamma-1} J_{0,z}^{\eta,\gamma,\delta} f(z) + a_1 - (1+\mu)\alpha} \right| < \beta, z \in E,$$

where

$$0 \leq \eta < 1, \eta - \delta < 3, \gamma - \delta < 3, 0 \leq \mu \leq 1, 0 \leq \alpha < 1, 0 < \beta \leq 1 \text{ and } \alpha < a_1.$$

Furthermore, by specializing the parameters $\alpha, \beta, \mu, \eta, \gamma, \delta, t$, we obtain the following subclasses studied by various authors,

- (1) $\varphi(\alpha, \beta, \mu, \eta, \eta, \delta, 1; 0) = P^*(\alpha, \beta, \mu, \eta)$ (Jochi [1]);
- (2) $\varphi(\alpha, \beta, \mu, 1, 1, \delta, 1; 0) = P^*(\alpha, \beta, \mu)$ (Owa and Aouf [3]);

The main purpose of this paper is to investigate coefficient-inequalites, distortion theorem and radius problem of functions in the class $\varphi(\alpha, \beta, \mu, \eta, \gamma, \delta, t; z_0)$. And, we obtain Hadmard product property and linear combination function.

2. A Coefficient Theorem

We begin by starting our first result as,

Theorem 2.1. A function $f(z)$ is in the class $\varphi(\alpha, \beta, \mu, \eta, \gamma, t, \delta; z_0)$ if and only if

$$(2.1) \quad \sum_{n=2}^{\infty} \left\{ (1+\mu\beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n - [(1-t) + tn] z_0^{n-1} \right\} a_n \leq (1+\mu)\beta(1-\alpha),$$

where $\Psi_n(\mu, \gamma, \delta)$ is in (1.7).

proof. Suppose that $f(z)$ is in the class $\varphi(\alpha, \beta, \mu, \eta, \gamma, t, \delta; z_0)$, so that condition (1.8) readily yields.

$$\left| \frac{\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} z^{\gamma-1} J_{0,z}^{\eta,\gamma,\delta} f(z) - a_1}{\mu \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} z^{\gamma-1} J_{0,z}^{\eta,\gamma,\delta} f(z) + a_1 - (1+\mu)\alpha} \right| < \beta.$$

Using Lemma 4, we obtain that

$$\left| \frac{-\sum_{n=2}^{\infty} \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n z^{n-1}}{\mu \left\{ a_1 - \sum_{n=2}^{\infty} \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n z^{n-1} \right\} + a_1 - (1+\mu)\alpha} \right| < \beta \quad (z \in E).$$

Since $|\Re(z)| \leq |z|$, for any z , we have

$$(2.2) \quad \Re \left\{ \frac{\sum_{n=2}^{\infty} \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n z^{n-1}}{(1+\mu)(a_1 - \alpha) - \mu \sum_{n=2}^{\infty} \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n z^{n-1}} \right\} < \beta.$$

Choose values of z on the real axis so that $\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} z^{\gamma-1} J_{0,z}^{\eta,\gamma,\delta} f(z)$ is real, upon clearing the denominator in (2.2) and letting $z \rightarrow 1$ through the real values, we get

$$(2.3) \quad \sum_{n=2}^{\infty} (1+\mu\beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n \leq (1+\mu)\beta(a_1 - \alpha).$$

Finally, substituting $a_1 = 1 + \sum_{n=2}^{\infty} [(1-t) + tn] a_n |z_0|^{n-1}$ in (2.3), we get (2.1).

Conversely, assume that the inequality (2.1) holds true. Consider

$$\begin{aligned} & \left| \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} z^{\gamma-1} J_{0,z}^{\eta,\gamma,\delta} f(z) - a_1 \right| - \\ & \beta \left| \mu \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} z^{\gamma-1} J_{0,z}^{\eta,\gamma,\delta} f(z) + a_1 - (1+\mu)\alpha \right| \\ & \leq \sum_{n=2}^{\infty} \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n |z|^{n-1} \\ & \quad - (1+\mu)\beta(a_1 - \alpha) + \beta\mu \sum_{n=2}^{\infty} \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n |z|^{n-1} \\ & \leq \sum_{n=2}^{\infty} (1+\mu\beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n - (1+\mu)\beta(a_1 - \alpha) \leq 0, \end{aligned}$$

by the hypothesis. Hence, a function $f(z)$ is in the class $\varphi(\alpha, \beta, \mu, \eta, \gamma, t, \delta; z_0)$. $\square$

Corollary 2.2. Let the function $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n$ defined by (1.3) be in the class $\varphi(\alpha, \beta, \mu, \eta, \gamma, \delta, t; z_0)$. Then

$$(2.4) \quad a_n \leq \frac{(1+\mu)\beta(1-\alpha)}{(1+\mu\beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n + (1-a_1)}.$$

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The assertion (2.1) of Theorem 2.1 is sharp extremal function being

$$(2.5) \quad f(z) = a_1 z - \frac{(1+\mu)\beta(1-\alpha)}{(1+\mu\beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n + 1 - a_1} z^n.$$

Where $a_1 = 1 + \sum_{n=2}^{\infty} [(1-t) + tn] a_n z_0^{n-1}$.

3. Distortion Theorem

Theorem 3.1. If a function $f(z)$ is in the class $\varphi(\alpha, \beta, \mu, \eta, \gamma, \delta, t; z_0)$ with $\eta > \gamma$, then

$$(3.1) \quad |f(z)| \geq a_1 |z| - \frac{(2-\gamma)(3-\eta+\delta)(1+\mu)\beta(a_1-\alpha)}{2(3-\gamma+\delta)(1+\mu\beta)} |z|^2,$$

and

$$(3.2) \quad |f(z)| \leq a_1 |z| + \frac{(2-\gamma)(3-\eta+\delta)(1+\mu)\beta(a_1-\alpha)}{2(3-\gamma+\delta)(1+\mu\beta)} |z|^2.$$

proof. In view of inequality (2.1) and the fact that $\Psi_n(\eta, \gamma, \delta)$ is non-decreasing for $\eta \geq \gamma$, we have

$$\begin{aligned} & (1+\mu)\beta(a_1-\alpha) \\ & \geq \sum_{n=2}^{\infty} (1+\mu\beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \frac{\Gamma(n+1)\Gamma(n-\gamma+\delta+2)}{\Gamma(n-\gamma+1)\Gamma(n-\eta+\delta+2)} a_n \\ & \geq \frac{2(3-\gamma+\delta)(1+\mu\beta)}{(2-\gamma)(3-\eta+\delta)} \sum_{n=2}^{\infty} a_n. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} |f(z)| & \geq a_1 |z| - \sum_{n=p+1}^{\infty} a_n |z|^n \geq a_1 |z| - |z|^2 \sum_{n=p+1}^{\infty} a_n \\ & \geq a_1 |z| - \frac{(2-\gamma)(3-\eta+\delta)(1+\mu)\beta(a_1-\alpha)}{2(3-\gamma+\delta)(1+\mu\beta)} |z|^2, \\ (3.3) \quad \sum_{n=2}^{\infty} a_n & \leq \frac{(2-\gamma)(3-\eta+\delta)(1+\mu)\beta(a_1-\alpha)}{2(3-\gamma+\delta)(1+\mu\beta)}, \end{aligned}$$

and we have

$$|f(z)| \geq a_1 |z| - \frac{(2-\gamma)(3-\eta+\delta)(1+\mu)\beta(a_1-\alpha)}{2(3-\gamma+\delta)(1+\mu\beta)} |z|^2.$$

Simillary,

$$|f(z)| \leq a_1 |z| + \frac{(2-\gamma)(3-\eta+\delta)(1+\mu)\beta(a_1-\alpha)}{2(3-\gamma+\delta)(1+\mu\beta)} |z|^2.$$

The proof is complete. $\square$

Theorem 3.2. *If a function $f(z)$ is in the class $\varphi(\alpha, \beta, \mu, \eta, \gamma, \delta, t; z_0)$, then*

$$(3.4) \quad \left| J_{0,z}^{\eta,\gamma,\delta} f(z) \right| \geq \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \left\{ a_1 |z|^{1-\gamma} - \frac{(1+\mu)\beta(a_1-\alpha)}{(1+\mu\beta)} |z|^{2-\gamma} \right\},$$

and

$$(3.5) \quad \left| J_{0,z}^{\eta,\gamma,\delta} f(z) \right| \leq \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \left\{ a_1 |z|^{1-\gamma} + \frac{(1+\mu)\beta(a_1-\alpha)}{(1+\mu\beta)} |z|^{2-\gamma} \right\},$$

for $z \in D_0$ where D_0 equals to E if $\gamma \leq 1$, and D_0 equals to E^* if $1 < \gamma < n$.

proof. By using in equality (1.8) and Theorem 2.1, we obtain that

$$\begin{aligned} \left| \frac{\Gamma(3-\gamma+\delta)}{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)} z^\gamma J_{0,z}^{\eta,\gamma,\delta} f(z) \right| &\geq a_1 |z| - \sum_{n=2}^{\infty} \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n |z|^n \\ &\geq a_1 |z| - |z|^2 \sum_{n=2}^{\infty} \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_n \geq a_1 |z| - |z|^2 \frac{(1+\mu)\beta(a_1-\alpha)}{(1+\mu\beta)}, \end{aligned}$$

which is equivalent to (3.4). Simillary, We obtain that

$$\left| J_{0,z}^{\eta,\gamma,\delta} f(z) \right| \leq \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \left\{ a_1 |z|^{1-\gamma} + \frac{(1+\mu)\beta(a_1-\alpha)}{(1+\mu\beta)} |z|^{2-\gamma} \right\}.$$

The proof is complete. $\square$

Corollary 3.3. *Let a function $f(z)$ be in the class $\varphi(\alpha, \beta, \mu, \eta, \gamma, \delta, t; z_0)$ with $\eta > \gamma$. Then, in view of Theorem 3.1, $f(z)$ is included in a disk with its center at origin and radius r given by*

$$(3.6) \quad r = a_1 + \frac{(2-\gamma)(3-\eta+\delta)}{2(3-\gamma+\delta)} \frac{(1+\mu)\beta(a_1-\alpha)}{(1+\mu\beta)},$$

and $J_{0,z}^{\eta,\gamma,\delta} f(z)$ is included in a disk with its center at origin and radius R given by

$$(3.7) \quad R = \frac{\Gamma(3-\gamma+\delta)}{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)} \left\{ a_1 + \frac{(1+\mu)\beta(a_1-\alpha)}{(1+\mu\beta)} \right\}.$$

4. Radius of convexity

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Theorem 4.1. Let $f(z)$ be in the class $\varphi(\alpha, \beta, \mu, \eta, \gamma, \delta, t; z_0)$. Then $f(z)$ is convex in the disk

(4.1)

$$|z| < r(\alpha, \beta, \mu, \eta, \gamma, \delta, t; z_0)$$

$$= \inf_{\substack{n \geq 2 \\ n \in N}} \left\{ \frac{(1 + \mu\beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n + [(1 + \mu)\beta(1 - \alpha) - 1] [(1 - t) + tn] z_0^{n-1}}{n^2 (1 + \mu)\beta(1 - \alpha)} \right\}^{\frac{1}{n-1}}$$

The result is sharp for the function given by

proof. It is sufficient to prove that $\left| \frac{zf''(z)}{f'(z)} \right| \leq 1$ for $r(\alpha, \beta, \mu, \eta, \gamma, \delta, t; z_0)$.

A simple calculation gives us

$$\left| \frac{zf''(z)}{f'(z)} \right| = \left| \frac{-\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{a_1 - \sum_{n=2}^{\infty} na_n z^{n-1}} \right| \leq \frac{\sum_{n=2}^{\infty} n(n-1)a_n |z|^{n-1}}{a_1 - \sum_{n=2}^{\infty} na_n |z|^{n-1}}.$$

Clearly $\left| \frac{zf''(z)}{f'(z)} \right| \leq 1$, if

$$(4.3) \quad \sum_{n=2}^{\infty} n(n-1)a_n |z|^{n-1} \leq a_1 - \sum_{n=2}^{\infty} na_n |z|^{n-1}.$$

Using $a_1 = 1 + \sum_{n=2}^{\infty} [(1-t)tn] a_1 z_0^{n-1}$ in (4.3), we are led to

$$(4.4) \quad \sum_{n=2}^{\infty} \{n^2 |z|^{n-1} - [(1-t) + tn] z_0^{n-1}\} a_n \leq 1.$$

By Theorem 2.1, we have

$$(4.5) \quad \sum_{n=2}^{\infty} \left\{ \frac{(1 + \mu\beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n - [(1-t) + tn] z_0^{n-1}}{(1 + \mu)\beta(1 - \alpha)} \right\} a_n \leq 1.$$

Hence (4.4) will hold, if

(4.6)

$$n^2 |z|^{n-1} - [(1-t) + tn] z_0^{n-1} \leq \frac{(1 + \mu\beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n - [(1-t) + tn] z_0^{n-1}}{(1 + \mu)\beta(1 - \alpha)},$$

or equivalently

$$(4.7) \quad |z|^{n-1} \leq \frac{(1 + \mu\beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n + [(1 + \mu)\beta(1 - \alpha) - 1] [(1-t) + tn] z_0^{n-1}}{n^2 (1 + \mu)\beta(1 - \alpha)}$$

which in turn implies the assertion of the theorem. $\square$

5. Property of Hadamard product

Let the function $f_j(z)$ ($j = 1, 2$) defined by $f_j(z) = a_{1,j}z - \sum_{n=2}^{\infty} a_{n,j}z^n$, we define the hadamard product $f(z) * g(z)$ by

$$(5.1) \quad (f_1 * f_2)(z) = a_{1,1}a_{1,2}z - \sum_{n=2}^{\infty} a_{n,1}a_{n,2}z^n.$$

Theorem 5.1. *Let the functions $f_j(z)$ ($j = 1, 2$) defined by (5.1) be in the class $\varphi(\alpha, \beta, \mu, \eta, \gamma, \delta, t; z_0)$ with $\mu > \gamma$. Then $f_1(z) * f_2(z)$ is in the class $\varphi(v, \beta, \mu, \eta, \gamma, \delta, t; z_0)$ where*

$$(5.2) \quad v = v(\alpha, \beta, \mu, \eta, \gamma, \delta) = a_{1,1}a_{1,2} - \frac{(2-\gamma)(3-\eta+\delta)(1+\mu)\beta(a_{1,1}-\alpha)(a_{1,2}-\alpha)}{2(3-\gamma+\delta)(1+\mu\beta)}.$$

proof. Suppose that $f_1(z)$ and $f_2(z)$ are in the class $\varphi(\alpha, \beta, \mu, \eta, \gamma, \delta, t; z_0)$, by using Theorem 2.1, we have

$$(5.3) \quad \sum_{n=2}^{\infty} \frac{(1+\mu\beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_{n,1}}{(1+\mu)\beta(a_{1,1}-\alpha)} \leq 1,$$

and

$$(5.4) \quad \sum_{n=2}^{\infty} \frac{(1+\mu\beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n a_{n,2}}{(1+\mu)\beta(a_{1,2}-\alpha)} \leq 1.$$

From (5.3) and (5.4), using Cauchy-Schwarze inequality, we have

$$(5.5) \quad \sum_{n=2}^{\infty} \frac{(1+\mu\beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n}{(1+\mu)\beta} \sqrt{\frac{a_{n,1}a_{n,2}}{(a_{1,1}-\alpha)(a_{1,2}-\alpha)}} \leq 1.$$

Hence, we find the largest v such that

$$(5.6) \quad \begin{aligned} & \sum_{n=2}^{\infty} \frac{(1+\mu\beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n}{(1+\mu)\beta(a_{1,1}a_{1,2}-v)} a_{n,1}a_{n,2} \\ & \leq \sum_{n=2}^{\infty} \frac{(1+\mu\beta) \frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} \Psi_n}{(1+\mu)\beta} \sqrt{\frac{a_{n,1}a_{n,2}}{(a_{1,1}-\alpha)(a_{1,2}-\alpha)}} \leq 1, \end{aligned}$$

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or equivalently

$$(5.7) \quad \sqrt{a_{n,1}a_{n,2}} \leq \frac{a_{1,1}a_{1,2} - v}{\sqrt{(a_{1,1} - \alpha)(a_{1,2} - \alpha)}}.$$

So, it is sufficient to find the largest v such that

$$(5.8) \quad \frac{(1 + \mu)\beta\sqrt{(a_{1,1} - \alpha)(a_{1,2} - \alpha)}}{(1 + \mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n} \leq \frac{a_{1,1}a_{1,2} - v}{\sqrt{(a_{1,1} - \alpha)(a_{1,2} - \alpha)}}.$$

Hence (5.8) yields $v \leq a_{1,1}a_{1,2} - \frac{(1+\mu)\beta(a_{1,1}-\alpha)(a_{1,2}-\alpha)}{(1+\mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_n}$. Since $\Psi_n(\mu, \gamma, \delta)$ is non-decreasing for $\mu \geq \gamma$, we have

$$v \leq a_{1,1}a_{1,2} - \frac{(2-\gamma)(3-\eta+\delta)(1+\mu)\beta(a_{1,1}-\alpha)(a_{1,2}-\alpha)}{2(3-\gamma+\delta)(1+\mu\beta)},$$

which proves the assertion of this theorem. $\square$

6. Linear combination of the function in the class $\varphi(\alpha, \beta, \mu, \eta, \gamma, \delta, t; z_0)$

Theorem 6.1. Let $T(\alpha, \beta, \mu, \eta, \gamma, \delta) = \frac{(1+\mu)\beta(1-\alpha)}{(1+\mu\beta)\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}\Psi_{n+1-a_1}}$, and let us put

$$(6.1) \quad f_n(z) = a_1z - T(\alpha, \beta, \mu, \eta, \gamma, \delta)z^n, n = 2, 3, \dots,$$

and $f_1(z) = a_1z$. Then $f(z) \in \varphi(\alpha, \beta, \mu, \eta, \gamma, \delta, t; z_0)$ if and only if

$$(6.2) \quad f(z) = \sum_{n=1}^{\infty} t_n f_n(z), z \in E,$$

where $\sum_{n=1}^{\infty} t_n = 1, t_n \geq 0$ for $n = 1, 2, 3, \dots$.

proof. Let $f(z) \in \varphi(\alpha, \beta, \mu, \eta, \gamma, \delta, t; z_0)$. Then, by Corollary 2.2, $|a_n| \leq T(\alpha, \beta, \mu, \eta, \gamma, \delta)$.

Let us put

$$(6.3) \quad t_n = T(\alpha, \beta, \mu, \eta, \gamma, \delta)^{-1} a_n, n = 2, 3, \dots,$$

and $t_1 = 1 - \sum_{n=2}^{\infty} t_n$. By assumption, we have $t_n \geq 0, n = 2, 3, \dots$, and $t_1 \geq 0$. Thus

$$(6.4) \quad \begin{aligned} \sum_{n=1}^{\infty} t_n f_n(z) &= t_1 f_1(z) + \sum_{n=2}^{\infty} t_n f_n(z) \\ &= \left(1 - \sum_{n=2}^{\infty} t_n\right) a_1 z + \sum_{n=2}^{\infty} t_n \{a_1 z - T(\alpha, \beta, \mu, \eta, \gamma, \delta) z^n\} \\ &= a_1 z - \sum_{n=2}^{\infty} a_n z^n = f(z). \end{aligned}$$

Conversely, Let us function $f(z)$ satisfy (6.2). Since

$$\begin{aligned}
 f(z) &= \sum_{n=1}^{\infty} t_n f_n(z) = t_1 f_1(z) + \sum_{n=2}^{\infty} t_n f_n(z) \\
 (6.5) \quad &= \left(1 - \sum_{n=2}^{\infty} t_n\right) a_1 z + \sum_{n=2}^{\infty} t_n \{a_1 z - T(\alpha, \beta, \mu, \eta, \gamma, \delta) z^n\}, \\
 &= a_1 z - \sum_{n=2}^{\infty} t_n T(\alpha, \beta, \mu, \eta, \gamma, \delta) z^n = a_1 z - \sum_{n=2}^{\infty} a_n z^n
 \end{aligned}$$

which proves the assertion of this theorem. $\square$

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