

EQUIVARIANT LOCAL INDEX AND SYMPLECTIC CUT

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1. INTRODUCTION

This is a survey article of [7]. In the joint work [1, 2, 3] with Fujita and Furuta we developed an index theory for Dirac-type operators on possibly non-compact Riemannian manifolds. We call the index in our theory the *local index* and also call its equivariant version the *equivariant local index*. As an application to Hamiltonian S^1 -actions on prequantizable closed symplectic manifolds we can show that the equivariant Riemann-Roch index is obtained as the sum of the equivariant local indices for the inverse images of the integer lattice points by the moment map. When the lattice point is a regular value of the moment map we can compute its equivariant local index, see [3, 6]. So the problem is how to compute the equivariant local index when the lattice point is a critical value. The purpose of this paper is to give a formula for the equivariant local index for the reduced space in a symplectic cut space which is a special case of critical lattice points (Theorem 3.4).

This paper is organized as follows. In Section 2 we briefly recall the equivariant local index in the case of the Hamiltonian S^1 -actions. After that we give the setting and the main theorem in Section 3.

Notation. In this paper we use the notation $\mathbb{C}_{(n)}$ for the irreducible representation of S^1 with weight n .

2. EQUIVARIANT LOCAL INDEX

Let (M, ω) be a prequantizable Hamiltonian S^1 -manifold and (L, ∇^L) an S^1 -equivariant prequantum line bundle on (M, ω) . We do not assume M is compact. Since all orbits are isotropic the restriction of (L, ∇^L) to each orbit is flat.

Definition 2.1. An orbit $\mathcal{O}$ is said to be *L-acyclic* if $H^0(\mathcal{O}; (L, \nabla^L)|_{\mathcal{O}}) = 0$.

Let V be an S^1 -invariant open set whose complement is compact and which contains only *L-acyclic* orbits. For these data we give the following theorem.

Theorem 2.2 ([1, 2, 3]). *There exists an element $\text{ind}_{S^1}(M, V; L) \in R(S^1)$ of the representation ring such that $\text{ind}_{S^1}(M, V; L)$ satisfies the following properties:*

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- (1) $\text{ind}_{S^1}(M, V; L)$ is invariant under continuous deformation of the data.
- (2) If M is closed, then $\text{ind}_{S^1}(M, V; L)$ is equal to the equivariant Riemann-Roch index $\text{ind}_{S^1}(M; L)$.
- (3) If M' is an S^1 -invariant open neighborhood of $M \setminus V$, then $\text{ind}_{S^1}(M, V; L)$ satisfies the following excision property

$$\text{ind}_{S^1}(M, V; L) = \text{ind}_{S^1}(M', M' \cap V; L|_{M'}).$$

- (4) $\text{ind}_{S^1}(M, V; L)$ satisfies a product formula.

We call $\text{ind}_{S^1}(M, V; L)$ the *equivariant local index*.

Example 2.3. For small positive real number $\varepsilon > 0$ which is less than 1 let $D_\varepsilon(\mathbb{C}_{(1)}) = \{z \in \mathbb{C}_{(1)} \mid |z| < \varepsilon\}$ be the 2-dimensional disc of radius ε . As $(L, \nabla^L) \rightarrow (M, \omega)$ we consider

$$\left(D_\varepsilon(\mathbb{C}_{(1)}) \times \mathbb{C}_{(m)}, d + \frac{1}{2}(zd\bar{z} - \bar{z}dz) \right) \rightarrow \left(D_\varepsilon(\mathbb{C}_{(1)}), \frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z} \right).$$

First let us detect non L -acyclic orbits. Suppose the orbit $\mathcal{O}$ through $z \in D_\varepsilon(\mathbb{C}_{(1)})$ has a non-trivial parallel section $s \in H^0(\mathcal{O}; (L, \nabla)|_{\mathcal{O}})$. Then s satisfies the following equation

$$0 = \nabla_{\partial_\theta}^L s = \frac{\partial s}{\partial \theta} - 2\pi\sqrt{-1}r^2 s,$$

where we use the polar coordinates $z = re^{2\pi\sqrt{-1}\theta}$. Hence s is of the form $s(\theta) = s_0 e^{2\pi\sqrt{-1}r^2\theta}$ for some non-zero constant s_0 . Since s is a global section on $\mathcal{O}$ s satisfies $s(0) = s(1)$. This implies $r = 0$.

Next, we put $V = D_\varepsilon(\mathbb{C}_{(1)}) \setminus \{0\}$ and let us compute $\text{ind}_{S^1}(M, V; L)$. We recall the definition of $\text{ind}_{S^1}(N, V; L)$. For $t \geq 0$ consider the following perturbation of the Spin^c Dirac operator $D: \Gamma(\wedge^{0,*} T^*M \otimes L) \rightarrow \Gamma(\wedge^{0,*} T^*M \otimes L)$ associated with the standard Hermitian structure on $M = D_\varepsilon(\mathbb{C}_{(1)})$

$$D_t = D + t\rho D_{S^1},$$

where ρ is a cut-off function of V and D_{S^1} is a first order formally self-adjoint differential operator of degree-one

$$D_{S^1}: \Gamma((\wedge^* T^* M^{0,1} \otimes L)|_V) \rightarrow \Gamma((\wedge^* T^* M^{0,1} \otimes L)|_V)$$

that satisfies the following conditions:

- (1) D_{S^1} contains only derivatives along orbits.
- (2) The restriction $D_{S^1}|_{\mathcal{O}}$ to an orbit $\mathcal{O}$ is the de Rham operator with coefficients in $L|_{\mathcal{O}}$.
- (3) For any S^1 -equivariant section u of the normal bundle $\nu_{\mathcal{O}}$ of $\mathcal{O}$ in M , D_{S^1} anti-commutes with the Clifford multiplication of u .

See [1, 2, 3] for more details. From the second condition and $\{0\}$ is the unique non L -acyclic orbit we can see $\ker(D_{S^1}|_{\mathcal{O}}) = 0$ for all orbits $\mathcal{O} \neq \{0\}$. Extend the complement of a neighborhood of 0 in $D_\varepsilon(\mathbb{C}_{(1)})$ cylindrically so that all the data are translationally invariant. Then we showed in [1, 2] that for a sufficiently large t D_t is Fredholm, namely, $\ker D_t \cap L^2$ is finite

dimensional and its super-dimension is independent of a sufficiently large t . So we define

$$\text{ind}_{S^1}(M, V; L) = \ker D_t^0 \cap L^2 - \ker D_t^1 \cap L^2$$

for a sufficiently large t . In this case, by the direct computation using the Fourier expansion of s with respect to θ , we can show that

$$\ker D_t^0 \cap L^2 \cong \mathbb{C}, \quad \ker D_t^1 \cap L^2 = 0,$$

and $\ker D_t^0 \cap L^2$ is spanned by a certain L^2 -function $a_0(r)$ on $D_\varepsilon(\mathbb{C}_{(1)})$ which depends only on $r = |z|$. Since the S^1 -action on $\ker D_t^0 \cap L^2$ is given by pull-back and the S^1 -action on the fiber is given by $\mathbb{C}_{(m)}$ we obtain

$$\begin{aligned} \text{ind}_{S^1}(M, V; L) &= \text{ind}_{S^1}(D_\varepsilon(\mathbb{C}_{(1)}), D_\varepsilon(\mathbb{C}_{(1)}) \setminus \{0\}; D_\varepsilon(\mathbb{C}_{(1)}) \times \mathbb{C}_{(m)}) \\ &= \mathbb{C}_{(-m)}. \end{aligned}$$

For more details see [1, Remark 6.10], or [5, Section 5.3.2].

It is well-known that the lift of S^1 -action on M to L defines the moment map $\mu: M \rightarrow \mathbb{R}$ by the Kostant formula

$$(2.1) \quad \mathcal{L}_X s = \nabla_X^L s + 2\pi\sqrt{-1}\mu s,$$

where s is a section of L , X is the vector field which generates the S^1 -action on (M, ω) , and $\mathcal{L}_X s$ is the Lie derivative which is defined by

$$\mathcal{L}_X s(x) = \frac{d}{d\theta} \Big|_{\theta=0} e^{-2\pi\sqrt{-1}\theta} s(e^{2\pi\sqrt{-1}\theta} x).$$

Lemma 2.4. *If an orbit $\mathcal{O}$ is not L -acyclic, namely, $H^0(\mathcal{O}; (L, \nabla^L)|_{\mathcal{O}}) \neq 0$, then, $\mu(\mathcal{O}) \in \mathbb{Z}$.*

If M is closed, then we have the following localization formula for the equivariant Riemann-Roch index.

Corollary 2.5. *Suppose M is closed. For $i \in \mu(M) \cap \mathbb{Z}$ let V_i be an S^1 -invariant open neighborhood of $\mu^{-1}(i)$ such that they are mutually disjoint, namely, $V_i \cap V_j \neq \emptyset$ for all $i \neq j$. Then,*

$$(2.2) \quad \text{ind}_{S^1}(M; L) = \bigoplus_{i \in \mu(M) \cap \mathbb{Z}} \text{ind}_{S^1}(V_i, V_i \cap V; L|_{V_i}).$$

3. THE SETTING AND THE MAIN THEOREM

Let (M, ω) be a Hamiltonian S^1 -space with moment map $\mu: M \rightarrow \mathbb{R}$. For a real number n the cut space $\overline{M}_{\mu \leq n}$ of (M, ω) by the symplectic cutting [4] is the reduced space of the diagonal S^1 -action on $(M, \omega) \times \left(\mathbb{C}_{(1)}, \frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z}\right)$, namely,

$$\overline{M}_{\mu \leq n} = \left\{ (x, z) \in (M, \omega) \times \left(\mathbb{C}_{(1)}, \frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z}\right) \mid \mu(x) + |z|^2 = n \right\} / S^1.$$

We denote the reduced space $\mu^{-1}(n)/S^1$ by M_n .

Proposition 3.1. (1) If S^1 acts on $\mu^{-1}(n)$ freely, then, $\overline{M}_{\mu \leq n}$ is a smooth Hamiltonian S^1 -space. The S^1 -action is given as

$$(3.1) \quad t[x, z] = [tx, z]$$

for $t \in S^1$ and $[x, z] \in \overline{M}_{\mu \leq n}$.

(2) Under the assumption in (1), the reduced space M_n and $\{x \in M \mid \mu(x) \leq n\}$ are symplectically embedded into $\overline{M}_{\mu \leq n}$ by $M_n \ni [x] \mapsto [x, 0] \in \overline{M}_{\mu \leq n}$ and $\{x \in M \mid \mu(x) \leq n\} \ni x \mapsto [x, \sqrt{n - \mu(x)}] \in \overline{M}_{\mu \leq n}$, respectively. In particular, $\overline{M}_{\mu \leq n}$ can be identified with the disjoint union $\{x \in M \mid \mu(x) \leq n\} \amalg M_n$ and with this identification M_n is fixed by the S^1 -action (3.1).

Suppose that (M, ω) is equipped with a prequantum line bundle $(L, \nabla^L) \rightarrow (M, \omega)$ and the S^1 -action lifts to (L, ∇^L) in such a way that μ satisfies the Kostant formula (2.1).

Proposition 3.2. If n is an integer and the S^1 -action on $\mu^{-1}(n)$ is free, then $\overline{M}_{\mu \leq n}$ is prequantizable. In this case a prequantum line bundle $(\overline{L}, \nabla^{\overline{L}})$ on $\overline{M}_{\mu \leq n}$ is given by

$$(\overline{L}, \nabla^{\overline{L}}) = ((L, \nabla^L) \otimes \mathbb{C}_{(n)}) \boxtimes \left(\mathbb{C}_{(1)} \times \mathbb{C}_{(0)}, d + \frac{1}{2}(zd\bar{z} - \bar{z}dz) \right) \Big|_{\Phi^{-1}(0)} / S^1,$$

where Φ is the moment map $\Phi: M \times \mathbb{C}_{(1)} \rightarrow \mathbb{R}$ associated to the lift of the diagonal S^1 -action which is written as $\Phi(x, z) = \mu(x) + |z|^2 - n$, and the lift of the S^1 -action (3.1) on $\overline{M}_{\mu \leq n}$ to $(\overline{L}, \nabla^{\overline{L}})$ is given by

$$(3.2) \quad t[u \otimes v \boxtimes (z, w)] = [(tu) \otimes v \boxtimes (z, w)]$$

for $t \in S^1$ and $[u \otimes v \boxtimes (z, w)] \in \overline{L}$. The moment map $\overline{\mu}: \overline{M}_{\mu \leq n} \rightarrow \mathbb{R}$ associated with the lift (3.2) is written as $\overline{\mu}([x, z]) = \mu(x) = n - |z|^2$.

See [4] for more details.

Remark 3.3. We denote the restriction of $(\overline{L}, \nabla^{\overline{L}})$ to M_n by (L_n, ∇^{L_n}) . (L_n, ∇^{L_n}) is a prequantum line bundle on M_n . The S^1 -action (3.2) on L_n is given by the fiberwise multiplication with weight n . Recall that M_n is fixed by the S^1 -action (3.1). See Proposition 3.1.

Suppose that n be an integer and the S^1 -action on $\mu^{-1}(n)$ is free. Then, the cut space $\overline{M}_{\mu \leq n}$ becomes a prequantizable Hamiltonian S^1 -manifold and the S^1 -equivariant prequantum line bundle $(\overline{L}, \nabla^{\overline{L}})$ is given by Proposition 3.1 and Proposition 3.2.

Suppose also that $\mu^{-1}(n)$ is compact. We take a sufficiently small S^1 -invariant open neighborhood O of M_n in $\overline{M}_{\mu \leq n}$ so that the intersection $\overline{\mu}(O) \cap \mathbb{Z}$ consists of the unique point n . Then we can define the equivariant local index $\text{ind}_{S^1}(O, O \setminus M_n; \overline{L}|_O)$ of M_n in $\overline{M}_{\mu \leq n}$. We give the following formula for $\text{ind}_{S^1}(O, O \setminus M_n; \overline{L}|_O)$.

Theorem 3.4. Let (M, ω) , (L, ∇^L) , and μ be as above. Let n be an integer. Suppose S^1 acts on $\mu^{-1}(n)$ freely and $\mu^{-1}(n)$ is compact. Let O be a sufficiently small S^1 -invariant open neighborhood of M_n in $\overline{M}_{\mu \leq n}$ which satisfies $\overline{\mu}(O) \cap \mathbb{Z} = \{n\}$. Then, the equivariant local index is given as

$$\text{ind}_{S^1}(O, O \setminus M_n; \overline{L}|_O) = \text{ind}(M_n; L_n) \mathbb{C}_{(n)},$$

where $\text{ind}(M_n; L_n)$ is the Riemann-Roch number of M_n .

Remark 3.5. By replacing $\mathbb{C}_{(1)}$ with $\mathbb{C}_{(-1)}$ in the above construction we obtain the other cut space $\overline{M}_{\mu \geq n} = \{(x, z) \in M \times \mathbb{C}_{-1} : \mu(x) - |z|^2 = n\} / S^1$. Theorem 3.4 also holds for $\overline{M}_{\mu \geq n}$.

The outline of the proof of Theorem 3.4. By the definition of the symplectic cutting, the normal bundle ν of M_n in $\overline{M}_{\mu \leq n}$ is given by

$$\nu = \mu^{-1}(N) \times_{S^1} \mathbb{C}_{(1)}.$$

For a sufficiently small $\varepsilon > 0$ let $D_\varepsilon(\mathbb{C}_{(1)}) = \{z \in \mathbb{C}_{(1)} : |z| < \varepsilon\}$ be the open disc of radius ε . We put $D_\varepsilon(\nu) = \mu^{-1}(n) \times_{S^1} D_\varepsilon(\mathbb{C}_{(1)})$, and define an S^1 -action on $D_\varepsilon(\nu)$ by

$$(3.3) \quad t[x, z] = [tx, z].$$

Let $p: D_\varepsilon(\nu) \rightarrow M_n$ be the natural projection. We define a complex line bundle $L_{D_\varepsilon(\nu)}$ on $D_\varepsilon(\nu)$ by

$$L_{D_\varepsilon(\nu)} = p^* L_n \otimes (\mu^{-1}(n) \times_{S^1} (D_\varepsilon(\mathbb{C}_{(1)}) \times \mathbb{C}_{(0)})),$$

and define an lift of the S^1 -action (3.3) to $L_{D_\varepsilon(\nu)}$ by

$$(3.4) \quad t((([x, z], [u \otimes v]) \otimes [x', z', w]) = ([tx, z], [(tu) \otimes v]) \otimes [tx', z', w]).$$

Then we can show that for a sufficiently small $\varepsilon > 0$ $L_{D_\varepsilon(\nu)}$ on $D_\varepsilon(\nu)$ is equivariantly identified with $\overline{L}$ restricted to certain neighborhood of M_n in $\overline{M}_{\mu \leq n}$. By using this identification and the equivariant version of the product formula [2, Theorem 8.8] we obtain

$$\begin{aligned} & \text{ind}_{S^1}(O, O \setminus M_n; \overline{L}|_O) \\ &= \text{ind}_{S^1}(D_\varepsilon(\nu), D_\varepsilon(\nu) \setminus M_n; L_{D_\varepsilon(\nu)}) \\ &= \text{ind}_{S^1}(M_n; L_n \otimes \mu^{-1}(n) \times_{S^1} \text{ind}_{S^1}(D_\varepsilon(\mathbb{C}_{(1)}), D_\varepsilon(\mathbb{C}_{(1)}) \setminus \{0\}; D_\varepsilon(\mathbb{C}_{(1)}) \times \mathbb{C}_{(0)})). \end{aligned}$$

Note that the product formula for the S^1 -equivariant local index holds since the S^1 -action preserves all the data. See [3, Section 6.2] for more details. From Example 2.3 the equivariant local index $\text{ind}_{S^1}(D_\varepsilon(\mathbb{C}_{(1)}), D_\varepsilon(\mathbb{C}_{(1)}) \setminus \{0\}; D_\varepsilon(\mathbb{C}_{(1)}) \times \mathbb{C}_{(0)})$ is equal to $\mathbb{C}_{(0)}$. By definition, L_n is naturally identified with the restriction of $\overline{L}$ to M_n . With this identification we can see that the restriction of the S^1 -action (3.2) to $L_n \rightarrow M_n$ is nothing but the fiberwise multiplication of t^{-n} . Since the S^1 -action on $\text{ind}_{S^1}(M_n; L_n)$ is defined by the pull-back, the S^1 -action on $\text{ind}_{S^1}(M_n; L_n)$ is given by the multiplication of t^n as we mentioned in Remark 3.3. This proves the theorem. $\square$

Example 3.6 (Complex projective space). As $(L, \nabla) \rightarrow (M, \omega)$ we adopt

$$\left((\mathbb{C}_{(1)})^m \times \mathbb{C}_{(0)}, d + \frac{1}{2} \sum_{i=1}^m (z_i d\bar{z}_i - \bar{z}_i dz_i) \right) \rightarrow \left((\mathbb{C}_{(1)})^m, \frac{\sqrt{-1}}{2\pi} \sum_{i=1}^m dz_i \wedge d\bar{z}_i \right).$$

For $n = 1$ the obtained $\overline{M}_{\mu \leq n}$, $\overline{L}$, and M_n are $\mathbb{C}P^m$, $\mathcal{O}(1)$, and $\mathbb{C}P^{m-1}$, respectively. The induced S^1 -actions on $\mathbb{C}P^m$ and $\mathcal{O}(1)$ are given by

$$(3.5) \quad \begin{aligned} t[z_0 : z_1 : \cdots : z_m] &= [z_0 : tz_1 : \cdots : tz_m], \\ t[z_0 : z_1 : \cdots : z_m, w] &= [z_0 : tz_1 : \cdots : tz_m, w]. \end{aligned}$$

The moment map $\bar{\mu}$ associated to the S^1 -action (3.5) is given by $\bar{\mu}([z_0 : \cdots : z_m]) = \sum_{i=1}^m |z_i|^2$. For $k = 0, 1$ let O_k be a sufficient small S^1 -invariant open neighborhood of $\bar{\mu}^{-1}(k)$. Then the equivariant local index $\text{ind}_{S^1}(O_k, O_k \setminus \bar{\mu}^{-1}(k); \bar{L}|_{O_k})$ is defined and By Corollary 2.2 the equivariant Riemann-Roch index $\text{ind}_{S^1}(\bar{M}_{\mu \leq n}, \bar{L})$ satisfies following equality

$$(3.6) \quad \begin{aligned} \text{ind}_{S^1}(\bar{M}_{\mu \leq n}, \bar{L}) &= \text{ind}_{S^1}(O_0, O_0 \setminus \bar{\mu}^{-1}(0); \bar{L}|_{O_0}) \\ &\quad + \text{ind}_{S^1}(O_1, O_1 \setminus \bar{\mu}^{-1}(1); \bar{L}|_{O_1}). \end{aligned}$$

The left hand side is computed as

$$(3.7) \quad \text{ind}_{S^1}(\bar{M}_{\mu \leq n}, \bar{L}) = \text{ind}_{S^1}(\mathbb{CP}^m, \mathcal{O}(1)) = \mathbb{C}_{(0)} \oplus m\mathbb{C}_{(1)}.$$

For $k = 1$, since $\bar{\mu}^{-1}(1) = M_n$, by Theorem 3.4 $\text{ind}_{S^1}(O_1, O_1 \setminus \bar{\mu}^{-1}(1); \bar{L}|_{O_1})$ is given as

$$(3.8) \quad \begin{aligned} \text{ind}_{S^1}(O_1, O_1 \setminus \bar{\mu}^{-1}(1); \bar{L}|_{O_1}) &= \text{ind}_{S^1}(O_1, O_1 \setminus M_n; \bar{L}|_{O_1}) \\ &= \text{ind}(\mathbb{CP}^{m-1}; \mathcal{O}(1))\mathbb{C}_{(1)} \\ &= m\mathbb{C}_{(1)}. \end{aligned}$$

For $k = 0$, it is easy to see that $\bar{\mu}^{-1}(0) = \{[z_0 : 0 : \cdots : 0]\}$ and $(\bar{L}, \nabla^{\bar{L}})|_{[z_0:0:\cdots:0]} \cong (\mathbb{C}_{(0)}, d + \frac{1}{2}(\bar{z}dz - z d\bar{z}))$. We can take O_0 in such a way that O_0 is identified with a sufficiently small open disc $D = \{(z_1, \dots, z_m) \in \mathbb{C}^m : \sum_{i=1}^m |z_i|^2 \leq \varepsilon\}$ with S^1 -action $t(z_1, \dots, z_m) = (tz_1, \dots, tz_m)$. Then, by (3.6), (3.7), and (3.8) we obtain the following formula

$$\text{ind}_{S^1}(D, D \setminus \{0\}; D \times \mathbb{C}_0) = \mathbb{C}_{(0)}.$$

In the case of $m = 1$ this formula can be obtained in [1, Remark 6.10] and [5, Section 5.3.2].

Example 3.7 (Exceptional divisor). Let n and $(L, \nabla) \rightarrow (M, \omega)$ be as in Example 3.6. Then the obtained cut space $\bar{M}_{\mu \geq n}$ is the blow-up $\tilde{\mathbb{C}}^m$ of the origin in $\mathbb{C}^m$, and M_n and L_n are the exceptional divisor $\mathbb{CP}^{m-1}$ and $\mathcal{O}(n)$, respectively. We take a sufficiently small invariant open neighborhood O of M_n . Then, by Theorem 3.4 the equivariant local index $\text{ind}_{S^1}(O, O \setminus M_n; \bar{L}|_O)$ is given by

$$\text{ind}_{S^1}(O, O \setminus M_n; \bar{L}|_O) = \text{ind}(\mathbb{CP}^{m-1}; \mathcal{O}(n))\mathbb{C}_{(n)} = \binom{m-1+n}{m-1}\mathbb{C}_{(n)}.$$

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