

Coefficient Estimates for Certain Classes of Analytic Functions

Shigeyoshi Owa and Junichi Nishiwaki

Abstract

For some real $\alpha (\alpha > 1)$, two subclasses $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ of analytic functions $f(z)$ with $f(0) = 0$ and $f'(0) = 1$ in $\mathbb{U}$ are introduced. The object of the present paper is to discuss the coefficient estimates for functions $f(z)$ belonging to the classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$.

1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{M}(\alpha)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy

$$\operatorname{Re} \left\{ \frac{z f'(z)}{f(z)} \right\} < \alpha \quad (z \in \mathbb{U})$$

for some $\alpha (\alpha > 1)$. And let $\mathcal{N}(\alpha)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy

$$\operatorname{Re} \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} < \alpha \quad (z \in \mathbb{U})$$

for some $\alpha (\alpha > 1)$. Then, we see that $f(z) \in \mathcal{N}(\alpha)$ if and only if $z f'(z) \in \mathcal{M}(\alpha)$.

Remark 1.1. For $1 < \alpha \leq \frac{4}{3}$, the classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ were introduced by Uralegaddi, Ganigi and Sarangi [2].

We easily see that

Example 1.1. (i) $f(z) = z(1 - z)^{2(\alpha-1)} \in \mathcal{M}(\alpha)$.

(ii) $g(z) = \frac{1}{2\alpha - 1} \{1 - (1 - z)^{2\alpha-1}\} \in \mathcal{N}(\alpha)$.

2 Coefficient estimates for functions

We try to derive sufficient conditions for $f(z)$ which are given by using coefficient inequalities.

Theorem 2.1. *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} \{(n-k) + |n+k-2\alpha|\} |a_n| \leq 2(\alpha-1)$$

for some k ($0 \leq k \leq 1$) and some α ($\alpha > 1$), then $f(z) \in \mathcal{M}(\alpha)$.

Proof. Let us suppose that

$$\sum_{n=2}^{\infty} \{(n-k) + |n+k-2\alpha|\} |a_n| \leq 2(\alpha-1) \quad (1)$$

for $f(z) \in \mathcal{A}$.

It suffices to show that

$$\left| \frac{\frac{zf'(z)}{f(z)} - k}{\frac{zf'(z)}{f(z)} - (2\alpha - k)} \right| < 1 \quad (z \in \mathbb{U}).$$

We note that

$$\begin{aligned} \left| \frac{\frac{zf'(z)}{f(z)} - k}{\frac{zf'(z)}{f(z)} - (2\alpha - k)} \right| &= \left| \frac{1 - k + \sum_{n=2}^{\infty} (n-k)a_n z^{n-1}}{1 + k - 2\alpha + \sum_{n=2}^{\infty} (n+k-2\alpha)a_n z^{n-1}} \right| \\ &\leq \frac{1 - k + \sum_{n=2}^{\infty} (n-k)|a_n||z|^{n-1}}{2\alpha - 1 - k - \sum_{n=2}^{\infty} |n+k-2\alpha||a_n||z|^{n-1}} \\ &< \frac{1 - k + \sum_{n=2}^{\infty} (n-k)|a_n|}{2\alpha - 1 - k - \sum_{n=2}^{\infty} |n+k-2\alpha||a_n|}. \end{aligned}$$

The last expression is bounded above by 1 if

$$1 - k + \sum_{n=2}^{\infty} (n-k)|a_n| \leq 2\alpha - 1 - k - \sum_{n=2}^{\infty} |n+k-2\alpha||a_n|$$

which is equivalent to our condition

$$\sum_{n=2}^{\infty} \{(n-k) + |n+k-2\alpha|\} |a_n| \leq 2(\alpha-1)$$

of the theorem. This completes the proof of the theorem.

If we take $k = 1$ and some $\alpha \left(1 < \alpha \leq \frac{3}{2}\right)$ in Theorem 2.1, then we have

Corollary 2.1. *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} (n - \alpha) |a_n| \leq \alpha - 1$$

for some $\alpha \left(1 < \alpha \leq \frac{3}{2}\right)$, then $f(z) \in \mathcal{M}(\alpha)$.

Example 2.1. The function $f(z)$ given by

$$f(z) = z + \sum_{n=2}^{\infty} \frac{4(\alpha - 1)}{n(n+1)(n-k+|n+k-2\alpha|)} z^n$$

belongs to the class $\mathcal{M}(\alpha)$.

For the class $\mathcal{N}(\alpha)$, we have

Theorem 2.2. *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} n(n-k+1+|n+k-2\alpha|) |a_n| \leq 2(\alpha - 1) \quad (2)$$

for some $k (0 \leq k \leq 1)$ and some $\alpha (\alpha > 1)$, then $f(z)$ belongs to the class $\mathcal{N}(\alpha)$.

Corollary 2.2. *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} n(n - \alpha) |a_n| \leq \alpha - 1$$

for some $\alpha \left(1 < \alpha \leq \frac{3}{2}\right)$, then $f(z) \in \mathcal{N}(\alpha)$.

Example 2.2. The function

$$f(z) = z + \sum_{n=2}^{\infty} \frac{4(\alpha - 1)}{n^2(n+1)(n-k+|n+k-2\alpha|)} z^n$$

belongs to the class $\mathcal{N}(\alpha)$.

Further, denoting by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ the subclasses of $\mathcal{A}$ consisting of all starlike functions of order α , and of all convex functions of order α , respectively, we derive

Theorem 2.3. If $f(z) \in \mathcal{A}$ satisfies the coefficient inequality (1) for some α $\left(1 < \alpha \leq \frac{k+2}{2} \leq \frac{k+2}{2}\right)$ then $f(z) \in \mathcal{S}^* \left(\frac{4-3\alpha}{3-2\alpha}\right)$. If $f(z) \in \mathcal{A}$ satisfies the coefficient inequality (2) for some α $\left(1 < \alpha \leq \frac{k-2}{2} \leq \frac{3}{2}\right)$ then $f(z) \in \mathcal{K} \left(\frac{4-3\alpha}{3-2\alpha}\right)$.

Proof. For some α $\left(1 < \alpha \leq \frac{k+2}{2} \leq \frac{3}{2}\right)$, we see that the coefficient inequality (1) implies that

$$\sum_{n=2}^{\infty} (n - \alpha) |a_n| \leq \alpha - 1.$$

It is well-known that if $f(z) \in \mathcal{A}$ satisfies

$$\sum_{n=2}^{\infty} \frac{n - \beta}{1 - \beta} |a_n| \leq 1$$

for some β ($0 \leq \beta < 1$), then $f(z) \in \mathcal{S}^*(\beta)$ by Silverman [1]. Therefore, we have to find the smallest positive β such that

$$\sum_{n=2}^{\infty} \frac{n - \beta}{1 - \beta} |a_n| \leq \sum_{n=2}^{\infty} \frac{n - \alpha}{\alpha - 1} |a_n| \leq 1.$$

This gives that

$$\beta \leq \frac{(2 - \alpha)n - \alpha}{n - 2\alpha + 1} \quad (3)$$

for all $n = 2, 3, 4, \dots$. Noting that the right hand side of the inequality (3) is increasing for n , we conclude that

$$\beta \leq \frac{4 - 3\alpha}{3 - 2\alpha},$$

which proves that $f(z) \in \mathcal{S}^* \left(\frac{4-3\alpha}{3-2\alpha}\right)$. Similarly, we can show that if $f(z) \in \mathcal{A}$ satisfies (2), then $f(z) \in \mathcal{K} \left(\frac{4-3\alpha}{3-2\alpha}\right)$. □

Our result for the coefficient estimates of functions $f(z) \in \mathcal{M}(\alpha)$ is contained in

Theorem 2.4. If $f(z) \in \mathcal{M}(\alpha)$, then

$$|a_n| \leq \frac{\prod_{j=2}^n (j + 2\alpha - 4)}{(n - 1)!} \quad (n \geq 2). \quad (4)$$

Proof. Let us define the function $p(z)$ by

$$p(z) = \frac{\alpha - \frac{zf'(z)}{f(z)}}{\alpha - 1}$$

for $f(z) \in \mathcal{M}(\alpha)$. Then $p(z)$ is analytic in $\mathbb{U}$, $p(0) = 1$ and $\operatorname{Re}(p(z)) > 0$ ($z \in \mathbb{U}$). Therefore, if we write

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

then $|p_n| \leq 2$ ($n \geq 1$). Since

$$\alpha f(z) - z f'(z) = (\alpha - 1)p(z)f(z),$$

we obtain that

$$(1 - n)a_n = (\alpha - 1)(p_{n-1} + a_2 p_{n-2} + a_3 p_{n-3} + \cdots + a_{n-1} p_1).$$

If $n = 2$, then $-a_2 = (\alpha - 1)p_1$ implies that

$$|a_2| = (\alpha - 1)|p_1| \leq 2\alpha - 2.$$

Thus the coefficient estimate (4) holds true for $n = 2$. Next, suppose that the coefficient estimate

$$|a_k| \leq \frac{\prod_{j=2}^k (j + 2\alpha - 4)}{(k - 1)!}$$

is true for all $k = 2, 3, 4, \dots, n$. Then we have that

$$-na_{n+1} = (\alpha - 1)(p_n + a_2 p_{n-1} + a_3 p_{n-2} + \cdots + a_n p_1),$$

so that

$$\begin{aligned} n|a_{n+1}| &\leq (2\alpha - 2)(1 + |a_2| + |a_3| + \cdots + |a_n|) \\ &\leq (2\alpha - 2) \left(1 + (2\alpha - 2) + \frac{(2\alpha - 2)(2\alpha - 1)}{2!} + \cdots + \frac{\prod_{j=2}^n (j + 2\alpha - 4)}{(n - 1)!} \right) \\ &= (2\alpha - 2) \left(\frac{(2\alpha - 1)2\alpha(2\alpha + 1) \cdots (2\alpha + n - 4)}{(n - 2)!} + \frac{(2\alpha - 2)(2\alpha - 1)2\alpha \cdots (2\alpha + n - 4)}{(n - 1)!} \right) \\ &= \frac{\prod_{j=2}^{n+1} (j + 2\alpha - 4)}{(n - 1)!}. \end{aligned}$$

Thus, the coefficient estimate (4) holds true for the case of $k = n + 1$. Applying the mathematical induction for the coefficient estimate (4), we complete the proof of the theorem. □

For the functions $f(z)$ belonging to the class $\mathcal{N}(\alpha)$, we also have

Theorem 2.5. *If $f(z) \in \mathcal{N}(\alpha)$, then*

$$|a_n| \leq \frac{\prod_{j=2}^n (j + 2\alpha - 4)}{n!} \quad (n \geq 2).$$

Remark 2.1. We can not show that Theorem 2.4 and Theorem 2.5 are sharp. If we prove that Theorem 2.4 is sharp, then the sharpness of Theorem 2.5 follows.

References

- [1] H.Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. **51**(1975), 109 – 116.
- [2] B.A.Uralegaddi, M.D.Ganigi and S.M.Sarangi, *Univalent functions with positive coefficients*, Tamkang J. Math. **25**(1994), 225 – 230.

*Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502
Japan*