

Remarks on Sakaguchi Functions

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1. Introduction

Let A denote the class of functions f which are analytic in the unit disc $D=\{z: |z|<1\}$, with

$$(1.1) \quad f(0)=0 \text{ and } f'(0)=1.$$

We denote some subclasses of A as follows:

$$(1.2) \quad S^* = \left\{ f \in A: \operatorname{Re} \frac{z f'(z)}{f(z)} > 0, z \in D \right\},$$

$$(1.3) \quad S = \left\{ f \in A: \operatorname{Re} \frac{z f'(z)}{f(z) - f(-z)} > 0, z \in D \right\}$$

and

$$(1.4) \quad R(\alpha) = \left\{ f \in A: \operatorname{Re} \frac{f(z)}{z} > \alpha, z \in D \right\}$$

where $0 \leq \alpha < 1$.

S^* is the usual class of starlike functions, and S is the class of Sakaguchi functions introduced by Sakaguchi in [2]. For relations between these two classes only the following result is known.

Theorem A (Sakaguchi [2]). $f(z) \in S$ if and only if

$$(1.5) \quad \frac{f(z) - f(-z)}{2} \in S^*.$$

For $R(\alpha)$, Wu posed the following conjecture in [3].

Conjecture

If $f(z) \in S$, then $f(z) \in R\left(\frac{1}{2}\right)$.

And the present author in [4] showed by the counter-example

$$(1.6) \quad f(z) = z + \frac{3}{5}z^2 + \frac{1}{15}z^3$$

that the conjecture is not true.

In this short paper we give two examples which show that

$$(1.7) \quad S^* \not\subset S$$

and

$$(1.8) \quad S \not\subset S^*.$$

2. Examples

Example 1. An example which shows the relation (1.7).

$$(2.1) \quad f(z) = z + \frac{4}{5}z^2 + \frac{1}{5}z^3.$$

If we let

$$\frac{zf'(z)}{f(z)} = \frac{5+8z+3z^2}{5+4z+z^2} = \frac{u+iv}{s+it}$$

and $z = re^{i\theta}$, then we have

$$s = 5 + 4r \cos \theta + r^2 \cos 2\theta, \quad t = 4r \sin \theta + r^2 \sin 2\theta$$

$$u = 5 + 8r \cos \theta + 3r^2 \cos 2\theta, \quad v = 8r \sin \theta + 3r^2 \sin 2\theta.$$

and it is evident that $\operatorname{Re} [zf'/f] > 0$ if and only if

$$su + tv > 0.$$

Since we easily deduce

$$su + tv = 40r^2 \cos^2 \theta + 20r(3 + r^2) \cos \theta + 25 + 12r^2 + 3r^4$$

$$\geq 25 - 60r + 52r^2 - 20r^3 + 3r^4 \quad (\cos \theta = -1)$$

$$= (1 - r)(25 - 35r + 17r^2 - 3r^3),$$

where

$$25 - 35r + 17r^2 - 3r^3 > 0 \quad (0 \leq r < 1).$$

we deduce $su + tv > 0$, which shows that $f \in S^*$.

On the other hand, if we put $z_0 = -\frac{3}{5} + \frac{4}{5}i$, then we have

$$\operatorname{Re} \frac{z_0 f'(z_0)}{f(z_0) - f(-z_0)} < 0,$$

which shows that $f \notin S$.

Example 2. An example for the relation (1.8).

$$(2.2) \quad f(z) = \frac{1}{2}z + z^2 + \log \frac{2+z}{2}$$

To show that the above function f belongs to S , we use the following theorem due to Miller and Mocanu.

Theorem B (Miller and Mocanu [1]). If $f(z) \in A$ satisfies

$$(2.3) \quad \left| \frac{zf''(z)}{f'(z)} \right| < 2 \quad (z \in D)$$

then $f(z)$ belongs to S^* .

If we let

$$g(z) = \frac{f(z) - f(-z)}{2},$$

then we obtain

$$\left| \frac{z g''(z)}{g'(z)} \right| = \left| \frac{8z^2}{(4-z^2)(8-z^2)} \right| < \frac{8|z|^2}{(4-|z|^2)(8-|z|^2)} < \frac{8}{21} \quad (z \in D).$$

Hence from Theorem B, we deduce $g(z) \in S^*$. Therefore, by using Theorem A, we have $f(z) \in S$.

On the other hand, if we put $z_0 = -5/8 \in D$, then we have

$$\operatorname{Re} \frac{z_0 f'(z_0)}{f(z_0)} = -0.047\ldots < 0,$$

which yields $f \notin S^*$.

References

- [1] S.S. Miller and P.T. Mocanu, On some class of first order differential subordinations, Michigan Math. J., 32(1985), 185-195.
- [2] K. Sakaguchi, On a certain univalent mapping, J. Math. Soc. Japan 11(1959), 72-75.
- [3] Z. Wu, On classes of Sakaguchi functions and Hadamard products, Sci. Sinica 30(1987), 128-135.
- [4] R. Yamakawa, Notes for Sakaguchi functions, (1991, preprint).