

A random walk with random obstacles and a tagged particle of an infinite hard core particle system in $\mathbf{R}^d$

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Let $\mathfrak{M}$ be the set of all countable subsets η of $\mathbf{R}^d$ satisfying $N_K(\eta) < \infty$ for any compact subset K , where $N_A(\eta)$ is the number of points of η in $A \subset \mathbf{R}^d$ ($d \geq 2$). We regard $\eta \in \mathfrak{M}$ as a non-negative integer valued Radon measure on $\mathbf{R}^d$: $\eta(\cdot) = \sum_{x \in \eta} \delta_x(\cdot)$ and accordingly equip $\mathfrak{M}$ with the vague topology, where δ_x denotes the δ -measure at x . We define σ -fields $\mathcal{B}(\mathfrak{M})$ and $\mathcal{B}_K(\mathfrak{M})$ by

$$\mathcal{B}(\mathfrak{M}) = \sigma(N_A; A \in \mathcal{B}(\mathbf{R}^d)),$$

and

$$\mathcal{B}_K(\mathfrak{M}) = \sigma(N_A; A \in \mathcal{B}(\mathbf{R}^d), A \subset K).$$

The σ -field $\mathcal{B}(\mathfrak{M})$ coincides with the topological Borel field of $\mathfrak{M}$.

For any $\eta \in \mathfrak{M}$ we define a measurable kernel $q_\eta(x, dy)$ on $\mathbf{R}^d \times \mathcal{B}(\mathbf{R}^d)$ by

$$q_\eta(x, dy) = p(|x - y|) \chi(x|\eta) \chi(y|\eta) dy,$$

where $p(\cdot)$ is a non-negative function on $[0, \infty)$ satisfying

$$(p.1) \quad \int_{\mathbf{R}^d} dx p(|x|) = 1,$$

$$(p.2) \quad \int_{\mathbf{R}^d} dx |x|^2 p(|x|) < \infty,$$

$$(p.3) \quad \{\alpha \in [0, \infty) : p(\alpha) > 0\} = [0, h), \quad \text{for some } h \in (0, \infty],$$

$$(p.4) \quad \text{ess. inf}\{p(\alpha) : \alpha \in [0, c)\} > 0 \quad \text{for any } c \in (0, h),$$

and for any $\eta \in \mathfrak{M}$ and $x \in \mathbf{R}^d$

$$\chi(x|\eta) = \exp\left\{-\sum_{y \in \eta} \Psi(|x - y|)\right\}.$$

Here Ψ is a given measurable function on $[0, \infty)$ which is bounded from below and satisfies

$$(\Psi.1) \quad \Psi(\alpha) = \infty \quad \text{if and only if } \alpha \in [0, r),$$

$$(\Psi.2) \quad \Psi(\alpha) = 0 \quad \text{if } \alpha \in [r_0, \infty),$$

for some positive constants r and r_0 with $r \leq r_0$.

Let $C_\infty(\mathbf{R}^d)$ be the space of continuous functions φ on $\mathbf{R}^d$ such that $\varphi(x) \rightarrow 0$ as $|x| \rightarrow \infty$. We denote by $(\Omega, \mathcal{F}, P_\eta, x(t))$ the right continuous Markov process starting from 0 with generator

$$L_\eta \varphi(x) = \int_{\mathbf{R}^d} q_\eta(x, dy) \{\varphi(y) - \varphi(x)\}, \quad \varphi \in C_\infty(\mathbf{R}^d).$$

For any probability measure ν on $\mathcal{M}$ we write $P_\nu = \int_{\mathcal{M}} \nu(d\eta) P_\eta$. We call the process $(\Omega, \mathcal{F}, P_\nu, x(t))$ a random walk with random obstacles.

Denote the r -neighborhood of $A \subset \mathbf{R}^d$ by $U_r(A)$ and abbreviate $U_r(\{x\})$ to $U_r(x)$. For $x \in \mathbf{R}^d$ and $\eta \in \mathcal{M}$ put

$$C(x, \eta) = \begin{cases} A_{x, \eta} \setminus \overline{U_r(\eta)}, & x \notin \overline{U_r(\eta)}, \\ \emptyset, & x \in \overline{U_r(\eta)}, \end{cases}$$

where $A_{x, \eta}$ is the connected component of $U_{\frac{1}{2}}(U_r(\eta)^c)$ containing x . We call the set $C(x, \eta)$ the cluster containing x for η . Define a measurable subset $\mathcal{M}^*$ of $\mathcal{M}$ by

$$\mathcal{M}^* = \{\eta \in \mathcal{M} : |C(0, \eta)| = \infty\}.$$

For a probability measure on $\mathcal{M}$ satisfying $\mu(\mathcal{M}^*) > 0$, we define

$$\mu^*(d\eta) = \frac{\mathbb{I}_{\mathcal{M}^*}(\eta)}{\mu(\mathcal{M}^*)} \mu(d\eta),$$

where $\mathbb{I}_A$ stands for the indicator function for a set A .

We study the asymptotic behavior of $(x(t), P_{\mu^*})$ in the case where μ is a Gibbs state. We introduce terminologies for Gibbs states. Let Φ be a real valued measurable function on $[0, \infty)$ which is bounded from below and satisfies the following condition ($\Phi.1$) called *regularity condition*:

$$(\Phi.1) \quad \int_{\mathbf{R}^d} dx |\exp(-\Phi(|x|)) - 1| < \infty.$$

Next we assume either one of the following conditions ($\Phi.2$) and ($\Phi.2'$):

$$(\Phi.2) \quad \Phi(\cdot) \geq 0,$$

$$(\Phi.2') \quad (i) \text{ There exists a positive number } r' \text{ such that}$$

$$\Phi(\alpha) = \infty, \quad \text{if and only if } \alpha \in [0, r'),$$

$$(ii) \text{ There exists a non-negative number } c_0 \text{ such that}$$

$$\sum_{i=1}^m \Phi(|x_i|) \geq -c_0$$

for all m and $x_1, x_2, \dots, x_m \in \mathbf{R}^d$ with $|x_i - x_j| \geq r'$ for $i \neq j$.

Φ is regarded as a pair potential which is rotation invariant and translation invariant. For $x_1, x_2, \dots, x_n \in \mathbf{R}^d$ and $\eta \in \mathfrak{M}$ we associate a potential energy

$$U(x_1, x_2, \dots, x_n | \eta) = \sum_{1 \leq i < j \leq n} \Phi(|x_i - x_j|) + \sum_{i=1}^n \sum_{y \in \eta} \Phi(|x_i - y|).$$

For any compact subset $K \subset \mathbf{R}^d$, we denote by $\mathfrak{M}(K)$ and $\mathfrak{M}(K, n)$ the set of all finite subsets of K and the set of all subsets of K having n points, respectively. An alternative description of $\mathfrak{M}(K, n)$ is given by

$$(1) \quad \mathfrak{M}(K, n) = \begin{cases} \{\emptyset\}, & \text{if } n = 0, \\ (K^n)' / S_n, & \text{if } n \geq 1, \end{cases}$$

where $(K^n)' = \{(x_1, x_2, \dots, x_n) \in K^n : x_i \neq x_j \text{ if } i \neq j\}$ and S_n is the symmetric group of degree n . By means of the factorization (1) we introduce a measure $\lambda_{K,z}$ on $\mathfrak{M}(K) = \bigcup_{n=0}^{\infty} \mathfrak{M}(K, n)$ (direct sum) such that

$$\lambda_{K,z}(\emptyset) = 1,$$

and

$$\lambda_{K,z}(A) = \frac{z^n}{n!} \int_A dx_1 dx_2 \cdots dx_n,$$

for a Borel set A in $\mathfrak{M}(K, n)$, $n \geq 1$, where $z \geq 0$ and $\tilde{A}$ is a preimage of A by the factor mapping in the factorization (1.3).

Now, we are going to define a Gibbs state.

Definition 1.1. A probability measure μ on $\mathfrak{M}$ is called a Gibbs state with respect to the activity $z \geq 0$ and the potential Φ , if for any compact subset K of $\mathbf{R}^d$,

$$\mu(\cdot | \mathcal{B}_{K^c}(\mathfrak{M}))(\eta) = \mu_{K,\eta,z}(\cdot), \quad \mu\text{-a.s. } \eta,$$

where $\mu_{K,\eta,z}$ is the probability measure on $\mathfrak{M}(K)$ defined by

$$\begin{aligned} \mu_{K,\eta,z}(d\mathbf{x}) &= \frac{1}{Z_{K,\eta,z}} \exp\{-U(\mathbf{x} | \eta \cap K^c)\} \lambda_{K,z}(d\mathbf{x}), \\ Z_{K,\eta,z} &= \int_{\mathfrak{M}(K)} \lambda_{K,z}(d\mathbf{x}) \exp\{-U(\mathbf{x} | \eta \cap K^c)\}. \end{aligned}$$

Denote by $\mathcal{G}(z, \Phi)$ the set of all Gibbs states with respect to the activity $z \geq 0$ and the potential Φ , and by $\mathcal{G}_0(z, \Phi)$ the set of all elements of $\mathcal{G}(z, \Phi)$ which are translation invariant.

Remark 1.1. (i) The set $\mathcal{G}_0(z, \Phi)$ is convex and any element of $\mathcal{G}_0(z, \Phi)$ is represented by the extremal points of $\mathcal{G}_0(z, \Phi)$, which are characterized by their ergodicity under translation (see [1]). We denote the set of all extremal points of $\mathcal{G}_0(z, \Phi)$ by $\text{ex}\mathcal{G}_0(z, \Phi)$. If $\|\mathcal{G}(z, \Phi) = 1$ and $\mu \in \mathcal{G}(z, \Phi)$, then μ is rotation invariant, translation invariant and ergodic under translation.

(ii) There exists a positive constant $z_1 > 0$ such that if $z \in (0, z_1)$ and $\mu \in \mathcal{G}(z, \Phi)$, then $\mu(\mathcal{M}^*) > 0$. In particular, $z_1 = \infty$ in case $h = \infty$.

Now, we shall state our first main result.

Theorem 1.1. *There exists $z_2 \in (0, z_1]$ such that if $z \in (0, z_2)$ and $\mu \in \text{ex}\mathcal{G}(z, \Phi)$, then the process $\varepsilon x(\frac{t}{\varepsilon})$ on $(\Omega, \mathcal{F}, P_\mu)$ converges to $D^*B(t)$ as $\varepsilon \rightarrow 0$ in distribution with respect to J_1 -topology on Skorohod's function space $\mathbf{D}[0, \infty)$, where $B(t)$ is a d -dimensional Brownian motion and D^* is a positive definite $d \times d$ -matrix. In particular, $z_2 = \infty$ in case $h = \infty$.*

In the previous paper [2] we studied a system of infinitely many hard balls with the same diameter r moving discontinuously in $\mathbf{R}^d$. We denote the configuration space of hard balls by $\mathcal{X}$:

$$\mathcal{X} = \{\xi = \{x_i\} : |x_i - x_j| \geq r, i \neq j\},$$

the position of a ball being represented by its center. The space $\mathcal{X}$ is a compact subset of $\mathcal{M}$ with the vague topology.

Let $\mathbf{C}(\mathcal{X})$ be the space of all real valued continuous functions on $\mathcal{X}$ and $\mathbf{C}_0(\mathcal{X})$ be the set of functions of $\mathbf{C}(\mathcal{X})$ each of which depends only on the configurations in some compact set K . The system is described by the $\mathcal{X}$ -valued Markov process ξ_t whose generator is the smallest closed extension of the operator $\bar{K}$ on $\mathbf{C}_0(\mathcal{X})$ given by

$$\bar{K}f(\xi) = \sum_{x \in \xi} \int_{\mathbf{R}^d} dy \{f(\xi^{x,y}) - f(\xi)\} p(|x - y|) \chi(y | \xi \setminus \{x\}), \quad f \in \mathbf{C}_0(\mathcal{X}),$$

where

$$\xi^{x,y} = \begin{cases} (\xi \setminus \{x\}) \cup \{y\} & \text{if } x \in \xi, y \notin \xi, \\ \xi & \text{otherwise.} \end{cases}$$

We study the behavior of a tagged particle in the process. In order to follow the motion of the tagged particle it is convenient to regard the process ξ_t as a Markov process $(y(t), \zeta_t)$ on the locally compact space $\mathbf{R}^d \times \mathcal{X}_0$, where

$$\mathcal{X}_0 = \{\zeta \in \mathcal{X} : \zeta \cap U_r(0) = \emptyset\}.$$

$y(t)$ is the position of the tagged particle and ζ_t is the entire configuration seen from the tagged particle. We can see that ζ_t is a Markov process whose generator $\bar{K}$ is

the smallest closed extension of the operator on $C_0(\mathfrak{X}_0)$ given by

$$\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2,$$

$$\mathcal{K}_1 f(\zeta) = \int_{\mathbf{R}^d} du \{f(\tau_{-u}\zeta) - f(\zeta)\} p(|u|) \chi(u|\zeta),$$

$$\mathcal{K}_2 f(\zeta) = \sum_{x \in \zeta} \int_{U_r(0)^c} dy \{f(\zeta^{x,y}) - f(\zeta)\} p(|x-y|) \chi(y|\zeta \setminus \{x\}), \quad f \in C_0(\mathfrak{X}_0),$$

where $C(\mathfrak{X}_0)$ and $C_0(\mathfrak{X}_0)$ are defined by the same way as $C(\mathfrak{X})$ and $C_0(\mathfrak{X})$, respectively. We denote by S_t the semigroup associated with generator $\mathcal{K}$ and by $(\Omega, \mathcal{F}, P_\nu^0, \zeta_t)$ the associated process with initial distribution ν .

For any $\mu \in \mathcal{G}(z, \Psi)$ we define

$$\mu_0(d\eta) = \frac{\chi(0|\eta)}{c_3} \mu(d\eta),$$

where $c_3 = \int_{\mathfrak{M}} \chi(0|\eta) \mu(d\eta)$. In [2] we proved that there exists $z_3 \in (0, \infty)$ such that if $z \in (0, z_3)$ and if $\|\mathcal{G}(z, \Psi) = 1$ and $\mu \in \mathcal{G}(z, \Psi)$, then $(P_{\mu_0}^0, \zeta_t)$ is an ergodic reversible Markov process.

The process $y(t)$ is driven by the process ζ_t in the following way. Let $A \in \mathcal{B}(\mathbf{R}^d)$ and let $\Xi(A)$ be the measurable subset of $\mathfrak{X}_0 \times \mathfrak{X}_0$ defined by

$$\Xi(A) = \{(\eta, \zeta) \in (\mathfrak{X}_0 \times \mathfrak{X}_0) \setminus \Delta : \zeta = \tau_{-u}\eta \text{ for some } u \in A\},$$

where

$$\Delta = \{(\zeta, \zeta) : \zeta \in \mathfrak{X}_0\} \cup \{(\zeta \in \mathfrak{X}_0 : \zeta = \tau_{-u}\zeta \text{ for some } u \in \mathbf{R}^d \setminus \{0\})^2\}.$$

Define a σ -finite random measure N by

$$N((0, t] \times A) = \sum_{s \in (0, t]} \mathbf{1}_{\Xi(A)}(\eta_{s-}, \eta_s), \quad t > 0.$$

Then,

$$y(t) = y(0) + \int_0^t \int_{\mathbf{R}^d} N(ds du) u.$$

Our second main result is the following theorem.

Theorem 1.2. *If $z \in (0, z_2 \wedge z_3)$ and if $\|\mathcal{G}(z, \Psi) = 1$ and $\mu \in \mathcal{G}(z, \Psi)$, then the process $\varepsilon y(\frac{t}{\varepsilon})$ on $(\Omega, \mathcal{F}, P_{\mu_0}^0)$ converges to $\sigma_0 B(t)$ as $\varepsilon \rightarrow 0$ in distribution with respect to J_1 -topology on Skorohod's function space $\mathbf{D}[0, \infty)$, where σ_0 is a positive constant.*

REFERENCES

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