

t -QUANTIZED CARTAN MATRIX AND R-MATRICES FOR CUSPIDAL MODULES OVER QUIVER HECKE ALGEBRAS

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ABSTRACT. As every simple module of a quiver Hecke algebra appears as the image of the R-matrix defined on the convolution product of certain cuspidal modules, knowing the $\mathbb{Z}$ -invariants of the R-matrices between cuspidal modules is quite significant. In this paper, we prove that the (q, t) -Cartan matrix specialized at $q = 1$ of *any* finite type, called the t -quantized Cartan matrix, inform us of the invariants of R-matrices. To prove this, we use combinatorial AR-quivers associated with Dynkin quivers and their properties as crucial ingredients.

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1. INTRODUCTION

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, Δ its Dynkin diagram, $C = (c_{i,j})_{i,j \in I}$ its Cartan matrix. In [8], Frenkel-Reshetikhin introduced the two-parameters deformation $C(q, t)$ of the Cartan matrix C to define a deformation $\mathcal{W}_{q,t}(\mathfrak{g})$ of the $\mathcal{W}$ -algebra $\mathcal{W}$ associated with $\mathfrak{g}$. Interestingly, it comes to light that $\mathcal{W}_{q,t}(\mathfrak{g})$ *interpolates* the representation theories of the quantum affine algebra $U'_q(\widehat{\mathfrak{g}})$ of untwisted type and its Langlands dual $U'_q(\widehat{\mathfrak{g}}^L)$ in the following sense. Let $\mathcal{C}_{\widehat{\mathfrak{g}}}$ be the category of finite dimensional $U_q(\widehat{\mathfrak{g}})$ -modules and $K(\mathcal{C}_{\widehat{\mathfrak{g}}})$ its

Date: 2023, February 15.

The research of M. Kashiwara was supported by Grant-in-Aid for Scientific Research (B) 20H01795, Japan Society for the Promotion of Science.

The research of S.-j. Oh was supported by the Ministry of Education of the Republic of Korea and the National Research Foundation of Korea (NRF-2022R1A2C1004045).

Grothendieck ring. Then the specialization of $\mathcal{W}_{q,t}(\mathfrak{g})$ at $t = 1$ recovers $K(\mathcal{C}_{\mathfrak{g}})$ while the one at $q = \exp(\pi i/r)$ is expected to recover $K(\mathcal{C}_{\mathfrak{g}^L})$, where r is the racing number of $\mathfrak{g}^L$ [9, 7, 6].

In the representation theory of $U'_q(\widehat{\mathfrak{g}})$ over the base field $\mathbf{k} = \overline{\mathbb{Q}(q)}$, the R-matrices play a central role as every simple module in $\mathcal{C}_{\mathfrak{g}}$ appears as the image of the R-matrix defined on a certain tensor product of a special family of simple modules, called the *fundamental modules* [1, 27, 57]. Let us explain this in a more precise way. The R-matrices are constructed as intertwining operators between modules in $\mathcal{C}_{\mathfrak{g}}$ satisfying the Yang-Baxter equation. For a simple module $M \in \mathcal{C}_{\mathfrak{g}}$, there exists a unique finite family $\{(i_k, a_k) \in I \times \mathbf{k}^\times\}_{1 \leq k \leq r}$ and the corresponding fundamental modules $\{V_{i_k}(a_k)\}_{1 \leq k \leq r}$ such that M is an image of the composition of R-matrices

$$\mathbf{r} : V_{i_r}(a_r) \otimes \cdots \otimes V_{i_2}(a_2) \otimes V_{i_1}(a_1) \rightarrow V_{i_1}(a_1) \otimes V_{i_2}(a_2) \otimes \cdots \otimes V_{i_r}(a_r),$$

where

$$\mathbf{r}_{t,s} : V_{i_t}(a_t) \otimes V_{i_s}(a_s) \rightarrow V_{i_s}(a_s) \otimes V_{i_t}(a_t)$$

denotes the R-matrix between $V_{i_t}(a_t)$ and $V_{i_s}(a_s)$,

$$\mathbf{r} = (\mathbf{r}_{2,1}) \circ \cdots \circ (\mathbf{r}_{r-1,1} \circ \cdots \circ \mathbf{r}_{r-1,r-2}) \circ (\mathbf{r}_{r,1} \circ \cdots \circ \mathbf{r}_{r,r-1}),$$

and a_s/a_t is *not* a root of the *denominator* of the *normalized R-matrix* $R_{V_{i_t}(1), V_{i_s}(1)z}^{\text{norm}}$, which is a polynomial in $\mathbf{k}[z]$ and denoted by $d_{i_t, i_s}(z)$. Furthermore, the denominator of $R_{V_{i_t}(1), V_{i_s}(1)z}^{\text{norm}}$ indicates whether $V_{i_t}(a_t) \otimes V_{i_s}(a_s)$ is simple or not (we refer [30] for more detail about the normalized R-matrix and its denominator). Thus the study of the R-matrices and denominators between fundamental modules is one of the first step to investigate the representation theory of quantum affine algebras. Surprisingly, it is proved that the matrix $(d_{i,j}(z))_{i,j \in I}$ can read from the inverse $\tilde{\mathbf{C}}_q(q) := \mathbf{C}_q(q)^{-1}$ of $\mathbf{C}_q(q) := \mathbf{C}(q, t)|_{t=1}$ by using the geometry of the graded quiver varieties in type *ADE* [11] and using the root system for a simply-laced type $\mathfrak{g}$ (see (1.1) below) in any finite type $\mathfrak{g}$ [17, 52, 14]. We remark here that $(d_{i,j}(z))_{i,j \in I}$ was explicitly calculated in [20, 5, 25, 47, 51, 11].

We call $\mathbf{C}_q(q)$ the *quantum Cartan matrix*. Note that $\tilde{\mathbf{C}}_q(q)$ is ubiquitously utilized in the representation theory of $U'_q(\widehat{\mathfrak{g}})$. For instances, it plays crucial roles in the q -character theory invented in [9, 7] and in the construction of the quantum Grothendieck ring $K_t(\mathcal{C}_{\mathfrak{g}})$ invented in [45, 56, 16]. As we mentioned, to compute $\tilde{\mathbf{C}}_q(q)$ of type $\mathfrak{g}$, we need to consider its corresponding simply-laced Lie algebra $\mathfrak{g}$ given in the below:

$$(1.1) \quad (\mathfrak{g}, \mathfrak{g}) = (ADE_n, ADE_n), (B_n, A_{2n-1}), (C_n, D_{n+1}), (F_4, E_6), (G_2, D_4).$$

Then it is revealed that the nature of the representation theory of $U'_q(\widehat{\mathfrak{g}})$ is simply-laced even though $\mathfrak{g}$ is not simply-laced. For instances, the quantum Grothendieck rings $K_t(\mathcal{C}_{\mathfrak{g}}^{\mathcal{Q}})$ of the *heart subcategories* $\mathcal{C}_{\mathfrak{g}}^{\mathcal{Q}}$ of $\mathcal{C}_{\mathfrak{g}}$ are isomorphic to the integral form of the *unipotent quantum coordinate ring* $A_q(\mathfrak{n})$ of $U_q(\mathfrak{g})$ [17, 34, 51], and the blocks of $\mathcal{C}_{\mathfrak{g}}$ is parameterized by the root lattice $\mathbf{Q}_{\mathfrak{g}}$ of $\mathfrak{g}$ [4, 33, 14]. Here, (A) the heart subcategory $\mathcal{C}_{\mathfrak{g}}^{\mathcal{Q}}$ depends on the choice of a $\mathcal{Q}$ -datum $\mathcal{Q} = (\Delta_{\mathfrak{g}}, \sigma, \xi)$ of $\mathfrak{g}$, which consists of (i) the Dynkin diagram $\Delta_{\mathfrak{g}}$ of type $\mathfrak{g}$, (ii) the Dynkin diagram automorphism σ which yields $\Delta_{\mathfrak{g}}$ as the σ -orbits of $\Delta_{\mathfrak{g}}$ and (iii) a height

function $\xi : I \rightarrow \mathbb{Z}$ satisfying certain conditions ([14, Definition 3.5]), (B) $\mathcal{C}_{\mathfrak{g}}^{\mathcal{Q}}$ is characterized by the smallest subcategory of $\mathcal{C}_{\mathfrak{g}}$ (i) containing $\mathbf{V}_{\mathcal{Q}} := \{V_i((-q)^p)\}$, where (i, p) ranges over a certain subset of $I \times \mathbb{Z}$ which is the set of vertices $(\Gamma_{\mathcal{Q}})_0$ of *combinatorial AR-quiver* $\Gamma_{\mathcal{Q}}$ of $\mathcal{Q}$, and (ii) stable by taking subquotients, extensions and tensor products [17, 34, 51].

The *quiver Hecke algebra* R , introduced by Khovanov-Lauda [39] and Rouquier [54] independently, is a $\mathbb{Z}$ -graded $\mathbf{k}$ -algebra, whose finite-dimensional graded module category $R\text{-gmod}$ also categorifies the integral form of $A_q(\mathbf{n})$ of the quantum group $U_q(\mathfrak{g})$ associated with a symmetrizable Kac-Moody algebra $\mathfrak{g}$. When $\mathfrak{g}$ is symmetric and the base field $\mathbf{k}$ is of characteristic 0, the self-dual simple modules in $R\text{-gmod}$ categorifies the distinguished basis $\mathbf{B}^{\text{up}}$ [58, 55], which is called the *upper global basis* (equivalent to Lusztig's *dual canonical basis*). For a finite simple Lie algebra $\mathfrak{g}$, every simple module can be obtained as the image of R-matrices of a certain ordered *convolution product* $\circ$ of *cuspidal modules* (§ 8.1), where the cuspidal modules categorify the dual PBW-vectors [37, 44] (see also [41, 18]). Let us explain this more precisely. For a *commutation class* $\mathbf{c}$ of reduced expressions of $\underline{w}_0$ of the longest element w_0 of Weyl group $\mathbf{W}_{\mathfrak{g}}$ (§ 2.2), one can define the associated set of cuspidal modules $\mathbf{S}_{\mathbf{c}} = \{S_{\mathbf{c}}(\beta)\}_{\beta \in \Phi_{\mathfrak{g}}^+}$, which consists of *affreal* simple modules (Definition 7.6) and is parameterized by the set of positive roots $\Phi_{\mathfrak{g}}^+$. Then, every simple R -module M can be obtained as the image of a distinguished homomorphism, which is also called the R-matrix and satisfy the Yang-Baxter equation (see [21] for more detail), defined an ordered convolution product of modules in $\mathbf{S}_{\mathbf{c}}$. Namely, there exists a unique $\underline{m} = (m_{\ell}, \dots, m_1) \in \mathbb{Z}_{\geq 0}^{\Phi_{\mathfrak{g}}^+}$ such that M is the image of the composition of R-matrices (§ 8.1)

$$\mathbf{r} : S_{\mathbf{c}}(\beta_{\ell})^{\circ m_{\ell}} \circ \dots \circ S_{\mathbf{c}}(\beta_2)^{\circ m_2} \circ S_{\mathbf{c}}(\beta_1)^{\circ m_1} \rightarrow S_{\mathbf{c}}(\beta_1)^{\circ m_1} \circ S_{\mathbf{c}}(\beta_2)^{\circ m_2} \circ \dots \circ S_{\mathbf{c}}(\beta_{\ell})^{\circ m_{\ell}},$$

where

$$\mathbf{r}_{t,s} : S_{\mathbf{c}}(\beta_t) \circ S_{\mathbf{c}}(\beta_s) \rightarrow S_{\mathbf{c}}(\beta_s) \circ S_{\mathbf{c}}(\beta_t)$$

denotes the R-matrix between $S_{\mathbf{c}}(\beta_t)$ and $S_{\mathbf{c}}(\beta_s)$, and

$$\mathbf{r} = (\mathbf{r}_{2,1}^{\circ m_2 m_1}) \circ \dots \circ (\mathbf{r}_{r-1,1}^{\circ m_{r-1} m_1} \circ \dots \circ \mathbf{r}_{r-1,r-2}^{\circ m_{r-1} m_{r-2}}) \circ (\mathbf{r}_{r,1}^{\circ m_r m_1} \circ \dots \circ \mathbf{r}_{r,r-1}^{\circ m_r m_{r-1}}).$$

Here the homogeneous degree $\Lambda(S_{\mathbf{c}}(\beta_t), S_{\mathbf{c}}(\beta_s))$ ($t > s$) of the R-matrix $\mathbf{r}_{t,s}$ coincides with $-(\beta_t, \beta_s)$. As in quantum affine algebra cases, for each $\mathbf{r}_{t,s}$, we have the $\mathbb{Z}_{\geq 0}$ -invariant of the R-matrix

$$\mathfrak{d}(S_{\mathbf{c}}(\beta_t), S_{\mathbf{c}}(\beta_s)) := \frac{1}{2}(\Lambda(S_{\mathbf{c}}(\beta_t), S_{\mathbf{c}}(\beta_s)) + \Lambda(S_{\mathbf{c}}(\beta_s), S_{\mathbf{c}}(\beta_t)))$$

indicating whether $S_{\mathbf{c}}(\beta_t) \circ S_{\mathbf{c}}(\beta_s)$ is simple or not. Note that, in general, computing $\mathfrak{d}(M, N)$ is quite difficult, when M and N are simple R -modules such that one of them is affreal.

In [21], Kang-Kashiwara-Kim constructed the *generalized Schur-Weyl functor* $\mathcal{F}_J$ from the category $R^J\text{-gmod}$ of the quiver Hecke algebra R^J to the category $\mathcal{C}_{\mathfrak{g}}$. Here the quiver Hecke algebra R^J and the monoidal functor $\mathcal{F}_J$ are determined by the *duality datum* ([32, Section 4]), a set of real simple modules $\{V^j\}_{j \in J}$ in $\mathcal{C}_{\mathfrak{g}}$, and the denominators of R-matrices $(\mathbf{r}_{V^i, V^j})_{i,j \in J}$. When the duality datum *arises* from a Q-datum $\mathcal{Q}$ for $\mathfrak{g}$, the quiver Hecke algebra $R^{\mathcal{Q}}$ is associated with $\mathfrak{g}$ and the functor $\mathcal{F}_{\mathcal{Q}}$ gives an equivalence of categories $\mathcal{C}_{\mathfrak{g}}^{\mathcal{Q}}$ and $R^{\mathfrak{g}}\text{-mod}$ for each pair $(\mathfrak{g}, \mathfrak{g})$ in (1.1) and any Q-datum $\mathcal{Q}$ of $\mathfrak{g}$ [20, 23, 34, 51, 10, 46,

[32]. Note that $\mathcal{F}_Q$ (i) preserves R-matrices in both categories, their invariants and (ii) sends $\mathcal{C}_{\mathfrak{g}}$ to the set of fundamental modules V_Q to $S_Q := \{S_Q(\beta) := S_{[Q]}(\beta)\}_{\beta \in \Phi_{\mathfrak{g}}^+}$, where $[Q]$ is a commutation class of w_0 adapted to Q . Hence we can obtain information about $\mathfrak{d}(S_Q(\alpha), S_Q(\beta))$ in $R^{\mathfrak{g}}\text{-gmod}$ from $\tilde{\mathbf{C}}_{\mathfrak{q}}(q)$ of $\mathfrak{g}$. Thus, roughly speaking, we can understand $\tilde{\mathbf{C}}_{\mathfrak{q}}(q)$ as the denominator formulae among the cuspidal modules in S_Q .

To sum up, the quantum Cartan matrices $\mathbf{C}_{\mathfrak{q}}(q)$, which are the specialization of $\mathbf{C}(q, t)$ at $t = 1$, or rather their inverses $\tilde{\mathbf{C}}_{\mathfrak{q}}(q)$ control the representation theories of quantum affine algebras $U'_q(\widehat{\mathfrak{g}})$ for all $\mathfrak{g}$ and the ones of quiver Hecke algebras $R^{\mathfrak{g}}$ for simply-laced types $\mathfrak{g}$:

$$(1.2) \quad \begin{array}{ccc} & & \text{controls} \rightsquigarrow \mathcal{C}_{\mathfrak{g}} \quad \text{for any } \mathfrak{g} \\ \mathbf{C}(q, t) & \xrightarrow{t=1} & \mathbf{C}_{\mathfrak{q}}(q) \\ & & \rightsquigarrow \text{controls} \quad \text{Rep}(R^{\mathfrak{g}}) \quad \text{only for } \mathfrak{g}. \end{array}$$

In this paper, *regardless of type*, we prove that (a) the matrix $\mathbf{C}(q, t)|_{q=1}$, denoted by $\mathbf{C}_{\mathfrak{t}}(t)$ and called the *t-quantized Cartan matrix*, and (b) a (combinatorial) AR-quiver Γ_Q associated with any Dynkin quiver $Q = (\Delta_{\mathfrak{g}}, \xi)$ inform us of the $\mathbb{Z}$ -invariants of R-matrices and related properties for pairs of cuspidal modules $(S_Q(\alpha), S_Q(\beta))$ over the quiver Hecke algebra $R^{\mathfrak{g}}$, where $[Q]$ is the commutation class of $w_0 \in W_{\mathfrak{g}}$ adapted to Q (Definition 2.4). Since $\mathbf{C}_{\mathfrak{t}}(t)$ can be obtained from $\mathbf{C}_{\mathfrak{q}}(q)$ by just replacing q with t when $\mathfrak{g}$ is simply-laced, we can say that $\mathbf{C}_{\mathfrak{t}}(t)$ controls $\mathcal{C}_{\widehat{\mathfrak{g}}}$ and hence we can complete the left wing of (1.2) as follows:

$$(1.3) \quad \begin{array}{ccccc} \text{Rep}(R^{\mathfrak{g}}) & & & & \text{controls} \rightsquigarrow \mathcal{C}_{\mathfrak{g}} \quad \text{for any } \mathfrak{g} \\ & \nwarrow \text{controls} & & & \\ & \mathbf{C}_{\mathfrak{t}}(t) & \xleftarrow{q=1} & \mathbf{C}(q, t) & \xrightarrow{t=1} & \mathbf{C}_{\mathfrak{q}}(q) \\ & \nwarrow \text{controls} & & & \nwarrow \text{controls} \\ \mathcal{C}_{\widehat{\mathfrak{g}}} & & & & \text{Rep}(R^{\mathfrak{g}}) \quad \text{only for } \mathfrak{g}. \end{array}$$

As we explained, when $\mathfrak{g}$ is of simply-laced type and Q is a Dynkin quiver of the same type, $\mathbf{C}_{\mathfrak{t}}(t) = \mathbf{C}_{\mathfrak{q}}(q)|_{q=t}$ indicates whether $S_Q(\alpha) \circ S_Q(\beta)$ is simple or not. More precisely, via the bijection $I \times \mathbb{Z} \supset \widehat{\Delta}_0 \xrightarrow{\phi_Q} \Phi_{\mathfrak{g}}^+ \times \mathbb{Z}$, we have

$$\mathfrak{d}(S_Q(\beta) \circ S_Q(\alpha)) = \text{the coefficient of } t^{|p-s|-1} \text{ in } \tilde{\mathbf{C}}_{\mathfrak{q}}(t)_{i,j},$$

where $\phi_Q^{-1}(\alpha, 0) = (i, p)$ and $\phi_Q^{-1}(\beta, 0) = (j, s)$. That is, by only reading (I) relative positions of α and β in Γ_Q and (II) the coefficients of $\tilde{\mathbf{C}}_{\mathfrak{q}}(t)_{i,j}$, we can obtain the $\mathbb{Z}$ -invariant $\mathfrak{d}(S_Q(\beta), S_Q(\alpha))$. This kind of strategy was achieved in [48, 49, 50, 52] by developing statistics on Γ_Q , which also depends only on the relative positions of α and β in Γ_Q .

We apply the aforementioned strategy to non simply-laced type $\mathfrak{g}$ also. For the purpose, in the previous paper [35] of the authors, we extended the notion for Dynkin quivers Q to the non simply-laced types. We then constructed the combinatorial AR-quiver Γ_Q for Q of any type $\mathfrak{g}$, which generalizes the classical AR-quiver Γ_Q of the path algebra $\mathbb{C}Q$ for a simply-laced type Q in an aspect of combinatorics. More precisely, we proved that (i) there are reduced expressions $\underline{w}_0$'s of w_0 adapted to Q and such reduced expressions form the commutation

class $[Q]$, and (ii) we introduced the combinatorial AR-quiver Γ_Q which realizes the *convex partial order* $\prec_Q$ on the set of positive roots $\Phi_{\mathfrak{g}}^+$ (see (2.4) and Theorem 4.1). We also understand Γ_Q as a *heart full-subquiver* by constructing the *repetition quiver* $\widehat{\Delta} = (\widehat{\Delta}_0, \widehat{\Delta}_1)$, which depends *only* on $\mathfrak{g}$ and whose set of vertices $\widehat{\Delta}_0$ is in bijection with $\Phi_{\mathfrak{g}}^+ \times \mathbb{Z}$. We remark here that, when $\mathfrak{g}$ is of simply-laced type, $\widehat{\Delta}$ is isomorphic to the AR-quiver of the bounded derived category of the path algebra $\mathbb{C}Q$ [15] and $\widehat{\Delta}_0$ is a subset of $I \times \mathbb{Z}$. One of the main theorems in [35] is that the inverse $\widetilde{\mathbf{B}}(t)$ of $\mathbf{B}(t) := \mathbf{C}_{\mathfrak{t}}(t) \times \mathbf{D}^{-1}$ can be read from any AR-quiver Γ_Q of the same type $\mathfrak{g}$. Here $\mathbf{D} = \text{diag}(\mathbf{d}_i \mid i \in I)$ is a diagonal matrix symmetrizing the Cartan matrix $\mathbf{C}$.

In this paper, we first show that $\widetilde{\mathbf{B}}(t)$ of type $\mathfrak{g}$ can be obtained from $\widetilde{\mathbf{B}}(t)$ of type $\mathfrak{g}$ via the Dynkin diagram folding σ on $\Delta_{\mathfrak{g}}$ yielding $\Delta_{\mathfrak{g}}$ (Theorem 3.7). Since $\widetilde{\mathbf{B}}(t)$ of type $\mathfrak{g}$ can be read from any Γ_Q of the same type $\mathfrak{g}$, Theorem 3.7 tells us that combinatorial properties of Γ_Q of type $\mathfrak{g}$ are related to the ones of AR-quivers of simply-laced type. In Section 5, we show that the labeling of Γ_Q of type $\mathfrak{g} = B_n, C_n, F_4$ and G_2 via $\Phi_{\mathfrak{g}}^+$ can be obtained from the one of Γ_Q of type $\mathfrak{g}' = D_{n+1}, A_{2n-1}, E_6$ and D_4 via $\Phi_{\mathfrak{g}'}^+$ by simple surgery, respectively:

$$(1.4) \quad (\mathfrak{g}, \mathfrak{g}') = (C_n, A_{2n-1}), (B_n, D_{n+1}), (F_4, E_6), (G_2, D_4).$$

Note that $\mathfrak{g}$ is realized as a non-trivial Lie subalgebra of $\mathfrak{g}'$ via the Dynkin diagram folding σ ([19, Proposition 7.9]). Based on the combinatorial results in Section 5, we show that the statistics, defined in [48, 49, 50] for simply-laced type, is also well-defined and does not depend on the choice of Dynkin quivers for any type. Hence we can obtain the set of polynomials, called the *degree polynomials* $(\mathfrak{d}_{i,j}(t))_{i,j \in I} \subset \mathbb{Z}_{\geq 0}[t]$ of any type $\mathfrak{g}$ (Proposition 6.1). Then we show in Theorem 6.6 that

$$(1.5) \quad \mathfrak{d}_{i,j}(t) + \delta_{i,j^*} \mathbf{d}_i t^{\mathbf{h}-1} = \widetilde{\mathbf{B}}_{i,j}(t) + t^{\mathbf{h}} \widetilde{\mathbf{B}}_{i,j^*}(t),$$

when $\mathfrak{g}$ is of classical type or type E_6 (see Remark 4.29 for the remained types). Here $\mathbf{h}$ is the Coxeter number of $\mathfrak{g}$ and $*$: $I \rightarrow I$ is an involution on I induced by the longest element w_0 .

Main Theorem (Theorem 8.6). Let Q be a Dynkin quiver of any type $\mathfrak{g}$. Then, for any $\alpha, \beta \in \Phi_{\mathfrak{g}}^+$, we have

$$(1.6) \quad \mathfrak{d}(S_Q(\alpha), S_Q(\beta)) = \text{the coefficient of } t^{|p-s|-1} \text{ in } \widetilde{\mathbf{B}}_{i,j}(t),$$

where $\phi_Q(i, p) = (\alpha, 0)$ and $\phi_Q(j, s) = (\beta, 0)$. Furthermore, we have the following:

- (a) The head of $S_Q(\alpha) \circ S_Q(\beta)$ is simple and isomorphic to a convolution product of mutually commuting cuspidal modules in $\mathbf{S}_Q$, under the assumption that $\beta \not\prec_Q \alpha$.
- (b) If the coefficient of $t^{|p-s|-1}$ in $\mathfrak{d}_{i,j}(t)$ is strictly larger than $\max(\mathbf{d}_i, \mathbf{d}_j)$, then the composition length $\ell(S_Q(\alpha) \circ S_Q(\beta))$ of $S_Q(\alpha) \circ S_Q(\beta)$ is strictly larger than 2.
- (c) If the coefficient of $t^{|p-s|-1}$ in $\mathfrak{d}_{i,j}(t)$ is equal to $\max(\mathbf{d}_i, \mathbf{d}_j)$, $\ell(S_Q(\alpha) \circ S_Q(\beta))$ is 2.
- (d) If the coefficient of $t^{|p-s|-1}$ in $\mathfrak{d}_{i,j}(t)$ is equal to 0, $S_Q(\alpha) \circ S_Q(\beta)$ is simple.

We prove **Main Theorem** by using (i) the combinatorial properties of AR-quivers investigated in the sections 4, 5, (ii) the study on dual PBW-vectors and their categorifications

in [3] and (iii) the theories related to R-matrices developed in [24, 29] by authors and their collaborators.

Note that **Main Theorem** tells that, for a Dynkin quiver $Q = (\Delta, \xi)$ with a *source* i , we have

$$(1.7) \quad \mathfrak{d}(S_Q(\alpha), S_Q(\beta)) = \mathfrak{d}(S_{s_i Q}(s_i \alpha), S_{s_i Q}(s_i \beta)) \quad \text{if } \alpha \neq \alpha_i \text{ and } \beta \neq \alpha_i,$$

where α_i is the i -th simple root and $s_i Q = (\Delta, s_i \xi)$ is *another* Dynkin quiver obtained from Q (see (2.7)). Since S_Q and $S_{s_i Q}$ categorify two different families of dual PBW-vectors, which are connected via the braid group action of Lusztig and Saito, we can expect an existence of a functor proving (1.7). When $\mathfrak{g}$ is of simply-laced type, Kato constructed such a functor in [37, 38], called *Saito reflection functor* by using a geometric approach. We remark here that, comparing with simply-laced type, there is no fully-developed geometry for non simply-laced type $\mathfrak{g}$, and hence we can not apply a geometric approach in [58, 37, 38, 11] at this moment. Thus we adopt an algebro-combinatorial approach in this paper.

This paper is organized as follows. In Section 2, we review the necessary backgrounds of this paper. In particular, we recall the notion of Dynkin quiver and the statistics defined on the pairs of positive roots with respect to the convex partial orders on Φ^+ . In Section 3, We first review the results in [35] about the computation of $\tilde{\mathbf{B}}(t) = (\tilde{\mathbf{B}}_{i,j}(t))$ as a matrix with entries in $\mathbb{Z}((t))$. Then we prove that $\tilde{\mathbf{B}}(t)$ of non simply-laced type $\mathfrak{g}$ can be obtained from the one of simply-laced type $\mathfrak{g}$. In Section 4, we review the results in [48, 49, 50] about the statistics and the degree polynomials of AR-quivers Γ_Q 's when Q is of simply-laced type. In Section 5, we develop the BCFG-analogues of Section 4 and observe the similarities among Γ_Q 's of different types. In particular, we show that the labeling of AR-quivers of non simply-laced type via Φ^+ can be obtained from the one of simply-laced type by simple surgery. In Section 6, we define the notion of degree polynomials for each Dynkin quiver Q and show that it does not depend on the choice of Q and hence well-defined for any finite type $\mathfrak{g}$. Then we show that (1.5) holds for classical type and E_6 . We allocate a lot of parts of this paper to Section 4, 5 and 6 to exhibit many computations and examples. In Section 7, we recall the quiver Hecke algebra, the R-matrix and related $\mathbb{Z}$ -invariants related to R-matrix. Also we review the categorification of PBW-vectors via cuspidal modules. In Section 8, we prove **Main Theorem** by a case-by-case method based on the results in the previous sections. More precisely, we classify pairs (α, β) by relative positions in Γ_Q and hence statistics, and prove (1.6) according to the relative position and the type of $\mathfrak{g}$.

Convention 1. *Throughout this paper, we follow the following conventions.*

- (1) For a statement P , $\delta(P)$ is 1 or 0 whether P is true or not. In particular, we set $\delta_{i,j} := \delta(i = j)$ (Kronecker's delta).
- (2) For a finite set A , we denote by $|A|$ the number of elements in A .
- (3) We sometimes use $\blacktriangle$ for a Dynkin diagram of non-simply-laced type to distinguish it from a Dynkin diagram of simply-laced type.

2. BACKGROUNDS

In this section, we review the necessary backgrounds of this paper.

2.1. Finite Cartan data. Let I be an index set. A *Cartan datum* is a quintuple

$$(\mathbf{C}, \mathbf{P}, \Pi, \mathbf{P}^\vee, \Pi^\vee)$$

consisting of

- (1) a generalized symmetrizable Cartan matrix $\mathbf{C} = (c_{i,j})_{i,j \in I}$,
- (2) a free abelian group $\mathbf{P}$, called the *weight lattice*,
- (3) $\Pi = \{\alpha_i \mid i \in I\}$, called the set of *simple roots*,
- (4) $\mathbf{P}^\vee := \text{Hom}_{\mathbb{Z}}(\mathbf{P}, \mathbb{Z})$, called the *coweight lattice* and
- (5) $\Pi^\vee = \{h_i \mid i \in I\}$, called the set of *simple coroots*

satisfying

- (i) $\langle h_i, \alpha_j \rangle = c_{i,j}$ for $i, j \in I$,
- (ii) Π is linearly independent over $\mathbb{Q}$,
- (iii) for each $i \in I$, there exists $\varpi_i \in \mathbf{P}$, called the *fundamental weight*, such that $\langle h_i, \varpi_j \rangle = \delta_{i,j}$ for all $j \in I$ and
- (iv) there exists a $\mathbb{Q}$ -valued symmetric bilinear form $(\cdot, \cdot)$ on $\mathbf{P}$ such that $\langle h_i, \lambda \rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)}$ and $(\alpha_i, \alpha_i) \in \mathbb{Q}_{>0}$ for any i and $\lambda \in \mathbf{P}$.

We set $\mathfrak{h} := \mathbb{Q} \otimes_{\mathbb{Z}} \mathbf{P}^\vee$, $\mathbf{Q} := \bigoplus_{i \in I} \mathbb{Z} \alpha_i$ and $\mathbf{Q}^+ := \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$, $\mathbf{P}^+ := \{\lambda \in \mathbf{P} \mid \langle h_i, \lambda \rangle \geq 0 \text{ for any } i \in I\}$ and $\text{ht}(\beta) = \sum_{i \in I} k_i$ for $\beta = \sum_{i \in I} k_i \alpha_i \in \mathbf{Q}^+$. We denote by Φ the set of roots, by $\Phi^\pm$ the set of *positive roots* (resp. *negative roots*).

Definition 2.1.

- (a) The *multiplicity* $\text{mul}(\beta)$ of $\beta = \sum_{i \in I} n_i \alpha_i \in \mathbf{Q}^+$ is defined by

$$\text{mul}(\beta) := \max \{n_i \mid i \in I\}.$$

- (b) The *support* $\text{supp}(\beta)$ of $\beta = \sum_{i \in I} n_i \alpha_i \in \mathbf{Q}^+$ is defined by

$$\text{supp}(\beta) := \{i \in I \mid n_i \neq 0\}.$$

- (c) For $i \in I$ and $\beta = \sum_{i \in I} n_i \alpha_i \in \mathbf{Q}^+$, we set $\text{supp}_i(\beta) := n_i$.

- (d) For $\alpha, \beta \in \Phi$, we set

$$p_{\beta, \alpha} := \max \{p \in \mathbb{Z} \mid \beta - p\alpha \in \Phi\}.$$

Throughout this paper, we only consider a *finite Cartan datum* and realize the root lattice $\mathbf{Q} \subset \bigoplus_{s \in J} \mathbb{R} \epsilon_s$ with an orthonormal basis $\{\epsilon_s\}_{s \in J}$. For instance, if $\mathbf{Q}$ is of type A_n , $\mathbf{Q} \subset \bigoplus_{i=1}^{n+1} \mathbb{Z} \epsilon_i$ and $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leq i \leq n$.

For each finite Cartan datum, we take the non-degenerate symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{h}^*$ such that $(\alpha, \alpha) = 2$ for short roots α in Φ^+ . In this paper, a *Dynkin diagram* Δ associated with a Cartan datum is a graph with I as the set of vertices and the set of edges with edges between i and j such that $(\alpha_i, \alpha_j) < 0$. We assign $d_i := (\alpha_i, \alpha_i)/2 = (\alpha_i, \varpi_i) \in \mathbb{Z}_{>0}$ to each

(2.1)

$$(2.4) \quad \alpha \prec_{\mathbf{c}} \beta \quad \text{if and only if} \quad \alpha <_{w_0} \beta \quad \text{for any } \underline{w}_0 \in \mathbf{c}.$$

For $[\underline{w}_0]$ of w_0 and $\alpha \in \Phi^+$, we define $[\underline{w}_0]$ -residue of α , denoted by $\text{res}^{[\underline{w}_0]}(\alpha)$, to be $i_k \in I$ if $\beta_k^{w_0} = \alpha$ with $\underline{w}_0 = s_{i_1} \cdots s_{i_\ell}$. Note that this notion is well-defined; i.e., for any $\underline{w}'_0 = s_{j_1} \cdots s_{j_\ell} \in [\underline{w}_0]$ with $\beta_t^{w'_0} = \alpha$, we have $j_t = i_k$. Note that

$$(\alpha, \alpha) = (\alpha_i, \alpha_i) \quad \text{if } i = \text{res}^{[\underline{w}_0]}(\alpha).$$

For a reduced expression $\underline{w}_0 = s_{i_1} s_{i_2} \cdots s_{i_\ell}$, it is known that the expression $\underline{w}'_0 := s_{i_2} \cdots s_{i_\ell} s_{i_1}^*$ is also a reduced expression of w_0 . This operation is sometimes referred to as a *combinatorial reflection functor* and we write $r_{i_1} \underline{w}_0 = \underline{w}'_0$. Also it induces the operation on commutation classes of w_0 (i.e., $r_{i_1}[\underline{w}_0] = [r_{i_1} \underline{w}_0]$) is well-defined if there exists a reduced expression $\underline{w}'_0 = s_{j_1} s_{j_2} \cdots s_{j_\ell} \in [\underline{w}_0]$ such that $j_1 = i_1$. The relations $[\underline{w}] \stackrel{r}{\sim} [r_i \underline{w}]$ for $i \in I$ generate an equivalence relation, called the *reflection equivalent relation* $\stackrel{r}{\sim}$, on the set of commutation classes of w_0 . For $\underline{w}_0$ of w_0 , the family of commutation classes $[\underline{w}_0] := \{[\underline{w}'_0] \mid [\underline{w}'_0] \stackrel{r}{\sim} [\underline{w}_0]\}$ is called an *r-cluster point*.

2.3. Statistics. An element $\underline{m} = \{\underline{m}_\beta\}_{\beta \in \Phi^+}$ of $\mathbb{Z}_{\geq 0}^{\Phi^+}$ is called an *exponent*. In this subsection, we give several notions concerning exponents, which we need in the PBW description of simple R -modules (e.g., see Theorem 8.1 below).

For an exponent $\underline{m}$, we set $\text{wt}(\underline{m}) := \sum_{\beta \in \Phi^+} \underline{m}_\beta \beta \in \mathbb{Q}^+$.

Definition 2.2 (cf. [44, 50]). For a reduced expression $\underline{w}_0 = s_{i_1} \cdots s_{i_\ell}$ and a commutation class $\mathbf{c}$, we define the partial orders $<_{\underline{w}_0}^b$ and $<_{\mathbf{c}}^b$ on $\mathbb{Z}_{\geq 0}^{\Phi^+}$ as follows:

- (i) $<_{\underline{w}_0}^b$ is the bi-lexicographical partial order induced by $<_{\underline{w}_0}$. Namely, $\underline{m} <_{\underline{w}_0}^b \underline{m}'$ if
 - $\text{wt}(\underline{m}) = \text{wt}(\underline{m}')$.
 - there exists $\alpha \in \Phi^+$ such that $\underline{m}_\alpha < \underline{m}'_\alpha$ and $\underline{m}_\beta = \underline{m}'_\beta$ for any β such that $\beta <_{\underline{w}_0} \alpha$,
 - there exists $\eta \in \Phi^+$ such that $\underline{m}_\eta < \underline{m}'_\eta$ and $\underline{m}_\zeta = \underline{m}'_\zeta$ for any ζ such that $\eta <_{\underline{w}_0} \zeta$.
- (ii) For sequences $\underline{m}$ and $\underline{m}'$, we define $\underline{m} <_{\mathbf{c}}^b \underline{m}'$ if the following conditions are satisfied:

$$\underline{m} <_{\underline{w}_0}^b \underline{m}' \quad \text{for all } \underline{w}_0 \in \mathbf{c}.$$

We say that an exponent $\underline{m} = \{\underline{m}_\beta\}_{\beta \in \Phi^+} \in \mathbb{Z}_{\geq 0}^{\Phi^+}$ is *c-simple* if it is minimal with respect to the partial order $<_{\mathbf{c}}^b$. For a given *c-simple* exponent $\underline{s} = \{\underline{s}_\beta\}_{\beta \in \Phi^+} \in \mathbb{Z}_{\geq 0}^{\Phi^+}$, we call a cover¹ of $\underline{s}$ under $<_{\mathbf{c}}^b$ a *c-minimal exponent for $\underline{s}$* . The *c-degree* of an exponent $\underline{m}$, denoted by $\deg_{\mathbf{c}}(\underline{m})$, is the largest integer $k \geq 0$ such that

$$(2.5) \quad \underline{m}^{(0)} <_{\mathbf{c}}^b \underline{m}^{(1)} <_{\mathbf{c}}^b \cdots <_{\mathbf{c}}^b \underline{m}^{(k-1)} <_{\mathbf{c}}^b \underline{m}^{(k)} = \underline{m}.$$

A pair $\langle\langle \alpha, \beta \rangle\rangle$ of $\alpha, \beta \in \Phi^+$ is called a *c-pair* if $\beta \not\prec_{\mathbf{c}} \alpha$ (i.e., $\alpha <_{[\underline{w}_0]} \beta$ for some $\underline{w}_0 \in \mathbf{c}$). We regard a *c-pair* $\underline{p} := \langle\langle \alpha, \beta \rangle\rangle$ as an exponent by $\underline{p}_\alpha = \underline{p}_\beta = 1$ and $\underline{p}_\gamma = 0$ for $\gamma \neq \alpha, \beta$. When there exists a unique *c-simple* exponent $\underline{s}$ satisfying $\underline{s} \prec_{\mathbf{c}}^b \underline{p}$, we call $\underline{s}$ the *c-head* of $\underline{p}$ and denote it by $\text{hd}_{\mathbf{c}}(\underline{p})$ ².

¹Recall that a *cover* of x in a poset P with partial order $<$ is an element $y \in P$ such that $x < y$ and there does not exist $y' \in P$ such that $x < y' < y$.

²In [50], we have used *socle* instead of *head*, because we use a different convention in Theorem 8.1 below

We regard a positive root $\alpha \in \Phi^+$ as an exponent $\underline{m}$ by $\underline{m}_\beta = \delta_{\beta, \alpha}$.

Remark 2.3. The definition of $\text{dist}_{[\tilde{w}_0]}(\underline{p})$ for a $\mathbf{c}$ -pair $\underline{p}$ in [50] is different from $\text{deg}_{[\tilde{w}_0]}(\underline{p})$ in (2.5), which was defined as follows:

$$(2.6) \quad \underline{m}^{(0)} \prec_{\mathbf{c}}^{\mathbf{b}} \underline{p}^{(1)} \prec_{\mathbf{c}}^{\mathbf{b}} \cdots \prec_{\mathbf{c}}^{\mathbf{b}} \underline{p}^{(k-1)} \prec_{\mathbf{c}}^{\mathbf{b}} \underline{p}^{(k)} = \underline{p},$$

for $\mathbf{c}$ -pairs $\underline{p}^{(i)}$ and $\underline{m}^{(0)}$ is $\mathbf{c}$ -simple. However, for a Dynkin quiver Q of simply-laced type, the statistics $\text{deg}_{[Q]}(\underline{p})$ for every $\mathbf{c}$ -pair $\underline{p}$ still coincides with the statistics $\text{dist}_{[Q]}(\underline{p})$ in [50] when Q is not of type E_7 and E_8 (see the next subsection for Dynkin quivers).

2.4. Dynkin quivers. A *Dynkin quiver* Q of Δ is an oriented graph whose underlying graph is Δ (see Example 2.8 below). For each Dynkin quiver of Q , we can associate a function $\xi: \Delta_0 \rightarrow \mathbb{Z}$, called a *height function* of Q which satisfies:

$$\xi_i = \xi_j + 1 \quad \text{if } i \rightarrow j \text{ in } Q.$$

Conversely, for a Dynkin diagram Δ and a function $\xi: \Delta_0 \rightarrow \mathbb{Z}$ satisfying $|\xi_i - \xi_j| = 1$ for $i, j \in I$ with $d(i, j) = 1$, we can associate a Dynkin quiver Q in a canonical way. Thus we can identify a Dynkin quiver Q with a pair (Δ, ξ) . Note that, since Δ is connected, height functions of Q differ by integers.

For a Dynkin quiver $Q = (\Delta, \xi)$, the function $\xi^*: \Delta_0 \rightarrow \mathbb{Z}$ given by $(\xi^*)_i = \xi_{i^*}$ is also a height function. Thus we set $Q^* = (\Delta, \xi^*)$.

For a Dynkin quiver Q , we call $i \in \Delta_0$ a *source* of Q if $\xi_i > \xi_j$ for all $j \in \Delta_0$ with $d(i, j) = 1$. For a Dynkin quiver $Q = (\Delta, \xi)$ and its source i , we denote by $s_i Q$ the Dynkin quiver $(\Delta, s_i \xi)$ where $s_i \xi$ is the height function defined by

$$(2.7) \quad (s_i \xi)_j = \xi_j - 2\delta_{i,j}.$$

Definition 2.4. Let $Q = (\Delta, \xi)$ be a Dynkin quiver.

- (i) A reduced expression $\underline{w} = s_{i_1} \cdots s_{i_l}$ of an element of $\mathbf{W}_\Delta$ is said to be *adapted to Q* (or *Q -adapted*) if

$$i_k \text{ is a source of } s_{i_{k-1}} s_{i_{k-2}} \cdots s_{i_1} Q \text{ for all } 1 \leq k \leq l.$$

- (ii) A *Coxeter element* τ of $\mathbf{W}_\Delta$ is a product of all simple reflections; i.e., there exists a reduced expression $s_{i_1} \cdots s_{i_{|I|}}$ of τ such that $\{i_1, \dots, i_{|I|}\} = I$.
- (iii) We denote by $\mathbf{h}$ the Coxeter number of Δ which is the order of a Coxeter element.

Theorem 2.5 ([53, 35]). *For each Dynkin quiver $Q = (\Delta, \xi)$, there exists a Q -adapted reduced expression $\underline{w}_0$ of $w_0 \in \mathbf{W}_\Delta$, and the set of all Q -adapted reduced expressions is a commutation class of w_0 .*

We denote by $[Q]$ the commutation class of w_0 consisting of all Q -adapted reduced expressions. It is known that, for a fixed Dynkin diagram Δ , the set of commutation classes $\{[Q] \mid Q = (\Delta, \xi)\}$ forms an r -cluster point, denoted by $\llbracket \Delta \rrbracket$.

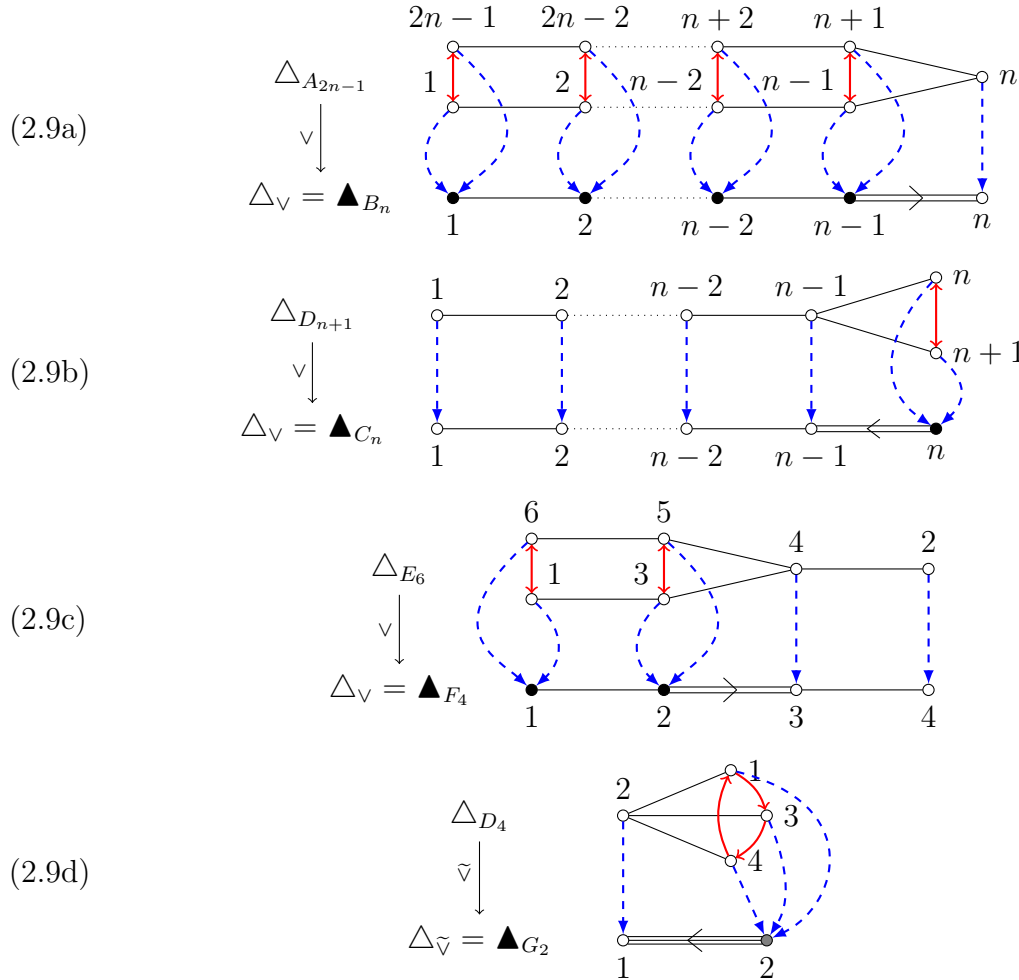
It is also well-known that all of reduced expressions of a Coxeter element τ form a single commutation class and they are adapted to some Dynkin quiver Q . Conversely, for each Dynkin quiver Q , there exists a unique Coxeter element τ_Q all of whose reduced expressions

are adapted to Q . All Coxeter elements are conjugate in W , and their common order is called the *Coxeter number* and denoted by h .

A bijection σ from Δ_0 to itself is said to be a *Dynkin diagram automorphism* if $\langle h_i, \alpha_j \rangle = \langle h_{\sigma(i)}, \alpha_{\sigma(j)} \rangle$ for all $i, j \in \Delta_0$. Throughout this paper, we assume that Dynkin diagram automorphisms σ satisfy the following condition:

$$(2.8) \quad \text{there is no } i \in \Delta_0 \text{ such that } d(i, \sigma(i)) = 1.$$

For a Dynkin diagram of type A_{2n-1} , D_n and E_6 , there exists a unique non-identity Dynkin diagram automorphism $\vee$ of order 2 (except D_4 -type in which case there are three automorphisms of order 2 and two automorphisms $\tilde{\vee}$ and $\tilde{\vee}^2$ of order 3) satisfying the condition in (2.8). The automorphisms $\vee$ and $\tilde{\vee}$ can be depicted as follows:



Definition 2.6. Let Δ be a simply-laced Dynkin diagram and σ a Dynkin diagram automorphism satisfying (2.8).

- (i) Let Δ_σ be the folding of Δ by σ . Namely, the set $(\Delta_\sigma)_0$ of vertices of Δ_σ is the set of σ -orbits. Let $\pi: \Delta_0 \rightarrow (\Delta_\sigma)_0$ be the projection. The edges of Δ_σ are given by the

Cartan matrix

$$c_{\pi(i), \pi(j)}^{\Delta_\sigma} = \sum_{j' \in \pi^{-1}\pi(j)} c_{i, j'}^\Delta.$$

Hence $d_{\pi(i)} = |\pi^{-1}\pi(i)|$.

- (ii) For a simple Lie algebra $\mathfrak{g}$ of simply-laced type associated to Δ , we denote by $\mathfrak{g}_\sigma$ the simple Lie algebra whose Dynkin diagram is Δ_σ .

Note that we have the following relations of Coxeter numbers:

$$(2.10) \quad h^{\Delta_\sigma} = h^\Delta.$$

Definition 2.7. Let $Q = (\Delta, \xi)$ be a Dynkin quiver with Δ and σ of a Dynkin diagram automorphism of Δ . We say that Q is σ -fixed if

$$\xi_{\sigma(i)} = \xi_i \quad \text{for all } i \in \Delta_0.$$

For a σ -fixed Dynkin quiver Q , we sometimes denote its height function by ${}^\sigma\xi$ instead of ξ , to emphasize.

Example 2.8. Here are several examples of σ -fixed Dynkin quivers Q .

$$(2.11a) \quad \begin{array}{ccccccc} \frac{3}{\circ} & \xrightarrow{\quad} & \frac{2}{\circ} & \xrightarrow{\quad} & \frac{1}{\circ} & \xleftarrow{\quad} & \frac{2}{\circ} & \xleftarrow{\quad} & \frac{3}{\circ} \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array} \quad \text{for } Q = (\Delta_{A_5}, {}^\vee\xi).$$

$$(2.11b) \quad \begin{array}{ccc} & & \frac{1}{\circ} \\ & & \swarrow \quad \searrow \\ \frac{3}{\circ} & \xrightarrow{\quad} & \frac{2}{\circ} \\ 1 & & 2 \end{array} \quad \text{for } Q = (\Delta_{D_4}, {}^\vee\xi).$$

$$(2.11c) \quad \begin{array}{ccccccc} & & \frac{2}{\circ} & & \frac{0}{\circ} & & \\ & & \uparrow \quad \downarrow & & \uparrow \quad \downarrow & & \\ \frac{3}{\circ} & \xrightarrow{\quad} & \frac{2}{\circ} & \xrightarrow{\quad} & \frac{1}{\circ} & \xleftarrow{\quad} & \frac{2}{\circ} & \xleftarrow{\quad} & \frac{3}{\circ} \\ 1 & & 3 & & 4 & & 5 & & 6 \end{array} \quad \text{for } Q = (\Delta_{E_6}, {}^\vee\xi).$$

$$(2.11d) \quad \begin{array}{ccc} & & \frac{1}{\circ} \\ & & \swarrow \quad \searrow \\ \frac{1}{\circ} & \xleftarrow{\quad} & \frac{2}{\circ} \\ 1 & & 2 \end{array} \quad \text{for } Q = (\Delta_{D_4}, \tilde{\vee}\xi).$$

Here

- (i) an underline integer $\ast$ is the value ξ_i at each vertex $i \in \Delta_0$,
- (ii) an arrow $\frac{\circ}{i} \xrightarrow{\quad} \frac{\circ}{j}$ means that $\xi_i > \xi_j$ and $d(i, j) = 1$.

The following lemma is obvious.

Lemma 2.9. Let Δ be a Dynkin diagram, and σ a Dynkin diagram automorphism of Δ . Then, the number of σ -fixed Dynkin quivers (as an oriented graph) is 2^r , where r is the number of edges of Δ_σ . In particular, the number of $\vee$ -fixed Dynkin quivers is

$$\begin{cases} 2^{n-1} & \text{if Dynkin is of type } A_{2n-1} \text{ or } D_{n+1}, \\ 8 & \text{if Dynkin is of type } E_6. \end{cases}$$

The number of $\tilde{\vee}$ -fixed Dynkin quivers of D_4 is 2.

3. t -QUANTIZED CARTAN MATRIX

In this section, we briefly recall the (q, t) -Cartan matrix $\mathbf{C}(q, t)$ introduced by Frenkel-Reshetikhin in [8], and the results about the relationship between $\mathbf{C}(1, t)^{-1}$ and Dynkin quivers in [17, 35]. Then we shall see that inverse of $\mathbf{C}(1, t)\mathbf{D}^{-1}$ of type Δ_σ can be obtained from the one of $\mathbf{C}(1, t)$ of type Δ via the *folding* by $\sigma \neq \text{id}$.

For an indeterminate x and $k \in \mathbb{Z}$, we set,

$$[k]_x := \frac{x^k - x^{-k}}{x - x^{-1}}.$$

For an indeterminate q and i , we set $q_i := q^{d_i}$. For instance, when $\mathfrak{g}$ is of type G_2 , we have $q_2 = q^3$.

For a given finite Cartan datum, we define the *adjacent matrix* $\mathcal{I} = (\mathcal{I}_{i,j})_{i,j \in I}$ of $\mathbf{C}$ by

$$(3.1) \quad \mathcal{I}_{i,j} = 2\delta_{i,j} - \mathbf{c}_{i,j} \quad \text{for } i, j \in I.$$

In [8], the (q, t) -deformation of Cartan matrix $\mathbf{C}(q, t) = (\mathbf{c}_{i,j}(q, t))_{i,j \in I}$ is introduced:

$$\mathbf{c}_{i,j}(q, t) := (q_i t^{-1} + q_i^{-1} t) \delta_{i,j} - [\mathcal{I}_{i,j}]_q.$$

The specialization of $\mathbf{C}(q, t)$ at $t = 1$, denoted by $\mathbf{C}_q(q) := \mathbf{C}(q, 1)$, is usually called the *quantum Cartan matrix*.

Definition 3.1. For each finite Cartan datum, we set

$$\mathbf{C}_t(t) := \mathbf{C}(1, t).$$

and call it *t-quantized Cartan matrix*. We also set

$$\underline{\mathbf{B}}(t) := \mathbf{C}_t(t)\mathbf{D}^{-1} = (\underline{\mathbf{B}}_{i,j}(t))_{i,j \in I} \quad \text{and} \quad \overline{\mathbf{B}}(t) := \mathbf{D}\mathbf{C}_t(t) = (\overline{\mathbf{B}}_{i,j}(t))_{i,j \in I}.$$

Hence we have

$$\underline{\mathbf{B}}_{i,j}(t) = \begin{cases} \mathbf{d}_i^{-1}(t + t^{-1}) & \text{if } i = j, \\ (\alpha_i^\vee, \alpha_j^\vee) & \text{if } i \neq j, \end{cases}$$

where $\alpha_i^\vee := \mathbf{d}_i^{-1} \alpha_i$ is the coroot.

Example 3.2. Note that, for simply-laced types, we have $\underline{\mathbf{B}}(t) = \mathbf{C}_t(t)$. The followings are $\underline{\mathbf{B}}(t)$ of non-simply-laced types:

$$\begin{aligned} \underline{\mathbf{B}}(t)_{B_n} &= \begin{pmatrix} \frac{t+t^{-1}}{2} & -\frac{1}{2} & 0 & 0 & \cdots & 0 \\ -\frac{1}{2} & \frac{t+t^{-1}}{2} & -\frac{1}{2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \cdots & 0 \\ 0 & \cdots & \cdots & -\frac{1}{2} & \frac{t+t^{-1}}{2} & -1 \\ 0 & \cdots & \cdots & 0 & -1 & t+t^{-1} \end{pmatrix}, \\ \underline{\mathbf{B}}(t)_{C_n} &= \begin{pmatrix} t+t^{-1} & -1 & 0 & 0 & \cdots & 0 \\ -1 & t+t^{-1} & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \cdots & 0 \\ 0 & \cdots & \cdots & -1 & t+t^{-1} & -1 \\ 0 & \cdots & \cdots & 0 & -1 & \frac{t+t^{-1}}{2} \end{pmatrix}, \end{aligned}$$

$$\underline{\mathbf{B}}(t)_{F_4} = \begin{pmatrix} \frac{t+t^{-1}}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{t+t^{-1}}{2} & -1 & 0 \\ 0 & -1 & t+t^{-1} & -1 \\ 0 & 0 & -1 & t+t^{-1} \end{pmatrix},$$

$$\underline{\mathbf{B}}(t)_{G_2} = \begin{pmatrix} t+t^{-1} & -1 \\ -1 & \frac{t+t^{-1}}{3} \end{pmatrix}.$$

Note that $\underline{\mathbf{B}}(t)|_{t=1} = \underline{\mathbf{B}} \in \mathrm{GL}_I(\mathbb{Q})$. We regard $\underline{\mathbf{B}}(t)$ as an element of $\mathrm{GL}_I(\mathbb{Q}(t))$ and denote its inverse by $\tilde{\underline{\mathbf{B}}}(t) := (\underline{\mathbf{B}}(t))^{-1} = (\tilde{\underline{\mathbf{B}}}_{i,j}(t))_{i,j \in I}$. Let

$$\tilde{\underline{\mathbf{B}}}_{i,j}(t) = \sum_{u \in \mathbb{Z}} \tilde{\mathbf{b}}_{i,j}(u) t^u$$

be the Laurent expansion of $\tilde{\underline{\mathbf{B}}}_{i,j}(t)$ at $t = 0$. Note that $\tilde{\underline{\mathbf{B}}}_{i,j}(t) = \tilde{\underline{\mathbf{B}}}_{j,i}(t)$ for all $i, j \in I$.

Recall that, for each Dynkin quiver $Q = (\Delta, \xi)$ of an arbitrary finite type, there exists a unique Coxeter element $\tau_Q \in W_\Delta$ such that all of its reduced expressions are adapted to Q .

Definition 3.3 ([17, 14, 35]). For a Dynkin quiver Q and $i, j \in I$, we define a function $\eta_{i,j}^Q: \mathbb{Z} \rightarrow \mathbb{Z}$ by

$$\eta_{i,j}^Q(u) := \begin{cases} (\varpi_i, \tau_Q^{(u+\xi_j-\xi_i-1)/2}(\gamma_j^Q)) & \text{if } u + \xi_j - \xi_i - 1 \in 2\mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\gamma_j^Q := (1 - \tau_Q)\varpi_j \in \Phi^+$.

Lemma 3.4 ([17, 35]). Let Q' be another Dynkin quiver of the same type of Q . Then we have

$$\eta_{i,j}^Q = \eta_{i,j}^{Q'} \quad \text{and hence } \eta_{i,j}^\Delta \text{ is well-defined.}$$

Theorem 3.5 ([17, 35]). For each $i, j \in I$ and $u \in \mathbb{Z}_{\geq 0}$, we have

$$\tilde{\mathbf{b}}_{i,j}(u) = \eta_{i,j}(u).$$

In other words, we can compute $\tilde{\underline{\mathbf{B}}}(t)$ by using Γ_Q of any Dynkin quiver Q as follows:

$$\tilde{\mathbf{b}}_{i,j}(u) = \begin{cases} (\varpi_i, \tau_Q^{(u+\xi_j-\xi_i-1)/2}(\gamma_j^Q)) & \text{if } u + \xi_j - \xi_i - 1 \in 2\mathbb{Z} \text{ and } u \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 3.6 ([17, 14, 13, 35]). The coefficients $\{\tilde{\mathbf{b}}_{i,j}(u) \mid i, j \in I, u \in \mathbb{Z}_{\geq 0}\}$ enjoy the following properties:

- (1) $\tilde{\mathbf{b}}_{i,j}(u) = 0$ for $u \leq 0$, and $\tilde{\mathbf{b}}_{i,j}(1) = \mathbf{d}_i \delta(i = j)$.
- (2) $\tilde{\mathbf{b}}_{i,j}(u-1) + \tilde{\mathbf{b}}_{i,j}(u+1) = \sum_{k; d(k,j)=1} -\langle h_k, \alpha_j \rangle \tilde{\mathbf{b}}_{i,k}(u)$ for $u \geq 1$.
- (3) $\tilde{\mathbf{b}}_{i,j}(u+h) = -\tilde{\mathbf{b}}_{i,j^*}(u)$ and $\tilde{\mathbf{b}}_{i,j}(u+2h) = \tilde{\mathbf{b}}_{i,j}(u)$ for $u \geq 0$.
- (4) $\tilde{\mathbf{b}}_{i,j}(h-u) = \tilde{\mathbf{b}}_{i,j^*}(u)$ for $0 \leq u \leq h$ and $\tilde{\mathbf{b}}_{i,j}(2h-u) = -\tilde{\mathbf{b}}_{i,j}(u)$ for $0 \leq u \leq 2h$.
- (5) $\tilde{\mathbf{b}}_{i,j}(u) \geq 0$ for $0 \leq u \leq h$ and $\tilde{\mathbf{b}}_{i,j}(u) \leq 0$ for $h \leq u \leq 2h$.

By Corollary 3.6 (3), it suffices to compute the $\widetilde{\mathbf{b}}_{i,j}(u)$ for $1 \leq u \leq \mathbf{h}$, to know $\widetilde{\mathbf{B}}_{i,j}(t)$ for any $i, j \in \Delta_0$. Thus we set

$$\widetilde{\mathbf{d}}_{i,j}(t) := \widetilde{\mathbf{B}}_{i,j}(t) + t^{\mathbf{h}} \widetilde{\mathbf{B}}_{i,j^*}(t) = \sum_{u=1}^{\mathbf{h}} \widetilde{\mathbf{b}}_{i,j}(u) t^u$$

and

$$\widetilde{\mathbf{d}}_{i,j}[k] := \widetilde{\mathbf{b}}_{i,j}(k-1) \quad \text{for } 0 \leq k \leq \mathbf{h}.$$

Then, we have

$$\widetilde{\mathbf{B}}_{i,j}(t) = \frac{\widetilde{\mathbf{d}}_{i,j}(t) - t^{2\mathbf{h}} \widetilde{\mathbf{d}}_{i,j}(t^{-1})}{1 - t^{2\mathbf{h}}}.$$

Theorem 3.7. *Let σ be a Dynkin diagram automorphism, Δ_σ the folding of Δ by σ , and $\pi: (\Delta)_0 \rightarrow (\Delta_\sigma)_0$ the projection (see Definition 2.6). Then we have the following relation:*

$$\widetilde{\mathbf{d}}_{i,j}^{\Delta_\sigma}(t) = \sum_{\substack{\widetilde{i} \in \pi^{-1}(i), \\ \widetilde{j} \in \pi^{-1}(j)}} \widetilde{\mathbf{d}}_{\widetilde{i},\widetilde{j}}^{\Delta}(t) \quad \text{for any } i, j \in (\Delta_\sigma)_0.$$

Proof. First let us remark the relations of $\mathbf{B}$ for Δ_σ and Δ :

$$(\mathbf{B}^{\Delta_\sigma})_{i,\pi(\widetilde{j})} = \mathbf{d}_i^{-1} \sum_{\widetilde{i} \in \pi^{-1}(i)} (\mathbf{B}^{\Delta})_{\widetilde{i},\widetilde{j}} \quad \text{for any } i \in (\Delta_\sigma)_0 \text{ and } \widetilde{j} \in \Delta_0.$$

Now set

$$\begin{aligned} f_{\widetilde{i},\widetilde{j}}^{\Delta} &= \left((\mathbf{B}^{\Delta})^{-1} \right)_{\widetilde{i},\widetilde{j}} = (1 + t^{\mathbf{h}})^{-1} \widetilde{\mathbf{d}}_{\widetilde{i},\widetilde{j}}^{\Delta}(t), \\ f_{i,j}^{\Delta_\sigma} &= \sum_{\widetilde{i} \in \pi^{-1}(i), \widetilde{j} \in \pi^{-1}(j)} f_{\widetilde{i},\widetilde{j}}^{\Delta} \end{aligned}$$

for $\widetilde{i}, \widetilde{j} \in \Delta_0$ and $i, j \in (\Delta_\sigma)_0$.

In order to see the theorem, it is enough to show that

$$(3.2) \quad \sum_{j \in (\Delta_\sigma)_0} (\mathbf{B}^{\Delta_\sigma})_{i,j} f_{j,k}^{\Delta_\sigma} = \delta_{i,k} \quad \text{for any } i, k \in (\Delta_\sigma)_0.$$

We have

$$\delta_{i,\widetilde{k}} = \sum_{\widetilde{j} \in (\Delta)_0} (\mathbf{B}^{\Delta})_{\widetilde{i},\widetilde{j}} f_{\widetilde{j},\widetilde{k}}^{\Delta}$$

for any $\widetilde{i}, \widetilde{k} \in \Delta_0$. Hence for $i \in (\Delta_\sigma)_0$ and $\widetilde{k} \in \Delta_0$, we have

$$\begin{aligned} \delta_{i,\pi(\widetilde{k})} &= \sum_{\widetilde{i} \in \pi^{-1}(i)} \delta_{\widetilde{i},\widetilde{k}} = \sum_{\widetilde{i} \in \pi^{-1}(i), \widetilde{j} \in (\Delta)_0} (\mathbf{B}^{\Delta})_{\widetilde{i},\widetilde{j}} f_{\widetilde{j},\widetilde{k}}^{\Delta} \\ &= \sum_{\widetilde{j} \in (\Delta)_0} \mathbf{d}_i (\mathbf{B}^{\Delta_\sigma})_{i,\pi(\widetilde{j})} f_{\widetilde{j},\widetilde{k}}^{\Delta}. \end{aligned}$$

Hence, for any $i, k \in (\Delta_\sigma)_0$, we have

$$\begin{aligned}
d_k \delta_{i,k} &= \sum_{\tilde{k} \in \pi^{-1}(k)} \delta_{i, \pi(\tilde{k})} = \sum_{\tilde{k} \in \pi^{-1}(k), \tilde{j} \in (\Delta)_0} d_i(\underline{\mathbb{B}}^{\Delta_\sigma})_{i, \pi(\tilde{j})} f_{\tilde{j}, \tilde{k}}^\Delta \\
&= \sum_{\tilde{k} \in \pi^{-1}(k), j \in (\Delta_\sigma)_0, \tilde{j} \in \pi^{-1}(j)} d_i(\underline{\mathbb{B}}^{\Delta_\sigma})_{i, j} f_{\tilde{j}, \tilde{k}}^\Delta \\
&= \sum_{j \in (\Delta_\sigma)_0} d_i(\underline{\mathbb{B}}^{\Delta_\sigma})_{i, j} f_{j, k}^{\Delta_\sigma}.
\end{aligned}$$

Thus we obtain (3.2). □

The polynomials $\tilde{\mathfrak{d}}_{i,j}(t)$ are calculated explicitly as follows:

Theorem 3.8 ([17, 11, 51, 35]). *Note that $\tilde{\mathfrak{d}}_{i,j}(t) = \tilde{\mathfrak{d}}_{j,i}(t)$ for $i, j \in I$.*

(1) *For Δ of type A_n , and $i, j \in I = \{1, \dots, n\}$, $\tilde{\mathfrak{d}}_{i,j}(t)$ is given as follows:*

$$(3.3) \quad \tilde{\mathfrak{d}}_{i,j}(t) = \sum_{s=1}^{\min(i, j, n+1-i, n+1-j)} t^{|i-j|+2s-1}.$$

(2) *For Δ of type D_{n+1} , and $i, j \in I = \{1, \dots, n, n+1\}$, $\tilde{\mathfrak{d}}_{i,j}(t)$ is given as follows:*

$$(3.4) \quad \tilde{\mathfrak{d}}_{i,j}(t) = \begin{cases} \sum_{s=1}^{\min(i, j)} (t^{|i-j|+2s-1} + \delta(\max(i, j) < n) t^{2n-i-j+2s-1}) & \text{if } \min(i, j) < n, \\ \sum_{s=1}^{\lfloor (n+\delta_{i,j})/2 \rfloor} t^{4s-1-2\delta(i, j)} & \text{otherwise.} \end{cases}$$

(3) *For Δ of type B_n or C_n , and $i, j \in I = \{1, \dots, n\}$, $\tilde{\mathfrak{d}}_{i,j}(t)$ is given as follows:*

$$(3.5) \quad \tilde{\mathfrak{d}}_{i,j}(t) = \max(d_i, d_j) \sum_{s=1}^{\min(i, j)} \left(t^{|i-j|+2s-1} + \delta(\max(i, j) < n) t^{2n-i-j+2s-1} \right).$$

(4) *For Δ of type E_6 and $i \leq j \in I = \{1, \dots, 6\}$, $\tilde{\mathfrak{d}}_{i,j}(t)$ is given as follows:*

$$\begin{aligned}
\tilde{\mathfrak{d}}_{1,1}(t) &= t + t^7, & \tilde{\mathfrak{d}}_{1,2}(t) &= t^4 + t^8, \\
\tilde{\mathfrak{d}}_{1,3}(t) &= t^2 + t^6 + t^8, & \tilde{\mathfrak{d}}_{1,4}(t) &= t^3 + t^5 + t^7 + t^9, \\
\tilde{\mathfrak{d}}_{1,5}(t) &= t^4 + t^6 + t^{10}, & \tilde{\mathfrak{d}}_{1,6}(t) &= t^5 + t^{11}, \\
\tilde{\mathfrak{d}}_{2,2}(t) &= t^1 + t^5 + t^7 + t^{11}, & \tilde{\mathfrak{d}}_{2,3}(t) &= t^3 + t^5 + t^7 + t^9, \\
\tilde{\mathfrak{d}}_{2,4}(t) &= t^2 + t^4 + 2t^6 + t^8 + t^{10}, & \tilde{\mathfrak{d}}_{3,3}(t) &= t^1 + t^3 + t^5 + 2t^7 + t^9, \\
\tilde{\mathfrak{d}}_{3,4}(t) &= t^2 + 2t^4 + 2t^6 + 2t^8 + t^{10}, & \tilde{\mathfrak{d}}_{3,5}(t) &= t^3 + 2t^5 + t^7 + t^9 + t^{11}, \\
\tilde{\mathfrak{d}}_{4,4}(t) &= t^1 + 2t^3 + 3t^5 + 3t^7 + 2t^9 + t^{11}, & \tilde{\mathfrak{d}}_{i,j}(t) &= t^h \tilde{\mathfrak{d}}_{i,j^*}(t^{-1}) = \tilde{\mathfrak{d}}_{j,i}(t) = t^h \tilde{\mathfrak{d}}_{j,i^*}(t^{-1}).
\end{aligned}$$

(5) *For E_7 and E_8 , see Appendix A.*

(6) For Δ of type F_4 and $i \leq j \in I = \{1, 2, 3, 4\}$, $\tilde{\mathfrak{d}}_{i,j}(t)$ is given as follows:

$$\begin{aligned} \tilde{\mathfrak{d}}_{1,1}(t) &= 2(t + t^5 + t^7 + t^{11}), & \tilde{\mathfrak{d}}_{1,2}(t) &= 2(t^2 + t^4 + 2t^6 + t^8 + t^{10}), \\ \tilde{\mathfrak{d}}_{1,3}(t) &= 2(t^3 + t^5 + t^7 + t^9), & \tilde{\mathfrak{d}}_{1,4}(t) &= 2(t^4 + t^8), \\ \tilde{\mathfrak{d}}_{2,2}(t) &= 2(t + 2t^3 + 3t^5 + 3t^7 + 2t^9 + t^{11}), & \tilde{\mathfrak{d}}_{2,3}(t) &= 2(t^2 + 2t^4 + 2t^6 + 2t^8 + t^{10}), \\ \tilde{\mathfrak{d}}_{2,4}(t) &= 2(t^3 + t^5 + t^7 + t^9), & \tilde{\mathfrak{d}}_{3,3}(t) &= t + 2t^3 + 3t^5 + 3t^7 + 2t^9 + t^{11}, \\ \tilde{\mathfrak{d}}_{3,4}(t) &= t^2 + t^4 + 2t^6 + t^8 + t^{10}, & \tilde{\mathfrak{d}}_{4,4}(t) &= t + t^5 + t^7 + t^{11}. \end{aligned}$$

(7) For Δ of type G_2 and $i \leq j \in I = \{1, 2\}$, $\tilde{\mathfrak{d}}_{i,j}(t)$ is given as follows:

$$\tilde{\mathfrak{d}}_{1,1}(t) = t + 2t^3 + t^5, \quad \tilde{\mathfrak{d}}_{1,2}(t) = 3(t^2 + t^4), \quad \tilde{\mathfrak{d}}_{2,2}(t) = 3\tilde{\mathfrak{d}}_{1,1}(t).$$

4. AUSLANDER-REITEN (AR) QUIVERS

In this section, we first recall (combinatorial) Auslander-Reiten quiver Γ_Q associated to each Dynkin quiver Q . Then we recall their combinatorial properties, including the simple algorithm for labeling them with the set of positive roots, and statistics in classical A_n and D_n -cases. In the next section, we will investigate the relationship among AR-quivers related to the Dynkin diagram folding σ .

4.1. Quivers. For each reduced expression $\underline{w}_0 = s_{i_1} \cdots s_{i_\ell}$ of $w_0 \in \mathbf{W}$, we associate a quiver $\Upsilon_{\underline{w}_0}$ as follows [52]:

- (i) The set of vertices is $\Phi^+ = \{\beta_k^{\underline{w}_0} \mid 1 \leq k \leq \ell\}$.
- (ii) We assign $(-\langle h_{i_k}, \alpha_{i_l} \rangle)$ -many arrows from $\beta_k^{\underline{w}_0}$ to $\beta_l^{\underline{w}_0}$ if and only if $1 \leq l < k \leq \ell$ and there is no j such that $l < j < k$ and $i_j \in \{i_k, i_l\}$.

We say that a total order $(\beta_1 < \cdots < \beta_\ell)$ of Φ^+ is a *compatible reading* of $\Upsilon_{\underline{w}_0}$ if we have $k < l$ whenever there is an arrow from β_l to β_k in $\Upsilon_{\underline{w}_0}$.

Theorem 4.1 ([52, 35]). *Let $\mathbf{c}$ be a commutation class.*

- (i) *Two reduced expressions $\underline{w}_0$ and $\underline{w}'_0$ of w_0 are commutation equivalent if and only if $\Upsilon_{\underline{w}_0} = \Upsilon_{\underline{w}'_0}$ as quivers. Hence $\Upsilon_{\mathbf{c}}$ is well-defined.*
- (ii) *For $\alpha, \beta \in \Phi^+$, $\alpha \preceq_{\mathbf{c}} \beta$ if and only if there exists a path from β to α in $\Upsilon_{\mathbf{c}}$. In other words, the quiver $\Upsilon_{\mathbf{c}}$ (forgetting the number of arrows) is the Hasse quiver of the partial order $\preceq_{\mathbf{c}}$.*
- (iii) *If $\alpha, \beta \in \Phi^+$ are not comparable with respect to $\preceq_{\mathbf{c}}$, then we have $(\alpha, \beta) = 0$.*
- (iv) *If $\Phi^+ = \{\beta_1 < \cdots < \beta_\ell\}$ is a compatible reading of $\Upsilon_{\mathbf{c}}$, then there is a unique reduced expression $\underline{w}_0 = s_{i_1} \cdots s_{i_\ell}$ in $\mathbf{c}$ such that $\beta_k = \beta_k^{\underline{w}_0}$ for any k .*

Definition 4.2 ([35]). For a Dynkin quiver $Q = (\Delta, \xi)$, the *repetition quiver* $\hat{\Delta} = (\hat{\Delta}_0, \hat{\Delta}_1)$ associated to Q is defined as follows:

$$(4.1) \quad \begin{aligned} \hat{\Delta}_0 &:= \{(i, p) \in \Delta_0 \times \mathbb{Z} \mid p - \xi_i \in 2\mathbb{Z}\}, \\ \hat{\Delta}_1 &:= \{(i, p) \xrightarrow{-(h_i, \alpha_j)} (j, p+1) \mid (i, p), (j, p+1) \in \hat{\Delta}_0, d(i, j) = 1\}. \end{aligned}$$

Here $(i, p) \xrightarrow{-\langle h_i, \alpha_j \rangle} (j, p+1)$ denotes that we assign $(-\langle h_i, \alpha_j \rangle)$ -many arrows from (i, p) to $(j, p+1)$. Note that the definition of $\widehat{\Delta}$ depend only on the parity of the height function of Q .

For any Dynkin quiver $Q = (\Delta, \xi)$, we have a bijection $\phi_Q: \widehat{\Delta}_0 \xrightarrow{\sim} \widehat{\Phi}_\Delta^+ := \Phi_\Delta^+ \times \mathbb{Z}$ as follows (see [17, 35]): set $\gamma_i^Q := (1 - \tau_Q)\varpi_i \in \Phi_\Delta^+$ and

$$(4.2) \quad \begin{aligned} & \text{(i) } \phi_Q(i, \xi_i) = (\gamma_i^Q, 0), \\ & \text{(ii) if } \phi_Q(i, p) = (\beta, u), \text{ then we define} \\ & \quad \text{(a) } \phi_Q(i, p \mp 2) = (\tau_Q^{\pm 1}(\beta), u) \text{ if } \tau_Q^{\pm 1}(\beta) \in \Phi_\Delta^+, \\ & \quad \text{(b) } \phi_Q(i, p \mp 2) = (-\tau_Q^{\pm 1}(\beta), u \pm 1) \text{ if } \tau_Q^{\pm 1}(\beta) \in \Phi_\Delta^-. \end{aligned}$$

We say that (i, p) with $\phi_Q(i, p) = (\beta, u)$ is the Q -coordinate of (β, u) in $\widehat{\Delta}$.

The combinatorial description γ_i^Q is given as follows [17, §2.2]:

$$(4.3) \quad \gamma_i^Q = \sum_{j \in B^Q(i)} \alpha_j,$$

where $B^Q(i)$ denotes the set of vertices j such that there exists a path in Q from j to i .

For each Dynkin quiver $Q = (\Delta, \xi)$, we also denote by $\Gamma_Q = ((\Gamma_Q)_0, (\Gamma_Q)_1)$ the full subquiver of $\widehat{\Delta}$ whose set of vertices is $\phi_Q^{-1}(\Phi_\Delta^+ \times 0)$. We call Γ_Q the *AR-quiver associated to Q* .

Theorem 4.3 ([17, 52, 35]). *Let $Q = (\Delta, \xi)$ be a Dynkin quiver.*

(i) *We have $\xi_i \equiv \xi_{i^*} - \mathbf{h} \pmod{2}$ for any $i \in I$ and*

$$(4.4) \quad (\Gamma_Q)_0 = \{(i, p) \in \widehat{\Delta}_0 \mid \xi_i \geq p > \xi_{i^*} - \mathbf{h}\}.$$

In particular, we have

$$\phi_Q(i, p) = (\tau_Q^{(\xi_i - p)/2}(\gamma_i^Q), 0) \quad \text{for any } (i, p) \in (\Gamma_Q)_0.$$

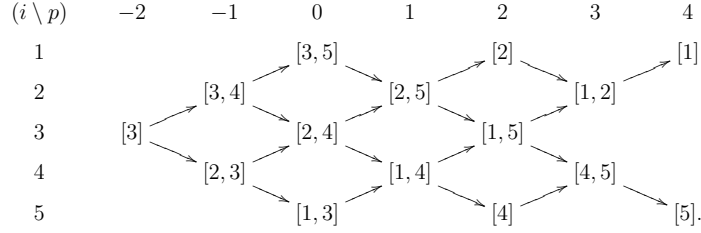
(ii) *If i is a source of Q , then we have*

$$\phi_{s_i Q}(i^*, \xi_i - \mathbf{h}) = \phi_Q(i, \xi_i) = (\alpha_i, 0) \text{ and } \Gamma_{s_i Q} = (\Gamma_Q \setminus \{(i, \xi_i)\}) \cup \{(i^*, \xi_i - \mathbf{h})\}.$$

(iii) *The map ϕ_Q induces a quiver isomorphism $\Gamma_Q \xrightarrow{\sim} \Upsilon_{[Q]}$. Recall that $[Q]$ is the commutation class of Q -adapted reduced expressions.*

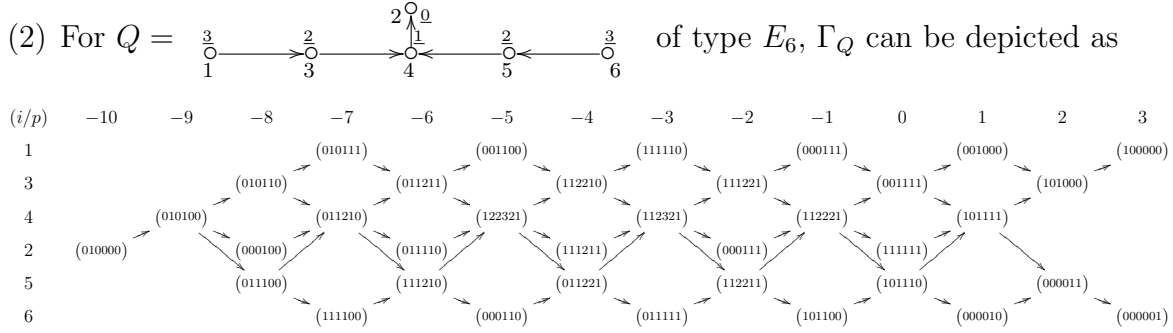
Example 4.4. Here are some examples of Γ_Q for Q of simply-laced types.

(1) For $Q = \begin{array}{c} \frac{4}{1} \longrightarrow \frac{3}{2} \longrightarrow \frac{2}{3} \longleftarrow \frac{3}{4} \longleftarrow \frac{4}{5} \end{array}$ of type A_5 , Γ_Q can be depicted as



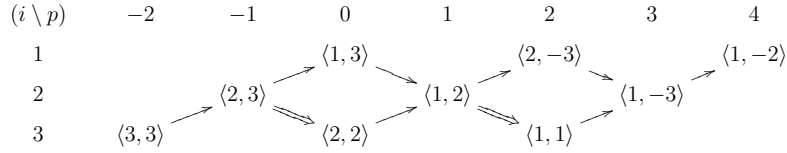
Here $[a, b] := \sum_{k=a}^b \alpha_k$ for $1 \leq a < b \leq 5$ and $[a] := \alpha_a$ for $1 \leq a \leq 5$.

(2) For $Q = \begin{array}{c} \frac{3}{1} \longrightarrow \frac{2}{3} \longrightarrow \frac{2}{4} \longleftarrow \frac{2}{5} \longleftarrow \frac{3}{6} \end{array}$ of type E_6 , Γ_Q can be depicted as



Here $(a_1 a_2 a_3 a_4 a_5 a_6) := \sum_{i=1}^6 a_i \alpha_i$.

Example 4.5. For $Q = \begin{array}{c} \frac{4}{1} \longrightarrow \frac{3}{2} \longrightarrow \frac{2}{3} \end{array}$ of type C_3 , Γ_Q can be depicted as



Here $\langle a, \pm b \rangle := \epsilon_a \pm \epsilon_b$ for $1 \leq a < b \leq 4$ and $\langle a, a \rangle := 2\epsilon_a$ for $1 \leq a \leq 3$.

Note the followings:

- (a) The vertices in Γ_Q are labeled by Φ_{Δ}^+ by the bijection $\phi_Q|_{(\Gamma_Q)_0}$.
- (b) For a source i of Q , $\Gamma_{s_i Q}$ can be obtained from Γ_Q by following way:
 - (i) Each $\beta \in \Phi^+ \setminus \{\alpha_i\}$ is located at (j, p) in $\Gamma_{s_i Q}$, if $s_i \beta$ is at (j, p) in Γ_Q .
 - (ii) α_i is located at the coordinate $(i^*, \xi_i - h)$ in $\Gamma_{s_i Q}$, while α_i was at (i, ξ_i) in Γ_Q .

We refer the operation in (4.5) (b) the *reflection* from Γ_Q to $\Gamma_{s_i Q}$.

Let Q be a Dynkin diagram except E -types. We say an arrow $(i, p) \rightarrow (j, p+1)$ in Γ_Q is a *downward* (resp. *upward*) arrow if $j > i$ (resp. $j < i$) by identifying vertices of Δ_0 with integers as in (2.1).

For Δ of E -types, we give a total order $<$ on $\{1, 2, \dots, 8\}$ as follows:

$$1 < 3 < 4 < 2 < 5 < 6 < 7 < 8.$$

Then we can define a downward (resp. upward) arrow in Γ_Q as in other cases (see Example 4.4 (2) for E_6 -case).

Definition 4.6.

- (i) A $[Q]$ -pair $\langle\langle \alpha, \beta \rangle\rangle \in (\Phi^+)^2$ with $\phi_Q(i, p) = (\alpha, 0)$ and $\phi_Q(j, s) = (\beta, 0)$ is *sectional* in Γ_Q if $d(i, j) = p - s$.
- (ii) A full subquiver ρ of Γ_Q is *sectional* if every $[Q]$ -pair $\langle\langle \alpha, \beta \rangle\rangle$ in ρ is sectional.
- (iii) A connected full subquiver ρ in Γ_Q is said to be *S-sectional* (resp. *N-sectional*) path if it is concatenation of downward (resp. upward) arrows.
- (iv) An *S-sectional* (resp. *N-sectional*) path ρ is said to be *N-path* (resp. *S-path*) if there is no longer connected sectional path in Γ_Q consisting of downward (resp. upward) arrows and containing ρ .

4.2. Classical AR-quiver Γ_Q . In this subsection, we review the labeling and the combinatorial properties of Γ_Q for Dynkin quivers Q of type *ADE*. Throughout this subsection, $Q = (\Delta, \xi)$ denotes a Dynkin quiver of type *ADE*.

Proposition 4.7 ([2, 20]). (1) Let Q be a Dynkin quiver of *ADE*-type, and $\mathfrak{h}$ the Coxeter number. For $i \in I$, let

$$r_i^Q := \frac{\mathfrak{h} + a_i^Q - b_i^Q}{2}$$

where a_i^Q is the number of arrows in Q between i and i^* directed toward i^* , b_i^Q is the number of arrows in Q between i and i^* directed toward i ($a_i^Q - b_i^Q = \xi_i - \xi_{i^*}$). Then we have $r_i^Q \in \mathbb{Z}_{\geq 0}$ and

$$(\Gamma_Q)_0 = \{(i, \xi_i - 2k) \mid 0 \leq k < r_i^Q\}.$$

(2) For $i, j \in \Delta_0$ and any Q , we have

$$(i, \xi_j - d(i, j)), (i, \xi_j - 2r_j^Q + d(i, j)) \in \Gamma_Q.$$

Note that, if Q is σ -fixed ($\sigma \neq \text{id}$), we have

$$(4.6) \quad r_i^Q = \mathfrak{h}/2 \quad \text{for all } i.$$

Proposition 4.8 ([50, Proposition 4.1]). Let Q be a Dynkin quiver Q of type *ADE* and let $\langle\langle \alpha, \beta \rangle\rangle$ be a $[Q]$ -pair sectional in Γ_Q . Then, we have

- (a) the $[Q]$ -pair $\langle\langle \alpha, \beta \rangle\rangle$ is $[Q]$ -simple,
- (b) $(\alpha, \beta) = 1$,
- (c) $\alpha - \beta \in \Phi$,
- (d) either $\langle\langle \alpha, \beta - \alpha \rangle\rangle$ is a $[Q]$ -minimal $[Q]$ -pair for β or $\langle\langle \alpha - \beta, \beta \rangle\rangle$ is a $[Q]$ -minimal $[Q]$ -pair for α .

Definition 4.9. For $i, j \in I$ and integer $k \in \mathbb{Z}_{\geq 0}$, let $\Phi_Q(i, j)[k]$ be the set of $[Q]$ -pairs $\langle\langle \alpha, \beta \rangle\rangle$ such that

$$|p - s| = k \quad \text{when } \{\phi_Q(i, p), \phi_Q(j, s)\} = \{(\alpha, 0), (\beta, 0)\}.$$

Lemma 4.10 ([50, Lemma 6.15]). *For any $\langle\langle\alpha, \beta\rangle\rangle, \langle\langle\alpha', \beta'\rangle\rangle \in \Phi_Q(i, j)[k]$, we have*

$$\deg_{[Q]}(\langle\langle\alpha, \beta\rangle\rangle) = \deg_{[Q]}(\langle\langle\alpha', \beta'\rangle\rangle).$$

For $i, j \in I$ and $k \in \mathbb{Z}_{>0}$, we define $\mathfrak{o}_k^Q(i, j)$ by

$$(4.7) \quad \mathfrak{o}_k^Q(i, j) := \begin{cases} \deg_{[Q]}(\langle\langle\alpha, \beta\rangle\rangle) & \text{if } \langle\langle\alpha, \beta\rangle\rangle \in \Phi_Q(i, j)[k], \\ 0 & \text{if } \Phi_Q(i, j)[k] = \emptyset. \end{cases}$$

It is well-defined by the above lemma.

Definition 4.11. Let t be an indeterminate. For $i, j \in I$ and a Dynkin quiver Q of type ADE, we define a polynomial $\mathfrak{d}_{i,j}^Q(z) \in \mathbb{Z}_{\geq 0}[t]$

$$\mathfrak{d}_{i,j}^Q(t) := \sum_{k \in \mathbb{Z}_{\geq 0}} \mathfrak{o}_k^Q(i, j) t^{k-1},$$

where

$$(4.8) \quad \mathfrak{o}_k^Q(i, j) := \max(\mathfrak{o}_k^Q(i, j), \mathfrak{o}_k^{Q^*}(i, j)).$$

Proposition 4.12 ([50, Proposition 6.18]). *For any Dynkin quivers Q, Q' of Δ , we have*

$$\mathfrak{o}_k^Q(i, j) = \mathfrak{o}_k^{Q'}(i, j).$$

Thus $\mathfrak{o}_k^\Delta(i, j)$ is well-defined.

Remark 4.13. In [50], $\mathfrak{o}_k^Q(i, j)$ was defined as $\max(\mathfrak{o}_k^Q(i, j), \mathfrak{o}_k^{Q^{\text{rev}}}(i, j))$, where Q^{rev} is a Dynkin quiver obtained from Q by reversing all arrows. However, by applying the argument in [50], one can check

$$\max(\mathfrak{o}_k^Q(i, j), \mathfrak{o}_k^{Q^{\text{rev}}}(i, j)) = \max(\mathfrak{o}_k^Q(i, j), \mathfrak{o}_k^{Q^*}(i, j)).$$

By Proposition 4.12, we can replace the superscript Q in $\mathfrak{d}_{i,j}^Q(t)$ with Δ . We call $\mathfrak{d}_{i,j}^\Delta(t)$ the *degree polynomial of Δ at (i, j)* . For $k \in \mathbb{Z}$, we set

$$\mathfrak{d}_{i,j}[k] := \mathfrak{o}_k^\Delta(i, j).$$

In the next § 4.2.1 and § 4.2.2 we recall the results in [48, 49, 50] on the combinatorial properties and statistics of AR-quivers of type A_n and D_n , and prove that $\widetilde{\mathfrak{d}}_{i,j}^\Delta(t) = \mathfrak{d}_{i,j}^\Delta(t) + \delta_{i,j^*} t^{h-1}$ in those types.

4.2.1. D_{n+1} -case. Let Q be a Dynkin quiver of type D_{n+1} . The simple roots $\{\alpha_i \mid 1 \leq i \leq n+1\}$ and $\Phi_{D_{n+1}}^+$ can be identified in $Q^+ \subset \bigoplus_{i=1}^{n+1} \mathbb{R}\epsilon_i$ as follows:

$$\begin{aligned}
 & \alpha_i = \epsilon_i - \epsilon_{i+1} \quad \text{for } i < n+1 \quad \text{and} \quad \alpha_{n+1} = \epsilon_n + \epsilon_{n+1}, \\
 & \Phi_{D_{n+1}}^+ = \left\{ \epsilon_i - \epsilon_j = \sum_{i \leq k < j} \alpha_k \mid 1 \leq i < j \leq n \right\} \\
 (4.9) \quad & \sqcup \left\{ \epsilon_i - \epsilon_{n+1} = \sum_{k=i}^n \alpha_k \mid 1 \leq i \leq n \right\} \sqcup \left\{ \epsilon_i + \epsilon_{n+1} = \sum_{k=i}^{n-1} \alpha_k + \alpha_{n+1} \mid 1 \leq i \leq n \right\} \\
 & \sqcup \left\{ \epsilon_i + \epsilon_j = \sum_{i \leq k < j} \alpha_k + 2 \sum_{j \leq k < n} \alpha_s + (\alpha_n + \alpha_{n+1}) \mid 1 \leq i < j < n+1 \right\}.
 \end{aligned}$$

Convention 2. For $2 \leq a \leq n+1$, we write ϵ_{-a} to denote $-\epsilon_a$. For $\beta = \epsilon_a + \epsilon_b \in \Phi$ ($1 \leq |a| < |b| \leq n+1$), we call ϵ_a and ϵ_b components of β .

Remark 4.14. Note the followings:

- (i) For each $1 \leq a \leq n$, there are exactly $(2n+1-a)$ -many positive roots having ϵ_a as their components.
- (ii) For each $2 \leq a \leq n$, there are exactly $(a-1)$ -many positive roots having ϵ_{-a} as their components.
- (iii) There are exactly n -many positive roots having $\epsilon_{\pm(n+1)}$ as their components.

Definition 4.15. A connected subquiver ρ in Γ_Q is called a *swing* if it consists of vertices and arrows in the following way: There exist positive roots $\alpha, \beta \in \Phi^+$ and $r, s \leq n$ such that

$$S_r \rightarrow S_{r+1} \rightarrow \cdots \rightarrow S_{n-1} \begin{array}{c} \nearrow \beta \\ \searrow \alpha \end{array} N_{n-1} \rightarrow N_{n-2} \rightarrow \cdots \rightarrow N_s, \text{ where}$$

- (a) $S_r \rightarrow S_{r+1} \rightarrow \cdots \rightarrow S_{n-1} \begin{array}{c} \nearrow \beta \\ \searrow \alpha \end{array}$ is an S -path ($\phi_Q^{-1}(S_l, 0) = (l, k)$ for some $k \in \mathbb{Z}$),
 - (b) $\begin{array}{c} \beta \searrow \\ \alpha \nearrow \end{array} N_{n-1} \rightarrow N_{n-2} \rightarrow \cdots \rightarrow N_s$ is an N -path ($\phi_Q^{-1}(N_l, 0) = (l, k')$ for some $k' \in \mathbb{Z}$),
 - (c) $\phi_Q^{-1}(\beta, 0) = (n, u)$ and $\phi_Q^{-1}(\alpha, 0) = (n+1, u)$ for some $u \in \mathbb{Z}$
- (see Example 4.20 below for examples of swings).

Proposition 4.16 ([48, Theorem 2.21]). For each swing ρ of Γ_Q , there exists a unique $1 \leq a \leq n$ satisfying the following properties:

- (i) Every vertex shares a component ϵ_a .

- Thus we say the swing as an a -swing and denote it by ρ_a .

Definition 4.17. We say that a S -path (resp. N -path) is *shallow* if it ends (resp. starts) at residue less than n .

- (i) ρ consists of $(a - 1)$ -many vertices,
- (ii) all vertices in ρ share ϵ_{-a} as their components.

In this paper, we sometimes understand elements in $W_{D_{n+1}}$ as automorphisms of the set

i.e., $s_i \epsilon_t = \epsilon_{s_i(t)}$ for t with $1 \leq |t| \leq n$. Note that $s_n s_{n+1} = s_{n+1} s_n$, $s_n s_{n+1}(n) = -n$ and $s_n s_{n+1}(n+1) = -n-1$.

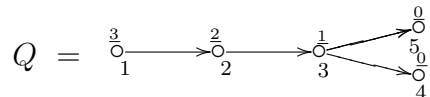
(a) If $\phi_Q^{-1}(\alpha, 0) = (n, k)$ (resp. $(n+1, k)$) for some $\alpha \in \Phi^+$, there exists $\beta \in \Phi^+$ such that $\phi_Q^{-1}(\beta, 0) = (n+1, k)$ (resp. (n, k)).

(b) For $\alpha, \beta \in \Phi^+$ with $\phi_Q^{-1}(\alpha, 0) = (n, k)$ and $\phi_Q^{-1}(\beta, 0) = (n+1, k)$, there exists a unique $1 \leq a \leq n$ such that

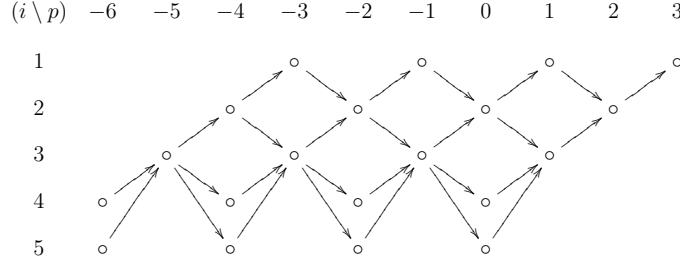
(c) $\tau_Q(\pm(n+1)) = \mp(n+1)$.

$$\tau_Q(n+1) = s_{i_1} \cdots s_{i_{u-1}} s_n s_{n+1}(n+1) = s_{i_1} \cdots s_{i_{u-1}}(-(n+1)) = -(n+1),$$

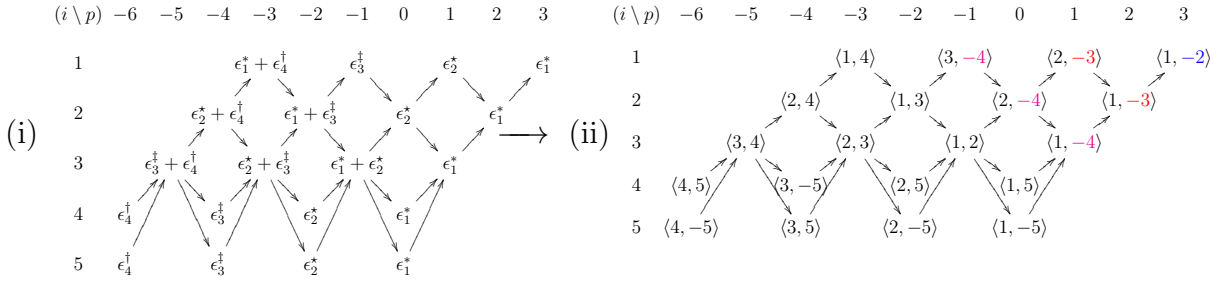
Example 4.20. Let us consider the following $\vee$ -fixed Dynkin quiver Q of type D_5 :



By (4.2), the Γ_Q without labeling are given as follows:



Using (i) Proposition 4.16 and (ii) Lemma 4.19 and Proposition 4.18 in order, we can complete the label of Γ_Q as follows:



Here, each symbol $*$ in (i) represents an a -swing in Proposition 4.16 and each color in (ii) represents an $(-a)$ -shallow path in Proposition 4.18.

Furthermore, in [50], the second named author proved the following proposition by using the labeling algorithm for $(\Gamma_Q)_0$ and considering all relative positions of $\langle\langle \alpha, \beta \rangle\rangle$'s.

Proposition 4.21 ([50, Proposition 4.5] (see also [3, Lemma 2.6])). *For any $\mathbf{c}$ -pair $\underline{p} = \langle\langle \alpha, \beta \rangle\rangle$ and any Dynkin quiver Q of type D , we have the followings:*

- (1) $\text{hd}_{[Q]}(\langle\langle \alpha, \beta \rangle\rangle)$ is well-defined. In particular, if $\alpha + \beta = \gamma \in \Phi^+$, then $\text{hd}_{[Q]}(\langle\langle \alpha, \beta \rangle\rangle) = \gamma$.
- (2) $\text{deg}_{[Q]}(\langle\langle \alpha, \beta \rangle\rangle) \leq 2$.

The relative positions for all $\mathbf{c}$ -pairs $\langle\langle \alpha, \beta \rangle\rangle$ with $\deg_Q(\langle\langle \alpha, \beta \rangle\rangle) \neq 0$ are classified and described, which do not depend on the choice of Q , as follows ([50]):

(4.10)

$\deg_Q(\langle\langle \alpha, \beta \rangle\rangle) = 1,$

and

(4.11)

$\deg_Q(\langle\langle \alpha, \beta \rangle\rangle) = 2.$

Here we only exhibit the cases when $\text{res}^{[Q]}(\alpha) \leq \text{res}^{[Q]}(\beta)$ and

- (1) we set $\underline{s} = \{s_1, \dots, s_r\}_{\beta \in \Phi^+} = \text{hd}_Q(\langle\langle \alpha, \beta \rangle\rangle)$,
- (2) β in (vii), (viii) and (ix) is one of two vertices $\circ$ and then s_2 in (vii) (resp. $\alpha + \beta$ in (viii), α in (ix)) is determined by the choice,
- (3) every exponent $\underline{m} \prec_Q^b \langle\langle \alpha, \beta \rangle\rangle$ is also described in the above pictures ($\prec_Q := \prec_{[Q]}$),
- (4) in (4.10), the positive roots with residue n or $n + 1$ appear in (v), (vi), (vii), (viii), (ix) (x) and (xi),
- (5) see Example 4.22 below for $\deg_Q(\langle\langle \alpha, \beta \rangle\rangle) = 2$ cases.

In each case, the value (α, β) does not depend on the choice of Q and is given as follows:

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(ix)	(x)
(α, β)	-1	0	-1	0	0	1	0	-1	-1	-1	0

TABLE 1. (α, β) for non $[Q]$ -simple $\mathbf{c}$ -pairs of type D_{n+1}

Example 4.22. Using Γ_Q in Example 4.20,

- (1) $\langle 1, -4 \rangle \prec_Q^b \langle \langle 1, -2 \rangle, \langle 2, -4 \rangle \rangle$ corresponds to (i),
- (2) $\langle \langle 2, -3 \rangle, \langle 1, -4 \rangle \rangle \prec_Q^b \langle \langle 1, -3 \rangle, \langle 2, -4 \rangle \rangle$ corresponds to (ii),
- (3) $\langle 1, 4 \rangle \prec_Q^b \langle \langle 1, -2 \rangle, \langle 2, 4 \rangle \rangle$ corresponds to (iii),
- (4) $\langle \langle 1, 4 \rangle, \langle 2, -3 \rangle \rangle \prec_Q^b \langle \langle 1, -3 \rangle, \langle 2, 4 \rangle \rangle$ corresponds to (iv),
- (5) $\langle \langle 1, 5 \rangle, \langle 1, -5 \rangle \rangle \prec_Q^b \langle \langle 1, -2 \rangle, \langle 1, 2 \rangle \rangle$ corresponds to (v),
- (6) $\{ \langle 2, -3 \rangle, \langle 1, 5 \rangle, \langle 1, -5 \rangle \}_{\beta \in \Phi^+} \prec_Q^b \langle \langle 1, -3 \rangle, \langle 1, 2 \rangle \rangle$ corresponds to (vi),
- (7) $\langle \langle 3, -4 \rangle, \langle 1, 5 \rangle \rangle \prec_Q^b \langle \langle 1, -4 \rangle, \langle 3, 5 \rangle \rangle$ and $\langle \langle 3, -4 \rangle, \langle 1, -5 \rangle \rangle \prec_Q^b \langle \langle 1, -4 \rangle, \langle 3, -5 \rangle \rangle$ correspond to (vii),
- (8) $\langle 1, 5 \rangle \prec_Q^b \langle \langle 1, -3 \rangle, \langle 3, 5 \rangle \rangle$ and $\langle 1, -5 \rangle \prec_Q^b \langle \langle 1, -3 \rangle, \langle 3, -5 \rangle \rangle$ correspond to (viii),
- (9) $\langle 1, 3 \rangle \prec_Q^b \langle \langle 1, 5 \rangle, \langle 3, -5 \rangle \rangle$ and $\langle 1, 3 \rangle \prec_Q^b \langle \langle 1, -5 \rangle, \langle 3, 5 \rangle \rangle$ correspond to (ix),
- (10) $\langle 1, 2 \rangle \prec_Q^b \langle \langle 1, -5 \rangle, \langle 2, 5 \rangle \rangle, \langle \langle 1, 5 \rangle, \langle 2, -5 \rangle \rangle, \langle \langle 1, 3 \rangle, \langle 2, -3 \rangle \rangle \prec_Q^b \langle \langle 1, -3 \rangle, \langle 2, 3 \rangle \rangle$ corresponds to (x),
- (11) $\langle \langle 1, 2 \rangle, \langle 3, -4 \rangle \rangle \prec_Q^b \{ \langle 1, 5 \rangle, \langle 2, -5 \rangle, \langle 3, -4 \rangle \}_{\beta \in \Phi^+} \prec_Q^b \langle \langle 1, -4 \rangle, \langle 2, 3 \rangle \rangle$ corresponds to (xi).

For Δ of type D_{n+1} , the degree polynomials $\mathfrak{d}_{i,j}^{D_{n+1}}(t)$ for $i, j \in \Delta_0$ are given in the following explicit form [48, 50, 11]:

$$(4.12) \quad \mathfrak{d}_{i,j}^{D_{n+1}}(t) + \delta_{i,j} t^{h-1} = \begin{cases} \frac{\sum_{s=1}^{\min(i,j)} (t^{|i-j|+2s-1} + t^{2n-i-j+2s-1})}{1 + \delta(\max(i,j) \geq n)} & \text{if } \min(i,j) < n, \\ \sum_{s=1}^{\lfloor (n+\delta_{i,j})/2 \rfloor} t^{4s-1-2\delta(i,j)} & \text{otherwise.} \end{cases}$$

Remark 4.23. The coefficient t^{s-1} of $\mathfrak{d}_{i,j}^\Delta(t)$ is 2 if and only if

$$(4.13) \quad 2 \leq i, j \leq n-1, \quad i+j \geq n+1, \quad h+2-i-j \leq s \leq i+j \quad \text{and} \quad s \equiv_2 i+j$$

(see [20, Lemma 3.2.4] also). Equivalently, if $\deg_Q(\langle \alpha, \beta \rangle) = 2$, the relative positions of the $\mathbf{c}$ -pairs $\langle \alpha, \beta \rangle$ in Γ_Q are described as in (4.11), where

(x) happens when $\alpha + \beta = \gamma = \underline{s}$ with $\text{mul}(\gamma) = 2$ and we have

$$\gamma \prec_Q^b \langle \eta_1, \eta_2 \rangle, \langle \tilde{\tau}_1, \tilde{\zeta}_1 \rangle, \langle \tilde{\tau}_2, \tilde{\zeta}_2 \rangle \prec_Q^b \langle \alpha, \beta \rangle$$

where $\{\tilde{\tau}_1, \tilde{\tau}_2\} = \{\tau_1, \tau_2\}$, $\{\tilde{\zeta}_1, \tilde{\zeta}_2\} = \{\zeta_1, \zeta_2\}$ and $\tilde{\tau}_i + \tilde{\zeta}_i = \gamma$ ($i = 1, 2$),

(xi) happens when $\text{hd}_Q(\langle \alpha, \beta \rangle) = \langle \gamma_1, \gamma_2 \rangle$ and we have

$$\langle \gamma_1, \gamma_2 \rangle \prec_Q^b \langle \eta_1, \eta_2 \rangle, \left\{ \tilde{\tau}_1, \tilde{\zeta}_1, \gamma_1 \right\}_{\beta \in \Phi^+}, \left\{ \tilde{\tau}_2, \tilde{\zeta}_2, \gamma_1 \right\}_{\beta \in \Phi^+} \prec_Q^b \langle \alpha, \beta \rangle$$

where $\{\tilde{\tau}_1, \tilde{\tau}_2\} = \{\tau_1, \tau_2\}$, $\{\tilde{\zeta}_1, \tilde{\zeta}_2\} = \{\zeta_1, \zeta_2\}$ and $\tilde{\tau}_i + \tilde{\zeta}_i = \gamma_1$ ($i = 1, 2$).

4.2.2. A_{2n-1} -case. Let Q be a Dynkin quiver of type A_{2n-1} . The simple roots and $\Phi_{A_{2n-1}}^+$ can be identified in $Q^+ \subset \bigoplus_{i=1}^{2n} \mathbb{Z}\epsilon_i$ as follows:

$$(4.14) \quad \begin{aligned} \alpha_i &= \epsilon_i - \epsilon_{i+1} \quad \text{for } 1 \leq i \leq 2n-1, \\ \Phi_{A_{2n-1}}^+ &= \left\{ \epsilon_i - \epsilon_j = \sum_{i \leq k < j} \alpha_k \mid 1 \leq i \leq j \leq 2n \right\}. \end{aligned}$$

Remark 4.24. Note that for $\vee$ -fixed Dynkin quiver $Q = (\Delta, \vee\xi)$ of type A_{2n-1} ,

- (1) there is no path p in Q passing through n ,
- (2) n is always sink or source of Q ,

since $\vee\xi_{n-1} = \vee\xi_{n+1}$ and $|\vee\xi_n - \vee\xi_{n\pm 1}| = 1$.

We identify $\beta \in \Phi_{A_{2n-1}}^+$ with a *segment* $[a, b]$ for $1 \leq a \leq b \leq 2n-1$, where $\beta = \sum_{k=a}^b \alpha_k$. For $[a, b]$, we call a the *first component* and b the *second component* of β . If β is simple, we sometimes write β as $[a]$ instead of $[a, a]$.

Proposition 4.25. [49, Theorem 1.11]

- (1) Every positive root in an N -path shares the same first component and every positive root in an S -path shares the same second component.
- (2) For each $1 \leq i \leq 2n-i$, there are exactly one N -path ϱ_i^N containing $(2n-i)$ vertices and each vertex in ϱ_i^N has i as its first component. Conversely, every positive root with the first component i appears in ϱ_i^N .
- (3) For each $1 \leq i \leq 2n-i$, there are exactly one S -path ϱ_i^S containing i vertices and each vertex in ϱ_i^S has i as its second component. Conversely, every positive root with the second component i appears in ϱ_i^S .

As in D_{n+1} -case, we can label the Γ_Q in terms of Φ^+ without computation $\beta_k^{w_0}$'s or the bijection $\phi_Q|_{\Gamma_Q}$ by using the result above and obtain the following proposition:

Proposition 4.26. [50, Proposition 4.5] (see also [3, Lemma 2.6]) For any $\mathbf{c}$ -pair $\underline{p} = \langle\langle \alpha, \beta \rangle\rangle$ and any Dynkin quiver Q of type A_n , we have the followings:

- (1) $\text{hd}_{[Q]}(\langle\langle \alpha, \beta \rangle\rangle)$ is well-defined. In particular, if $\alpha + \beta = \gamma \in \Phi^+$, then $\text{hd}_{[Q]}(\langle\langle \alpha, \beta \rangle\rangle) = \gamma$.
- (2) $\deg_{[Q]}(\langle\langle \alpha, \beta \rangle\rangle) \leq 1$.

Understanding the non-trivial Dynkin diagram automorphism $\vee$ on A_{2n-1} as an involution on $\{1, \dots, 2n\}$, $\vee$ normalizes $\mathbf{W}_{A_{2n-1}}$ via $\vee s_i \vee = s_{\vee(i)}$.

Lemma 4.27. For $\vee$ -fixed Dynkin quiver $Q = (\Delta, \vee\xi)$ of type A_{2n-1} and $i \in \Delta_0$, we have

$$\gamma_{\vee(i)}^Q = \vee(\gamma_i^Q).$$

Here we understand $\vee$ as an involution of $\Phi_{A_{2n-1}}^+$.

Proof. Recall $B^Q(i)$ in (4.3). By the facts that $\vee\xi_i = \vee\xi_{\vee(i)}$ and Remark 4.24, if there exists a path j to i in Q for $i, j \leq n$, then there exists a path $2n-i$ to $2n-j$ in Q . Hence $j \in B^Q(i) \iff 2n-j \in B^Q(2n-i)$ for $i, j \leq n$ and our assertion follows from (4.3). $\square$

For Δ of type A_{2n-1} with $h = 2n$, the degree polynomials $\mathfrak{d}_{i,j}^{A_{2n-1}}(t)$ for $i, j \in \Delta_0$ are given in the following explicit form [48, 50, 11]:

$$\mathfrak{d}_{i,j}^{A_{2n-1}}(t) + \delta_{i,j*} t^{h-1} = \sum_{s=1}^{\min(i,j,2n-i,2n-j)} t^{|i-j|+2s-1}.$$

Note that every non-zero coefficient of $\mathfrak{d}_{i,j}^{A_{2n-1}}(t)$ is 1.

4.3. Relation between $\mathfrak{d}_{i,j}(t)$ and $\tilde{\mathfrak{d}}_{i,j}(t)$. For a Dynkin diagram Δ of type A_n , D_n and E_6 , we have the following theorem:

Theorem 4.28 ([50]). *For Δ of type A_n , D_n and E_6 , and $i, j \in I$, we have*

$$(4.15) \quad \tilde{\mathfrak{d}}_{i,j}(t) = \mathfrak{d}_{i,j}(t) + \delta_{i,j*} t^{h-1}.$$

Remark 4.29. In Section 6, we will see that (4.15) also holds for B_n and C_n (see Theorem 6.6). However, in types E_7 , E_8 , F_4 and G_2 , (4.15) do not hold (see Remark 6.16 and Remark 6.18 below for F_4 and G_2 .)

5. LABELING OF AR-QUIVERS IN TYPE BCFG

For a Lie algebra $\mathfrak{g}$ of simply-laced type associated to Δ and a Dynkin diagram automorphism σ ($\neq \text{id}$) on $\Delta_{\mathfrak{g}}$, we denote by ${}^{\sigma}\mathfrak{g}$ the Lie subalgebra of $\mathfrak{g}$ such that it is of non simply-laced type [19, Proposition 7.9]:

(Δ, σ)	$(\Delta_{A_{2n-1}}, \vee)$	$(\Delta_{D_{n+1}}, \vee)$	$(\Delta_{E_6}, \vee)$	$(\Delta_{D_4}, \tilde{\vee})$
Type of ${}^{\sigma}\mathfrak{g}$	C_n	B_n	F_4	G_2

TABLE 2. ${}^{\sigma}\mathfrak{g}$ for each (Δ, σ) with $\sigma \neq \text{id}$

Note that the index ${}^{\sigma}I$ for ${}^{\sigma}\mathfrak{g}$ can be identified the image of $I_{\mathfrak{g}}$ under the following surjection $\bar{\sigma}$: For ${}^{\sigma}\mathfrak{g}$ of type B_n or C_n ,

$$\bar{\sigma}(i) = \begin{cases} n & \text{if } {}^{\sigma}\mathfrak{g} \text{ is of type } B_n \text{ and } i = n+1, \\ 2n-i & \text{if } {}^{\sigma}\mathfrak{g} \text{ is of type } C_n \text{ and } i \geq n+1, \\ i & \text{otherwise,} \end{cases}$$

and

$$\begin{aligned} \bar{\sigma}(1) = \bar{\sigma}(6) = 2, \quad \bar{\sigma}(3) = \bar{\sigma}(5) = 2, \quad \bar{\sigma}(4) = 2, \quad \bar{\sigma}(2) = 1, & \quad \text{if } {}^{\sigma}\mathfrak{g} \text{ is of type } F_4, \\ \bar{\sigma}(1) = \bar{\sigma}(3) = \bar{\sigma}(4) = 2, \quad \bar{\sigma}(2) = 1, & \quad \text{if } {}^{\sigma}\mathfrak{g} \text{ is of type } G_2. \end{aligned}$$

Then, for a σ -fixed Dynkin quiver $Q = (\Delta_{\mathfrak{g}}, {}^{\sigma}\xi)$, we can obtain a Dynkin quiver $\bar{Q} = (\Delta_{{}^{\sigma}\mathfrak{g}}, \bar{\xi})$ such that

$$(5.1) \quad \bar{\xi}_{\bar{\sigma}(i)} = {}^{\sigma}\xi_i.$$

for any $i \in I$. We write ${}^{\sigma}I = \{\bar{i} := \bar{\sigma}(i) \mid i \in I_{\mathfrak{g}}\}$.

Example 5.1. Here are $\overline{Q}$'s corresponding to Q 's in Example 2.8.

- (1) $\overline{Q} = \overset{\frac{3}{1}}{\circ} \longrightarrow \overset{\frac{2}{2}}{\circ} \longrightarrow \overset{\frac{1}{3}}{\circ}$ of type C_n for Q of type A_5 in Example 2.8 (2.11a).
- (2) $\overline{Q} = \overset{\frac{3}{1}}{\circ} \longrightarrow \overset{\frac{2}{2}}{\circ} \longrightarrow \overset{\frac{1}{3}}{\circ}$ of type B_3 for Q of type D_4 in Example 2.8 (2.11b).
- (3) $\overline{Q} = \overset{\frac{0}{1}}{\circ} \longleftarrow \overset{\frac{1}{2}}{\circ} \longleftarrow \overset{\frac{2}{3}}{\circ} \longleftarrow \overset{\frac{3}{4}}{\circ}$ of type F_4 for Q of type E_6 in Example 2.8 (2.11c).
- (4) $\overline{Q} = \overset{\frac{2}{1}}{\circ} \longrightarrow \overset{\frac{1}{2}}{\circ}$ of type G_2 for Q of type D_4 in Example 2.8 (2.11d).

In this section, we shall show that we can obtain $\Gamma_{\overline{Q}}$ from Γ_Q and hence the labeling of $\Gamma_{\overline{Q}}$ from the ones of Γ_Q . The following lemma holds for Dynkin quivers Q of any finite type, which generalizes (4.3):

Lemma 5.2. *For Dynkin quiver $Q = (\Delta, \xi)$ of any finite type and $i \in I$, we have*

$$\gamma_i^Q = \sum_{j \in B^Q(i)} \left(\prod_{k=1}^{l^j-1} - \langle h_{p_k^j}, \alpha_{p_{k+1}^j} \rangle \right) \alpha_j$$

where $B^Q(i)$ denotes the set of indices j such that there exists a path $\mathbf{p}^j = (j = p_1^j \rightarrow p_2^j \rightarrow \cdots \rightarrow p_{l^j-1}^j \rightarrow p_{l^j}^j = i)$ in Q . Here we understand $\left(\prod_{k=1}^{l^j-1} - \langle h_{p_k^j}, \alpha_{p_{k+1}^j} \rangle \right) = 1$ when $i = j$ and hence $l^j = 1$.

Proof. Set $i_m = i$. Note that we have

$$(5.2) \quad \gamma_{i_m}^Q = (1 - \tau_Q) \varpi_{i_m} = (s_{i_1} \cdots s_{i_{m-1}} - s_{i_1} \cdots s_{i_{m-1}} s_{i_m}) \varpi_{i_m} = s_{i_1} \cdots s_{i_{m-1}} (\alpha_{i_m}),$$

where $\tau_Q = s_{i_1} \cdots s_{i_{|\Delta_0|}}$ and $1 \leq m \leq |\Delta_0|$. Let $j \notin B^Q(i)$. Then there is no exponent $1 \leq t_1 < \cdots < t_l = m$ such that $d(i_{t_x}, i_{t_{x+1}}) = 1$. Hence j can not be contained in $\text{supp}(\gamma_i^Q)$ by (5.2).

Let $j \in B^Q(i)$. Since Δ is of finite type, there exists a unique path $\mathbf{p}^j = (j = p_1^j \rightarrow p_2^j \rightarrow \cdots \rightarrow p_{l^j-1}^j \rightarrow p_{l^j}^j = i)$ in Q . If the length of $\mathbf{p}^j$ is 2, we have $\xi_j = \xi_i + 1$ and the assertion follows from (5.2). By an induction on l , $p_2^j \in \text{supp}(\gamma_i^Q)$ and $\text{supp}_{p_2^j}(\gamma_i^Q) =$

$$\left(\prod_{k=2}^{l^j-1} - \langle h_{p_k^j}, \alpha_{p_{k+1}^j} \rangle \right). \text{ Since } \xi_j = \xi_{p_2^j} + 1 \text{ and } \tau_Q \text{ is adapted to } Q, \text{ we can guarantee that}$$

$$\text{supp}_j(\gamma_i^Q) \geq \left(\prod_{k=1}^{l^j-1} - \langle h_{p_k^j}, \alpha_{p_{k+1}^j} \rangle \right).$$

(i) Assume that there exists $j' \in \Delta_0$ such that $d(j, j') = 1$ and $\xi_{j'} = \xi_j + 1$. Then $s_{j'}$ appears earlier than s_j in every Q -adapted reduced expression of τ_Q .

(ii) Assume that there exists $j' \in \Delta_0$ such that $d(j, j') = 1$ and $\xi_{j'} = \xi_j - 1$. Then there is no path from j' to i . Thus, even though $s_{j'}$ appears later than s_j in every Q -adapted reduced expression of τ_Q , we have $j' \notin \text{supp}(\gamma_i^Q)$.

By (i) and (ii), we have $\text{supp}_j(\gamma_i^Q) = \left(\prod_{k=1}^{l^j-1} - \langle h_{p_k^j}, \alpha_{p_{k+1}^j} \rangle \right)$, as we desired. $\square$

Corollary 5.3. *For a Dynkin quiver $Q = (\Delta, \xi)$ of an arbitrary type and $i \in \Delta_0$, we have*

$$\alpha_i - \sum_{\substack{d(i,j)=1, \\ \xi_j - \xi_i = 1}} c_{j,i} \gamma_j^Q = \gamma_i^Q.$$

Convention 3. *In the rest of this section, we use*

- (a) ι, j for elements in the index set I of type ADE and i, j for elements in the index set I of type BCFG,
- (b) α, β, γ for positive roots of type ADE and α, β, γ for positive roots of type BCFG,
- (c) ϵ, ε for elements in $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Q}^+$ of type ADE and ϵ, ε for elements in $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Q}^+$ of type BCFG,

to avoid confusion.

For subsections 5.1 and 5.2 below, we shall investigate the combinatorial properties and the labeling algorithm for AR-quivers of type B_n and C_n , and observe the relationship with the ones for D_n and A_{2n-1} , respectively. The results in those subsections will be used for obtaining the degree polynomials in the next section crucially.

5.1. B_n -case. Note that for $\mathbf{g}$ of type D_{n+1} , its corresponding ${}^{\vee}\mathbf{g}$ is of type B_n . For $Q = (\blacktriangle_{B_n}, \xi)$ of type B_n , we denote by $\underline{Q} = (\Delta_{D_{n+1}}, {}^{\vee}\xi)$ the $\vee$ -fixed Dynkin quiver of type D_{n+1} such that

$$(5.3) \quad {}^{\vee}\xi_{\iota} = \xi_{\bar{\iota}} \quad \text{for } \iota \in \Delta_0.$$

Recall the notation $\bar{Q} = (\blacktriangle_{B_n}, \bar{\xi})$ of type B_n corresponding to $\vee$ -fixed Dynkin quiver $Q = (\Delta_{D_{n+1}}, {}^{\vee}\xi)$ of type D_{n+1} in (5.1).

The simple roots $\{\alpha_i\}$ and Φ^+ of type B_n can be identified in $\mathbb{Q}^+ \subset \bigoplus_{i=1}^n \mathbb{R}\epsilon_i$ as follows:

$$(5.4) \quad \begin{aligned} & \alpha_i = \epsilon_i - \epsilon_{i+1} \quad \text{for } i < n, \quad \alpha_n = \epsilon_n \quad \text{and} \quad \epsilon_i = \sqrt{2}\epsilon_i, \\ & \Phi_{B_n}^+ = \left\{ \epsilon_i - \epsilon_j = \sum_{k=i}^{j-1} \alpha_k \mid 1 \leq i < j \leq n \right\} \sqcup \left\{ \epsilon_i = \sum_{k=i}^n \alpha_k \mid 1 \leq i \leq n \right\} \\ & \quad \sqcup \left\{ \epsilon_i + \epsilon_j = \sum_{k=i}^{j-1} \alpha_k + 2 \sum_{s=j}^n \alpha_s \mid 1 \leq i < j \leq n \right\}. \end{aligned}$$

For $\beta \in \Phi_{B_n}^+$, we write

$$(5.5) \quad \beta = \begin{cases} \langle i, \pm j \rangle & \text{if } \beta = \epsilon_i \pm \epsilon_j \text{ for some } 1 \leq i < j \leq n, \\ \langle i \rangle & \text{if } \beta = \epsilon_i \text{ for some } 1 \leq i \leq n \end{cases}$$

Understanding W_{B_n} as bijections on $\{\pm 1, \dots, \pm n\}$, we have the following descriptions for simple reflections:

$$s_i(\pm i) = \pm(i+1) \quad \text{for } i < n \quad \text{and} \quad s_n(n) = -n.$$

Note that there is a surjection $\psi: \Phi_{D_{n+1}}^+ \rightarrow \Phi_{B_n}^+$ whose description can be described in two ways using ϵ_i 's, ϵ_i 's, α_i 's and α_i 's:

$$(5.6) \quad \psi(\epsilon_i) = \begin{cases} \epsilon_{\bar{i}} & \text{if } i \leq n, \\ 0 & \text{if } i = n+1, \end{cases} \quad (\text{equivalently, } \psi(\alpha_i) = \alpha_{\bar{i}})$$

and extends it linearly.

For each $i \in I_{B_n}$, let ι be the index of I_{D_n} such that $i = \iota$ as an integer. By (4.9) and (5.4), we have

$$\psi^{-1}(\epsilon_i) = \{\epsilon_i + \epsilon_{n+1}, \epsilon_i - \epsilon_{n+1}\} \quad \text{and} \quad \psi^{-1}(\epsilon_i \pm \epsilon_j) = \epsilon_i \pm \epsilon_j.$$

Proposition 5.4. *For a $\vee$ -fixed Dynkin quiver Q of type D_{n+1} , we can obtain $\Gamma_{\overline{Q}}$ from Γ_Q by the following simple surgery:*

- (1) Remove all vertices in Γ_Q located at residue $(n+1)$.
- (2) Replace all labels at residue n from $\langle \iota, \pm(n+1) \rangle$ to $\langle i \rangle$.

To prove Proposition 5.4, we need preparations:

Lemma 5.5. *For any $\vee$ -fixed Dynkin quiver Q of type D_{n+1} , we have*

$$\tau_Q(\iota) = \tau_{\overline{Q}}(i) \quad \text{for } 1 \leq \iota \leq n.$$

Proof. By Lemma 4.19 (c), $\tau_Q(n+1) = -(n+1)$. Since Q is $\vee$ -fixed, there exists a $1 \leq k < n+1$ reduced expression $s_{\iota_1} \dots s_{\iota_{n+1}}$ of τ_Q such that $\{\iota_k, \iota_{k+1}\} = \{n, n+1\}$. Since $\bar{\xi}_i = {}^\vee \xi_i$, $1 \leq \iota \leq n$, $\tau_{\overline{Q}}$ has a reduced expression $s_{\bar{\iota}_1} \dots s_{\bar{\iota}_{k-1}} s_{i_k}^\dagger s_{\bar{\iota}_{k+2}} \dots s_{\bar{\iota}_{n+1}}$ such that $i_k^\dagger = n$. Thus our assertion follows. $\square$

The following lemma is obvious:

Lemma 5.6. *For $Q = (\blacktriangle_{B_n}, \xi)$, we have*

$$\underline{s_i Q} = \iota(s_i) \underline{Q} \quad \text{for } 1 \leq i \leq n,$$

where

$$\iota(s_i) := \begin{cases} s_i & \text{if } i < n, \\ s_n s_{n+1} & \text{if } i = n. \end{cases}$$

Lemma 5.7. *For any $\vee$ -fixed Dynkin quiver Q of type D_{n+1} , we have*

$$\psi(\gamma_i^Q) = \gamma_i^{\overline{Q}} \quad \text{for } 1 \leq \iota \leq n \quad \text{and} \quad \psi(\gamma_{n+1}^Q) = \gamma_n^{\overline{Q}}.$$

Proof. By Lemma 5.2 and (5.6),

$$(5.7) \quad \begin{aligned} \psi(\gamma_i^Q) = & \sum_{j \in B^Q(i) \setminus \{n+1\}} \left(\prod_{k=1}^{l^j-1} - \langle h_{p_k^j}, \alpha_{p_{k+1}^j} \rangle \right) \alpha_j \\ & + \delta(n+1 \in B^Q(i)) \left(\prod_{k=1}^{l^{n+1}-1} - \langle h_{p_k^{n+1}}, \alpha_{p_{k+1}^{n+1}} \rangle \right) \alpha_n. \end{aligned}$$

For any $j \in B^{\bar{Q}}(i) \setminus \{n\}$, we have $|c_{ji}^{B^n}| \leq 1$. Then we have $\text{supp}_j(\gamma_i^{\bar{Q}}) = \text{supp}_j(\gamma_i^Q)$ for any $j \in B^Q(i) \setminus \{n\}$ by Lemma 5.2.

Assume that $n \in B^{\bar{Q}}(i)$. Then we have $n, n+1 \in B^Q(i)$ if $i < n$ and $n+1 \notin B^Q(n)$ if $i = n$. Since $-c_{n,n-1} = 2$, we have

$$\begin{aligned} \left(\prod_{k=1}^{l^n-1} - \langle h_{p_k^n}, \alpha_{p_{k+1}^n} \rangle \right) \alpha_n + \delta(n+1 \in B^Q(i)) \left(\prod_{k=1}^{l^{n+1}-1} - \langle h_{p_k^{n+1}}, \alpha_{p_{k+1}^{n+1}} \rangle \right) \alpha_n \\ = \left(\prod_{k=1}^{l^n-1} - \langle h_{p_k^n}, \alpha_{p_{k+1}^n} \rangle \right) \alpha_n, \end{aligned}$$

which completes our assertion. $\square$

Proof of Proposition 5.4. By Lemma 5.7, we can obtain $\{\gamma_i^{\bar{Q}}\}$ by using $\{\gamma_i^Q\}$ and the surjection ψ in (5.6). Then our assertion follows from Lemma 5.5 and (4.2). $\square$

Definition 5.8. A connected full subquiver $\bar{p}$ in $\Gamma_{\bar{Q}}$ is said to be *swing* if it is a concatenation of one N -path and one S -path whose intersection is located at $(n, p) \in (\Gamma_Q)_0$ for some $p \in \mathbb{Z}$: There exist a positive root $\beta \in \Phi^+$ and $r, s \leq n$ such that

$$S_r \rightarrow S_{r+1} \rightarrow \cdots \rightarrow S_{n-1} \rightarrow \beta \Rightarrow N_{n-1} \rightarrow N_{n-2} \rightarrow \cdots \rightarrow N_s \quad \text{where}$$

- $S_r \rightarrow S_{r+1} \rightarrow \cdots \rightarrow S_{n-1} \rightarrow \beta$ is an S -path ($\phi_Q^{-1}(S_l, 0) = (l, k)$ for $k \in \mathbb{Z}$),
- $\beta \Rightarrow N_{n-1} \rightarrow N_{n-2} \rightarrow \cdots \rightarrow N_s$ is an N -path ($\phi_Q^{-1}(N_l, 0) = (l, k')$ for $k' \in \mathbb{Z}$),
- β is located at (n, p) for some $p \in \mathbb{Z}$.

By Proposition 5.4, we have the B_n -analogue of Proposition 4.16 and Proposition 4.18 as follows:

Proposition 5.9.

- (1) For each swing in $\Gamma_{\bar{Q}}$, there exists a unique $1 \leq a \leq n$ satisfying the following properties:
 - (i) Every vertex shares a component ε_a .
 - (ii) If every vertex shares a component ε_a , it consists of exactly $(2n - a)$ -many vertices.
 - (iii) Every positive roots having ε_a as its component appears in the swing.
 - (iv) It starts at 1 or ends at 1.
 - (v) There are only n -swings.

Thus we say the swing as a -swing and denote it by $\bar{p}_a$.

(2) Let $\bar{\rho}$ be a shallow S (resp. N)-path in $\Gamma_{\bar{Q}}$. Then there exists a unique $2 \leq a \leq n$ such that

- (i) $\bar{\rho}$ consists of $(a - 1)$ -many vertices,
- (ii) all vertices in ρ share ε_{-a} as their components.

Furthermore, there are exactly $(n - 1)$ -many shallow X -paths ($X = S$ or N) in $\Gamma_{\bar{Q}}$. Thus we say the shallow X -path as $(-a)$ -path and denote it by $\bar{\rho}_{-a}$.

From Proposition 5.4 and Proposition 5.9, we have the following main result of this subsection:

Theorem 5.10. For each σ -fixed Q of type D_{n+1} and $k \in \mathbb{Z}$, we have

$$\psi(\tau_Q^k(\gamma_i^Q)) = \tau_{\bar{Q}}^k(\gamma_i^{\bar{Q}}) \quad \text{for } i < n$$

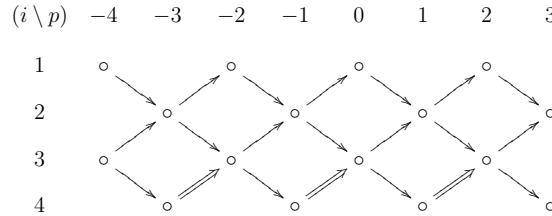
and

$$\psi(\tau_Q^k(\gamma_n^Q)) = \tau_{\bar{Q}}^k(\gamma_n^{\bar{Q}}) \quad \text{for } i = n \text{ or } n + 1.$$

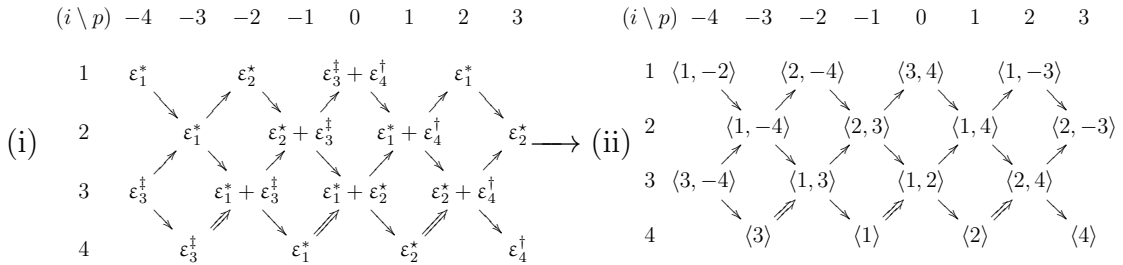
Example 5.11. Let us consider the following $\vee$ -fixed Dynkin quiver Q of type B_4 :

$$Q = \begin{array}{ccccccc} & \overset{2}{\circ} & \xleftarrow{3} & \overset{3}{\circ} & \xrightarrow{2} & \overset{2}{\circ} & \xleftarrow{3} & \overset{3}{\circ} \\ & 1 & & 2 & & 3 & & 4 \end{array}$$

By (4.4), the Γ_Q without labeling are given as follows:



By applying Proposition 5.9 (1) and (2) in order, we can complete the label of Γ_Q as follows:



5.2. C_n -**case**. Note that for $\mathfrak{g}$ of type A_{2n-1} , its corresponding ${}^\vee\mathfrak{g}$ is of type C_n . The simple roots $\{\alpha_i\}$ and $\Phi_{C_n}^+$ can be identified in $Q^+ \subset \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$ as follows:

$$(5.8) \quad \begin{aligned} \alpha_i &= \epsilon_i - \epsilon_{i+1} \quad \text{for } i < n \quad \text{and} \quad \alpha_n = 2\epsilon_n \\ \Phi_{C_n}^+ &= \left\{ \epsilon_i - \epsilon_j = \sum_{k=i}^{j-1} \alpha_k \mid 1 \leq i < j \leq n \right\} \sqcup \left\{ 2\epsilon_i = 2 \sum_{k=i}^{n-1} \alpha_k + \alpha_n \mid 1 \leq i \leq n \right\} \\ &\sqcup \left\{ \epsilon_i + \epsilon_j = \sum_{k=i}^{j-1} \alpha_k + 2 \sum_{s=j}^{n-1} \alpha_s + \alpha_n \mid 1 \leq i < j \leq n \right\}. \end{aligned}$$

For $\beta \in \Phi_{C_n}^+$, we write (cf. (5.5))

$$(5.9) \quad \beta = \begin{cases} \langle i, \pm j \rangle & \text{if } \beta = \epsilon_i \pm \epsilon_j \text{ for some } 1 \leq i < j \leq n, \\ \langle i, i \rangle & \text{if } \beta = 2\epsilon_i \text{ for some } 1 \leq i \leq n. \end{cases}$$

Note that $W_{B_n} \simeq W_{C_n}$. For a Dynkin quiver $Q = (\blacktriangle_{B_n}, \xi)$ of type B_n , we denote by $Q^{\text{tr}} = (\blacktriangle_{C_n}, \xi^{\text{tr}})$ the Dynkin quiver of type C_n such that

$$\xi_i^{\text{tr}} = \xi_i \quad \text{for } 1 \leq i \leq n.$$

Then we have a natural bijection $\psi^{\text{tr}}: \Phi_{B_n}^+ \rightarrow \Phi_{C_n}^+$ given by

$$\psi^{\text{tr}}(\epsilon_i \pm \epsilon_j) = \epsilon_i \pm \epsilon_j \quad (1 \leq i < j \leq n) \quad \text{and} \quad \phi^{\text{tr}}(\epsilon_i) = 2\epsilon_i.$$

Proposition 5.12. *For all $1 \leq i \leq n$, we have*

$$\psi^{\text{tr}}(\gamma_i^Q) = \gamma_i^{Q^{\text{tr}}}.$$

Proof. Note that $B^Q(i) = B^{Q^{\text{tr}}}(i)$ for all $1 \leq i \leq n$. Set

- (a) $a = \min(s \mid s \in B^Q(i))$,
- (b) $b = \max(s \mid s \in B^Q(i))$ if $i \neq n$ and $n \notin B^Q(i)$.

By Lemma 5.2, we have

$$(5.10) \quad \begin{aligned} \gamma_i^Q &= \begin{cases} \sum_{j \in B^Q(n)} \alpha_j & \text{if } i = n, \\ \sum_{j \in B^Q(i) \setminus \{n\}} \alpha_j + \delta(n \in B^Q(i)) 2\alpha_n & \text{if } i \neq n, \end{cases} \\ &= \begin{cases} \epsilon_a & \text{if } i = n, \\ \epsilon_a - \epsilon_{b+1} & \text{if } i \neq n \text{ and } n \notin B^Q(i), \\ \epsilon_a + \epsilon_n, & \text{if } i \neq n \text{ and } n \in B^Q(i), \end{cases} \end{aligned}$$

On the other hand, we have

$$(5.11) \quad \gamma_i^{Q^{\text{tr}}} = \begin{cases} \left(\sum_{j \in B^Q(n) \setminus \{n\}} 2\alpha_j \right) + \alpha_n & \text{if } i = n, \\ \sum_{j \in B^Q(i)} \alpha_j & \text{if } i \neq n, \end{cases}$$

$$= \begin{cases} 2\epsilon_a & \text{if } i = n, \\ \epsilon_a - \epsilon_{b+1} & \text{if } i \neq n \text{ and } n \notin B^Q(i), \\ \epsilon_a + \epsilon_n, & \text{if } i \neq n \text{ and } n \in B^Q(i), \end{cases}$$

which implies our assertion. $\square$

Note that every Q of type C_n is Q^{tr} for some Q of type B_n . Hence we have the following theorem:

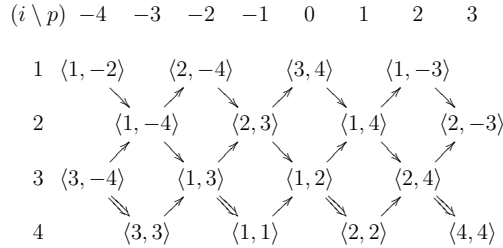
Proposition 5.13. *The labeling of $\Gamma_{Q^{\text{tr}}}$ in terms of notations $\langle i, \pm j \rangle$ and $\langle i, i \rangle$ can be obtained from the one of Γ_Q by replacing only $\langle i \rangle$ with $\langle i, i \rangle$ for $1 \leq i \leq n$.*

Proof. By the previous proposition, the replacement of notations works for $\{\gamma_i^Q\}$ and $\{\gamma_i^{Q^{\text{tr}}}\}$. Then our assertion follows from the fact that $W_{B_n} = W_{C_n}$ and the bijection ϕ in (4.2). $\square$

Example 5.14. Consider the Dynkin quiver Q of type B_4 in Example 5.11. Then its corresponding Q^{tr} of C_4 can be depicted as follows:

$$Q = \overset{2}{\underset{1}{\circ}} \xleftarrow{3} \overset{3}{\underset{2}{\circ}} \xrightarrow{2} \overset{2}{\underset{3}{\circ}} \xleftarrow{3} \overset{3}{\underset{4}{\circ}}$$

Using the labeling of Γ_Q in Example 5.11 and Proposition 5.13, we have



For each $i \in I_{C_n}$, let ι be the index of $I_{A_{2n-1}}$ such that $i = \iota$ as an integer. Note that there is a surjection $\psi: \Phi_{A_{2n-1}}^+ \rightarrow \Phi_{C_n}^+$ which is described as follows:

$$(5.12) \quad \psi(\alpha_i) = \alpha_{\bar{\iota}} \quad \left(\text{equivalently, } \psi(\epsilon_i) = \begin{cases} \epsilon_{\bar{\iota}} & \text{if } \iota \leq n, \\ -\epsilon_{2n+1-\bar{\iota}} & \text{if } \iota > n, \end{cases} \right)$$

and extends it linearly.

Then the inverse image of $\beta \in \Phi_{C_n}^+$ can be described as follows:

$$(5.13) \quad \begin{aligned} \psi^{-1}(2\epsilon_i) &= \epsilon_i - \epsilon_{2n+1-i} = \sum_{k=i}^{2n-i} \alpha_k \quad \text{for } 1 \leq i \leq n, \\ \psi^{-1}(\epsilon_i - \epsilon_j) &= \left\{ \epsilon_i - \epsilon_j = \sum_{k=i}^{j-1} \alpha_k, \vee(\epsilon_i - \epsilon_j) = \epsilon_{2n+1-j} - \epsilon_{2n+1-i} = \sum_{k=2n+1-j}^{2n-i} \alpha_k \right\}, \\ \psi^{-1}(\epsilon_i + \epsilon_j) &= \left\{ \epsilon_i - \epsilon_{2n+1-j} = \sum_{k=i}^{2n-j} \alpha_k, \vee(\epsilon_i - \epsilon_{2n+1-j}) = \epsilon_j - \epsilon_{2n+1-i} = \sum_{k=j}^{2n-i} \alpha_k \right\} \end{aligned}$$

for $i < j \leq n$.

Note that, for $\vee$ -fixed Dynkin quiver $Q = (\Delta, \vee\xi)$ of type A_{2n-1} , since there exists a reduced expression $s_{i_1} \cdots s_{i_{2n-1}}$ of τ_Q such that

$$(5.14) \quad \begin{cases} i_1 = n, & (i_{2s}, i_{2s+1}) = (k_s, \vee(k_s)) & \text{if } n \text{ is a source of } Q, \\ i_{2n-1} = n, & (i_{2s-1}, i_{2s}) = (k_s, \vee(k_s)) & \text{if } n \text{ is a sink of } Q, \end{cases}$$

for $1 \leq s \leq n-1$ and unique $k_s \in \{1, \dots, n-1\}$. Thus, for a $\vee$ -fixed Dynkin quiver Q of type A_{2n-1} , the Coxeter element τ_Q is also $\vee$ -fixed in the sense that

$$(5.15) \quad \vee \tau_Q \vee = \tau_Q.$$

Lemma 5.15. *For each $\vee$ -fixed Q of type A_{2n-1} and $k \in \mathbb{Z}$, we have*

$$(5.16) \quad \psi(\gamma_i^Q) = \psi(\gamma_{2n-i}^Q) = \gamma_i^{\bar{Q}} \quad \text{for } i \leq n.$$

Proof. The equation in (5.16) follow from Lemma 4.27 and the surjection ψ in (5.12).

Since n is a sink or source of Q , $\text{supp}(\gamma_i^Q) \subset \{1, \dots, n\}$ for $i < n$ and $\text{supp}(\gamma_i^Q) \subset \{n, \dots, 2n-1\}$ for $i > n$. Thus n is a sink or source of $\bar{Q}$ and $\text{supp}(\gamma_i^Q) \subset \{1, \dots, n\}$ for $i < n$ and

$$\psi(\gamma_i^Q) = \sum_{j \in B^Q(i)} \alpha_j.$$

Since $B^Q(i) = B^{\bar{Q}}(i)$ for $i < n$ as subsets of integers, the assertion for these cases follows by (5.11).

For $i = n$, if $i \in B^{\bar{Q}}(n)$, we have $i, 2n-i \in B^Q(n)$. Thus the assertion for $i = n$ also follows by (5.11). $\square$

For $\vee$ -fixed Dynkin quiver Q of type A_{2n-1} , the set of positive roots of residue n in Γ_Q form the set $\eta := \{[i, 2n-i] \mid 1 \leq i \leq n\}$, since γ_n^Q and τ_Q are $\vee$ -fixed and $r_n^Q = n$ (see (4.6)).

We label the set η by using ϕ_Q^{-1} as follows:

$$\eta_k = [a_k, 2n - a_k] = \epsilon_{a_k} - \epsilon_{2n+1-a_k} = \epsilon_{a_k} - \epsilon_{b_k} \in \Phi_{A_{2n-1}}^+ \quad (1 \leq k \leq n \text{ and } 1 \leq a_k \leq n)$$

where

$$(\eta_1, 0) = \phi_Q(n, \xi_n), \quad (\eta_2, 0) = \phi_Q(n, \xi_n - 2), \dots, (\eta_n, 0) = \phi_Q(n, \xi_n - 2(n-1)).$$

Then we have $\{a_1, \dots, a_n\} = \{1, \dots, n\}$ and $\{b_1, \dots, b_n\} = \{n+1, \dots, 2n\}$. It also implies

$$\tau_Q(\epsilon_{a_k}) = \epsilon_{a_{k+1}} \text{ and } \tau_Q(\epsilon_{b_k}) = \tau_Q(\epsilon_{2n+1-a_k}) = \epsilon_{b_{k+1}} \text{ for } 1 \leq i < n.$$

Recall that n is source or sink of Q .

- (1) If n is source of Q , $\epsilon_{a_1} - \epsilon_{b_1} = [n] = \epsilon_n - \epsilon_{n+1}$ and thus $\tau_Q(\epsilon_{a_n}) = \epsilon_{n+1}$.
- (2) If n is sink of Q , $\epsilon_{a_n} - \epsilon_{b_n} = [n] = \epsilon_n - \epsilon_{n+1}$ and thus $\tau_Q(\epsilon_n) = \epsilon_{2n+1-a_1}$ and $\tau_Q(\epsilon_{n+1}) = \epsilon_{a_1}$.

Proposition 5.16. *We have*

$$\tau_{\bar{Q}}(\epsilon_{a_k}) = \epsilon_{a_{k+1}} \quad \text{for } 1 \leq k < n \quad \text{and} \quad \tau_{\bar{Q}}(\epsilon_{a_n}) = -\epsilon_{a_1}$$

Proof. Recall the reduced expression $\tau_Q = s_{i_1} \cdots s_{i_{2n-1}}$ in (5.14).

(1) Assume n is a source of Q . Then $\tau_{\bar{Q}}$ has a reduced expression $s_n s_{i_2} s_{i_4} \cdots s_{i_{2n}}$. Since $a_k < n$ for $2 \leq k \leq n$, we have $\tau_Q^{k-1}(\epsilon_n) = \tau_Q^{k-1}(\epsilon_{a_1}) = \epsilon_{a_k} = s_n \epsilon_{a_k}$. Hence $\tau_{\bar{Q}}(\epsilon_{a_k}) = \epsilon_{a_{k+1}}$ for $1 \leq k < n$. For $k = n$, we have $\tau_Q(\epsilon_{a_n}) = \epsilon_{n+1} = s_n(\epsilon_n)$ which imply $s_{i_2} s_{i_4} \cdots s_{i_{2n}}(\epsilon_{a_n}) = \epsilon_n$. Then we $\tau_{\bar{Q}}(\epsilon_{a_n}) = s_n(\epsilon_n) = -\epsilon_n = -\epsilon_{a_1}$.

(2) Assume n is a sink of Q . Then $\tau_{\bar{Q}}$ has a reduced expression $s_{i_1} s_{i_3} \cdots s_{i_{2n-1}} s_n$. Note that $a_i \leq n$ for $1 \leq i < n$ and $a_n = n$. Then, for $1 \leq k \leq n$, we have $\tau_Q^{k-1}(\epsilon_{a_1}) = \tau_Q^{k-1}(\epsilon_{a_1}) = \epsilon_{a_k}$. Now let us compute $\tau_{\bar{Q}}(\epsilon_{a_n})$. By Lemma 5.15 and , we have

$$w_0(2\epsilon_{a_1}) = \tau_{\bar{Q}}^n(2\epsilon_{a_1}) = \tau_{\bar{Q}}^n(\gamma_n^{\bar{Q}}) = \tau_{\bar{Q}}(2\epsilon_{a_n}) = -\gamma_n^{\bar{Q}} = -2\epsilon_{a_1},$$

which implies $\tau_{\bar{Q}}(\epsilon_{a_n}) = -\epsilon_{a_1}$. □

Theorem 5.17. *For each $\vee$ -fixed Q of type A_{2n-1} and $k \in \mathbb{Z}$, we have*

$$\psi(\tau_Q^k(\gamma_i^Q)) = \psi\left(\tau_Q^k(\gamma_{2n-i}^Q)\right) = \tau_{\bar{Q}}^k(\gamma_i^{\bar{Q}}).$$

Proof. By Proposition 5.12, (5.12) and (5.15), we have

$$\psi(\tau_Q(\alpha_i)) = \tau_{\bar{Q}}(\psi(\alpha_i)) \quad \text{for all } 1 \leq i \leq 2n-1.$$

Since $\psi(\gamma_i^Q) = \psi(\gamma_{2n-i}^Q) = \gamma_i^{\bar{Q}}$ by Lemma 5.15, our assertion follows. □

5.3. F_4 and G_2 cases. (1) The set of positive roots $\Phi_{E_6}^+$ can be described as

$$\begin{aligned} & (000001), (000010), (000100), (011111), (101110), (010100), \\ & (001000), (100000), (000011), (000110), (011211), (112221), \\ & (111210), (011100), (101000), (000111), (011221), (112321), \\ & (122321), (112210), (111100), (010111), (001110), (111211), \\ & (011110), (101100), (010000), (001111), (111211), (011210), \\ & (112211), (111110), (101111), (010110), (001100), (111111), \end{aligned}$$

where $(a_1 a_2 a_3 a_4 a_5 a_6) := \sum_{i=1}^6 a_i \alpha_i$.

On the other hand, the set of positive roots $\Phi_{F_4}^+$ can be described as

$$\begin{aligned} & (0001), (0010), (0100), (1121), (0121), (1100), \\ & (0011), (0110), (1221), (1242), (1110), (0111), \\ & (1231), (1342), (2342), (1111), (0120), (1222), \\ & (1120), (1000), (1220), (1232), (0122), (1122), \end{aligned}$$

where $(b_1 b_2 b_3 b_4) := \sum_{i=1}^4 b_i \alpha_i$.

Then one can easily check that there exists a bijection $\psi: \Phi_{E_6}^+ \rightarrow \Phi_{F_4}^+$ given by

$$\alpha_i \mapsto \alpha_{\bar{i}} \quad \text{and extends it linearly.}$$

Note that the positive roots β in $\Phi_{F_4}^+$ such that $\psi^{-1}(\beta) = 1$ or 2, and

$$\{\beta \in \Phi_{F_4}^+ \mid \phi^{-1}(\beta) = 2\} = \left\{ \begin{aligned} & (0001), (0010), (1221), (0011), (1231), (1111) \\ & (0111), (0121), (1222), (1121), (1110), (0110) \end{aligned} \right\}.$$

(2) The positive roots $\Phi_{G_2}^+$ are listed as below:

$$\Phi_{G_2}^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$$

Then one can easily check that there exists a bijection $\tilde{\psi}: \Phi_{D_4}^+ \rightarrow \Phi_{G_2}^+$ given by

$$\alpha_i \mapsto \alpha_{\bar{i}} \quad \text{and extends it linearly.}$$

Now one can check the following theorem holds:

Theorem 5.18. *For each $\vee$ -fixed (resp. $\tilde{\vee}$ -fixed) $Q = (\Delta, \vee\xi)$ (resp. $Q = (\Delta, \tilde{\vee}\xi)$) of type E_6 (resp. D_4) and $k \in \mathbb{Z}$, we have*

$$\psi(\tau_Q^k(\gamma_i^Q)) = \tau_Q^k(\gamma_{\bar{i}}^{\bar{Q}}) \quad (\text{resp. } \tilde{\psi}(\tau_Q^k(\gamma_i^Q)) = \tau_Q^k(\gamma_{\bar{i}}^{\bar{Q}})) \quad \text{for } i \in \Delta_0.$$

6. DEGREE POLYNOMIALS

In this section, we generalize the definition of degree polynomial to include type $BCFG$ and compute the degree polynomials in those types by using the results in the previous sections. Recall the statistics reviewed in subsection 2.3. In this section, we almost skip the proofs because their arguments are almost similar to the ones in [50] which use the labeling algorithms described in Proposition 5.9 and Proposition 5.13 as crucial ingredients.

By (4.2) and the reflection operation in (4.5) (b), Lemma 4.10 and Proposition 4.12 can be generalized to Dynkin diagram Δ of any finite type:

Proposition 6.1.

- (a) *Let Q be a Dynkin quiver of any finite type. For any $\langle\langle\alpha, \beta\rangle\rangle, \langle\langle\alpha', \beta'\rangle\rangle \in \Phi_Q(i, j)[k]$, we have $\deg_{[Q]}(\langle\langle\alpha, \beta\rangle\rangle) = \deg_{[Q]}(\langle\langle\alpha', \beta'\rangle\rangle)$. Thus the integer $\mathfrak{o}_k^Q(i, j)$ is well-defined by (4.7). We then define $\mathfrak{o}_k^Q(i, j)$ by (4.8).*
- (b) *Let Δ be a Dynkin diagram of any finite type. For any $[Q], [Q'] \in \llbracket \Delta \rrbracket$, we have $\mathfrak{o}_k^Q(i, j) = \mathfrak{o}_k^{Q'}(i, j)$. Thus $\mathfrak{o}_k^\Delta(i, j)$ is well-defined.*

Proof. (a) By (4.2), (4.4) and the assumption, there exists $0 \leq a, a', b, b' < \mathfrak{h}/2$ such that $\tau_Q^a(\gamma_i^Q) = \alpha$, $\tau_Q^b(\gamma_j^Q) = \beta$, $\tau_Q^{a'}(\gamma_i^Q) = \alpha'$, $\tau_Q^{b'}(\gamma_j^Q) = \beta'$, $s := a - a' = b - b'$. Thus $\underline{m} \prec_Q^b \langle\langle \alpha, \beta \rangle\rangle$ if and only if $\tau_Q^{-s}(\underline{m}) \prec_Q^b \langle\langle \alpha', \beta' \rangle\rangle$. Here $\tau_Q^{-s}(\underline{m})$ denotes the canonical exponents of $\underline{m}$ obtained by applying τ_Q^{-s} to $\underline{m}$.

(b) It is enough to show that $\mathfrak{O}_k^Q(i, j) = \mathfrak{O}_k^{stQ}(i, j)$ for a Dynkin quiver $Q = (\Delta, \xi)$ with a source $t \in \Delta_0$. Since $\mathfrak{h}/2 > 2$, the second assertion follows from (a) and the reflection operation in (4.5) (b). $\square$

Definition 6.2. Let t be an indeterminate.

(1) For a Dynkin diagram Δ of any finite type and $i, j \in \Delta_0$, we define a polynomial $\mathfrak{d}_{i,j}^\Delta(t)$ as follows:

$$(6.1) \quad \mathfrak{d}_{i,j}^\Delta(t) := \sum_{k \in \mathbb{Z}_{\geq 0}} \max(\mathfrak{d}_i, \mathfrak{d}_j) \mathfrak{O}_k^\Delta(i, j) t^{k-1}.$$

We call $\mathfrak{d}_{i,j}^\Delta(t)$ the *degree polynomial of Δ at (i, j)* .

(2) For $k \in \mathbb{Z}$, we set

$$\mathfrak{d}_{i,j}[k] := \max(\mathfrak{d}_i, \mathfrak{d}_j) \mathfrak{O}_k^\Delta(i, j).$$

Remark 6.3. Definition 6.2 coincides with Definition 4.11 when Q is a Dynkin quiver of type ADE , since $\mathfrak{d}_i = 1$ for all $i \in \Delta_0$.

Lemma 6.4. Let $\underline{p} = \langle\langle \alpha, \beta \rangle\rangle$ be a $[Q]$ -pair in $(\Phi_\Delta^+)^2$. If $\deg_Q(\langle\langle \alpha, \beta \rangle\rangle) \neq 0$, we have

$$\deg_{\overline{Q}}(\langle\langle \psi(\alpha), \psi(\beta) \rangle\rangle) \neq 0.$$

Proof. Since $\deg_Q(\alpha, \beta) \neq 0$, there exists $\underline{m}$ such that $\underline{m} \prec_Q^b \underline{p}$. Let $\psi(\underline{m})$ be the exponent in Φ_Δ^+ , which can be understood as a canonical image of $\underline{m}$ via ψ . Then we have

$$\psi(\underline{m}) \prec_Q^b \langle\langle \psi(\alpha), \psi(\beta) \rangle\rangle$$

since $\eta_1 \prec_Q \eta_2$ implies $\psi(\eta_1) \prec_{\overline{Q}} \psi(\eta_2)$. Thus the assertion follows. $\square$

Remark 6.5. Note that even though $\alpha, \beta \in \Phi_\Delta^+$ is not comparable with respect to $\prec_Q$, sometimes $\psi(\alpha), \psi(\beta) \in \Phi_\Delta^+$ is comparable with respect to $\prec_{\overline{Q}}$. Moreover, there exists a pair $\langle\langle \alpha, \beta \rangle\rangle \in (\Phi_\Delta^+)^2$ such that $\deg_Q(\alpha, \beta) = 0$ but $\deg_{\overline{Q}}(\psi(\alpha), \psi(\beta)) \neq 0$. For instance, consider $\langle\langle [2, 5], [2, 3] \rangle\rangle$ of type A_5 in Example 4.4. Then we have

$$\deg_Q([2, 5], [2, 3]) = 0, \text{ while } \deg_{\overline{Q}}(\psi([2, 5]), \psi([2, 3])) = \deg_{\overline{Q}}(\langle 2, 3 \rangle, \langle 1, 2 \rangle) = 1,$$

since $\text{hd}_{\overline{Q}}(\langle 2, 3 \rangle, \langle 1, 2 \rangle) = (\langle 1, 3 \rangle, \langle 2, 2 \rangle)$ (see Γ_Q of type A_5 in Example 4.4 and $\Gamma_{\overline{Q}}$ of type C_3 in Example 4.5).

In the following subsections, we will give the proof of the following theorem by the case-by-case method (see Remark 4.29).

Theorem 6.6. Let Δ be a Dynkin diagram of classical type A_n, B_n, C_n, D_n and E_6 . Then we have

$$\widetilde{\mathfrak{d}}_{i,j}^\Delta(t) = \mathfrak{d}_{i,j}^\Delta(t) + \delta_{i,j*} \mathfrak{d}_i t^{\mathfrak{h}-1}.$$

Note that for cases when A_n, D_n and E_6 , the above theorem was proved in [48, 49, 50].

6.1. **B_n -case.** Throughout this subsection, $\triangle$ denotes the Dynkin diagram of type D_{n+1} and $\blacktriangle$ denotes the one of type B_n .

Theorem 6.7. *Let $\blacktriangle$ be the Dynkin diagram of type B_n . Then we have*

$$\mathfrak{d}_{i,j}^{\blacktriangle}(t) + \delta_{i,j} \mathbf{d}_i t^{h-1} = \tilde{\mathfrak{d}}_{i,j}^{\blacktriangle}(t).$$

By taking ψ on (4.10), we have the following pictures:

(6.2)

By taking ψ on (4.11), we have the following:

(6.3)

In each case, the value $(\alpha, \beta) = (\psi(\alpha), \psi(\beta))$ does not depend on the choice of Q and are given in Table 3:

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)	(ix)
(α, β)	-2	0	-2	0	0	2	0	-2	0	-2	0

TABLE 3. (α, β) for non $[Q]$ -simple $[Q]$ -pairs of type B_n

For the cases when $\langle\langle \alpha, \beta \rangle\rangle$ is a $[Q]$ -minimal $[Q]$ -pair for $\alpha + \beta \in \Phi^+$,

$$(6.4) \quad p_{\beta, \alpha} = 1 \quad \text{if } \langle\langle \alpha, \beta \rangle\rangle \text{ is of case (ix)' and } p_{\beta, \alpha} = 0, \text{ otherwise.}$$

Example 6.8. Using Γ_Q in Example 5.11,

- (1) $\langle 1, 2 \rangle \prec_Q^b \langle \langle 1, -3 \rangle, \langle 2, 3 \rangle \rangle$ corresponds to (i)',
- (2) $\langle \langle 1, -3 \rangle, \langle 2, 4 \rangle \rangle \prec_Q^b \langle \langle 2, -3 \rangle, \langle 1, 4 \rangle \rangle$ corresponds to (ii)',
- (3) $\langle 2, -4 \rangle \prec_Q^b \langle \langle 2, -3 \rangle, \langle 3, -4 \rangle \rangle$ corresponds to (iii)',
- (4) $\langle \langle 1, -3 \rangle, \langle 2, -4 \rangle \rangle \prec_Q^b \langle \langle 2, -3 \rangle, \langle 1, -4 \rangle \rangle$ corresponds to (iv)',
- (5) $2\langle 1 \rangle \prec_Q^b \langle \langle 1, -3 \rangle, \langle 1, 3 \rangle \rangle$ corresponds to (v)',
- (6) $\{\langle 3, 4 \rangle, 2\langle 1 \rangle\}_{\beta \in \Phi^+} \prec_Q^b \langle \langle 1, 4 \rangle, \langle 1, 3 \rangle \rangle$ corresponds to (vi)',
- (7) $\langle \langle 1 \rangle, \langle 2, 3 \rangle \rangle \prec_Q^b \langle \langle 1, 2 \rangle, \langle 3 \rangle \rangle$ correspond to (vii)',
- (8) $\langle 1 \rangle \prec_Q^b \langle \langle 1, -3 \rangle, \langle 3 \rangle \rangle$ correspond to (viii)',
- (9) $\langle 2, 3 \rangle \prec_Q^b \langle \langle 2 \rangle, \langle 3 \rangle \rangle$ correspond to (ix)',
- (10) $\langle 1, 2 \rangle \prec_Q^b \langle \langle 1, -3 \rangle, \langle 2, 3 \rangle \rangle, \langle \langle 2 \rangle, \langle 1 \rangle \rangle \prec_Q^b \langle \langle 2, -3 \rangle, \langle 1, 3 \rangle \rangle$ corresponds to (x)',
- (11) $\langle \langle 1, 2 \rangle, \langle 3, 4 \rangle \rangle \prec_Q^b \langle \langle 1 \rangle, \langle 2 \rangle, \langle 3, 4 \rangle \rangle, \langle \langle 2, 3 \rangle, \langle 1, 4 \rangle \rangle \prec_Q^b \langle \langle 2, 4 \rangle, \langle 1, 3 \rangle \rangle$ corresponds to (xi)'.

By the similarities between the combinatorics for Γ_Q and $\Gamma_{\overline{Q}}$, described in Proposition 4.16, 4.18 for Γ_Q and Proposition 5.9 for $\Gamma_{\overline{Q}}$, we can apply the same argument in [50] and obtain the following theorem:

Theorem 6.9. Let $Q = (\blacktriangle, \xi)$ be a Dynkin quiver of type B_n .

- (1) For any $[Q]$ -pair $\langle \alpha, \beta \rangle \in (\Phi_{B_n}^+)^2$, $\text{hd}_Q(\langle \alpha, \beta \rangle)$ is well-defined.
- (2) For a $[Q]$ -pair $\langle \alpha, \beta \rangle$, if $\deg_Q(\langle \alpha, \beta \rangle) \neq 0$, then the relative position of $\langle \alpha, \beta \rangle$ is one of (i)' $\sim$ (xi)' in (6.2) and (6.3).
- (3) Every exponent $\underline{m} \prec_Q^b \langle \alpha, \beta \rangle$ is also described in (6.2) and (6.3).
- (4) For $\langle \alpha, \beta \rangle \in (\Phi_{\blacktriangle}^+)^2$, assume $\deg_Q(\langle \alpha, \beta \rangle) \neq 0$. Then there exist $\alpha' \in \psi^{-1}(\alpha)$ and $\beta' \in \psi^{-1}(\beta)$ such that $\deg_{\underline{Q}}(\langle \alpha', \beta' \rangle) \neq 0$ and $\text{hd}_Q(\langle \alpha, \beta \rangle) = \psi(\text{hd}_{\underline{Q}}(\langle \alpha', \beta' \rangle))$. Furthermore, in this case, we have

$$\deg_Q(\langle \alpha, \beta \rangle) = \deg_{\underline{Q}}(\langle \alpha', \beta' \rangle).$$

Notice that, in cases (v) and (vi), the head of $\langle \alpha, \beta \rangle$ has component 2 in which phenomena does not happen in D_{n+1} -case.

Proof of Theorem 6.7. Note that for $1 \leq i, j \leq n$ with $\min(i, j) < n$, we have $\max(\mathbf{d}_i, \mathbf{d}_j) = 2$. By Theorem 6.9, we have

$$\mathfrak{d}_{i,j}^{\blacktriangle}(t) = 2 \times \mathfrak{d}_{i,j}^{\triangle}(t)$$

implying the assertion by the formulas (4.12) and (6.1).

In the case when $i = j = n$, by Theorem 6.9, and the facts that $\psi(\epsilon_i \pm \epsilon_{n+1}) = \epsilon_i$ and $\mathbf{d}_n = 1$, we have

$$\mathfrak{d}_{n,n}^{\blacktriangle}(t) = \left(\mathfrak{d}_{n,n+1}^{\triangle}(t) + \mathfrak{d}_{n,n}^{\triangle}(t) \right) = \sum_{s=1}^n t^{2s-1},$$

which completes the assertion. □

By observing (6.3), we have the following proposition:

Proposition 6.10. *For a Dynkin quiver Q of type B_n , let $\langle\langle\alpha, \beta\rangle\rangle$ be a $[Q]$ -pair such that $\deg_Q(\langle\langle\alpha, \beta\rangle\rangle) = 2$.*

(1) *If $\alpha + \beta = \gamma \in \Phi_{B_n}^+$, then there are two $[Q]$ -pairs $\langle\langle\alpha_i, \beta_i\rangle\rangle$ ($1 \leq i \leq 2$) of γ such that*

$$\text{hd}_Q(\langle\langle\alpha, \beta\rangle\rangle) = \gamma \prec_{[Q]}^b \langle\langle\alpha_i, \beta_i\rangle\rangle \prec_{[Q]}^b \langle\langle\alpha, \beta\rangle\rangle$$

(i) $\langle\langle\alpha_1, \beta_1\rangle\rangle, \langle\langle\alpha_2, \beta_2\rangle\rangle$ are not comparable with respect to $\prec_{[Q]}^b$,

(ii) $\{\alpha_2, \beta_2\} = \{\langle a \rangle, \langle b \rangle\}$ and $\text{res}_Q(\alpha_1), \text{res}_Q(\beta_1) < n$.

(2) *If $\alpha + \beta \notin \Phi_{B_n}^+$, then there are exponents $\langle\langle\alpha_1, \beta_1\rangle\rangle, \{\tau, \zeta, \gamma\}_{\beta \in \Phi^+}$ such that*

$$\text{hd}_Q(\langle\langle\alpha, \beta\rangle\rangle) \prec_{[Q]}^b \langle\langle\alpha_1, \beta_1\rangle\rangle, \{\tau, \zeta, \gamma\}_{\beta \in \Phi^+} \prec_{[Q]}^b \langle\langle\alpha, \beta\rangle\rangle$$

(i) $\langle\langle\alpha_1, \beta_1\rangle\rangle, \{\tau, \zeta, \gamma\}_{\beta \in \Phi^+}$ are not comparable with respect to $\prec_{[Q]}^b$,

(ii) $\text{res}^{[Q]}(\alpha_1), \text{res}^{[Q]}(\gamma), \text{res}^{[Q]}(\beta_1) < n$ and $\text{res}^{[Q]}(\tau), \text{res}^{[Q]}(\zeta) = n$.

6.2. C_n -case. Throughout this subsection, $\blacktriangle$ usually denotes the Dynkin diagram of type C_n . This subsection is devoted to prove the following theorem:

Theorem 6.11. *Let $\blacktriangle$ be the Dynkin diagram of type C_n . Then we have*

$$\mathfrak{d}_{i,j}^{\blacktriangle}(t) + \delta_{i,j} \max(\mathbf{d}_i, \mathbf{d}_j) t^{h-1} = \tilde{\mathfrak{d}}_{i,j}^{\blacktriangle}(t).$$

Recall the natural bijection $\psi^{\text{tr}}: \Phi_{B_n}^+ \rightarrow \Phi_{C_n}^+$ and Proposition 5.13, which tells that labeling of Γ_Q and $\Gamma_{Q^{\text{tr}}}$ for Dynkin quiver Q of type B_n are almost the same, when we use the notations $\langle a, b \rangle$, $\langle a \rangle$ and $\langle a, a \rangle$ for $1 \leq a \leq |b| \leq n$.

For $\alpha \in \Phi_{B_n}^+$, we set

$$\alpha^{\text{tr}} := \psi^{\text{tr}}(\alpha).$$

Using Proposition 5.9 for $\Gamma_{\overline{Q}}$ and the similar argument in [50], we obtain the following as in B_n -cases:

Theorem 6.12. *Let Q^{tr} a Dynkin quiver of type C_n .*

(1) *For any $[Q]$ -pair $\langle\langle\alpha, \beta\rangle\rangle \in (\Phi_{C_n}^+)^2$, $\text{hd}_{Q^{\text{tr}}}(\langle\langle\alpha, \beta\rangle\rangle)$ is well-defined.*

(2) *For $\langle\langle\alpha, \beta\rangle\rangle \in (\Phi_{B_n}^+)^2$, $\deg_Q(\langle\langle\alpha, \beta\rangle\rangle) \neq 0$ if and only if $\deg_{Q^{\text{tr}}}(\langle\langle\alpha^{\text{tr}}, \beta^{\text{tr}}\rangle\rangle) \neq 0$. Moreover, we have*

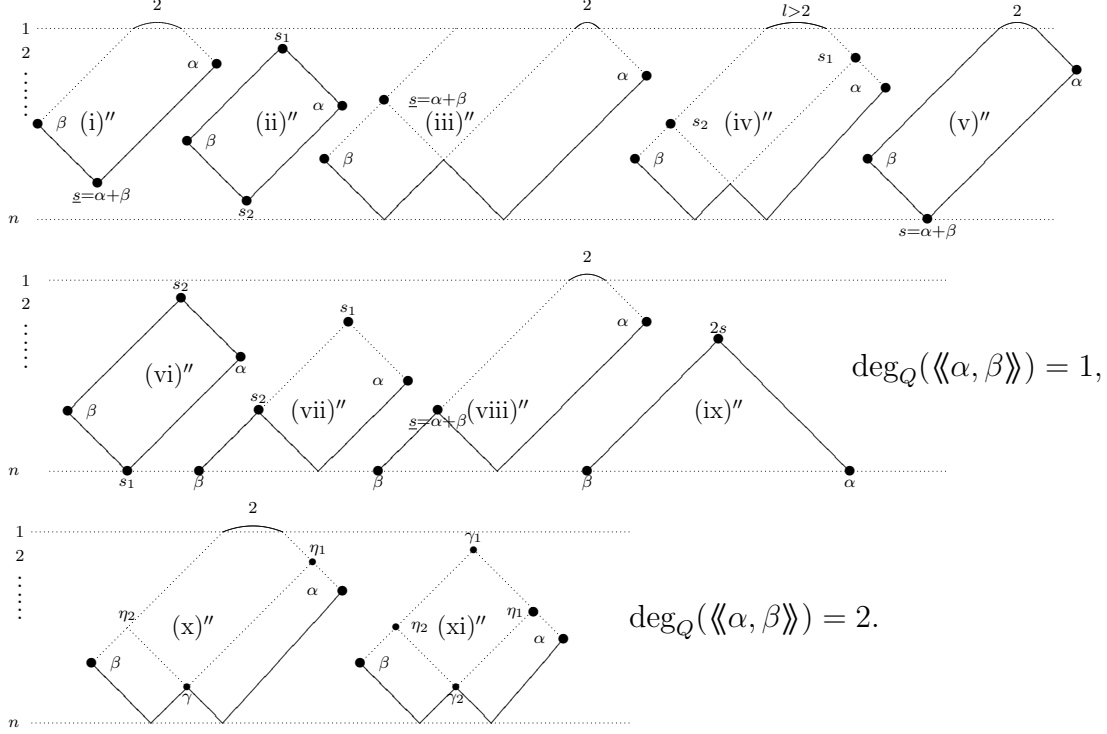
$$\deg_Q(\langle\langle\alpha, \beta\rangle\rangle) = \deg_{Q^{\text{tr}}}(\langle\langle\alpha^{\text{tr}}, \beta^{\text{tr}}\rangle\rangle).$$

Note that

$$\langle i \rangle = \varepsilon_i \text{ if } \blacktriangle \text{ is of type } B_n \text{ and } \langle i, i \rangle = 2\varepsilon_i \text{ if } \blacktriangle \text{ is of type } C_n.$$

By this difference, the descriptions for $\text{hd}_Q(\langle\langle\alpha, \beta\rangle\rangle)$ for $[Q]$ -pairs $\langle\langle\alpha, \beta\rangle\rangle \in \Phi_{C_n}^+$ and a Dynkin quiver Q of type C_n with $\deg_Q(\langle\langle\alpha, \beta\rangle\rangle) \neq 0$ are sometimes different from the corresponding ones in cases of type B_n . Here we present the $[Q]$ -pairs $\langle\langle\alpha, \beta\rangle\rangle \in \Phi_{C_n}^+$ for $\deg_Q(\langle\langle\alpha, \beta\rangle\rangle) \neq 0$

with description of $\text{hd}_Q(\langle\langle\alpha, \beta\rangle\rangle)$ as follows:



Here one can recognize that the differences happen in cases (vii)'' $\sim$ (xi)'', when we compare the cases for B_n .

In each case, the value (α, β) does not depend on the choice of Q and are given in Table 4:

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)	(xi)
(α, β)	-1	0	-1	0	0	1	0	-1	0	-1	0

TABLE 4. (α, β) for non $[Q]$ -simple $[Q]$ -pairs of type C_n

For the cases when a $[Q]$ -pair $\langle\langle\alpha, \beta\rangle\rangle$ is $[Q]$ -minimal for $\alpha + \beta \in \Phi^+$,

$$(6.5) \quad p_{\beta, \alpha} = 1 \quad \text{if } \langle\langle\alpha, \beta\rangle\rangle \text{ is of case (v)''} \quad \text{and} \quad p_{\beta, \alpha} = 0, \text{ otherwise.}$$

Example 6.13. Using Γ_Q in Example 5.14,

- (7) $\langle\langle 1, 3 \rangle, \langle 3, 4 \rangle \rangle \prec_Q^b \langle\langle 1, 4 \rangle, \langle 3, 3 \rangle \rangle$ correspond to (vii)'',
- (8) $\langle 1, 3 \rangle \prec_Q^b \langle\langle 1, 4 \rangle, \langle 3, -4 \rangle \rangle$ correspond to (viii)'',
- (9) $2\langle 3, 4 \rangle \prec_Q^b \langle\langle 3, 3 \rangle, \langle 4, 4 \rangle \rangle$ correspond to (ix)'',
- (10) $\langle 1, 3 \rangle \prec_Q^b \langle\langle 3, 4 \rangle, \langle 1, -4 \rangle \rangle \prec_Q^b \langle\langle 1, 4 \rangle, \langle 3, -4 \rangle \rangle$ corresponds to (x)'',
- (11) $\langle\langle 1, 2 \rangle, \langle 3, 4 \rangle \rangle \prec_Q^b \langle\langle 2, 3 \rangle, \langle 1, 4 \rangle \rangle \prec_Q^b \langle\langle 2, 4 \rangle, \langle 1, 3 \rangle \rangle$ corresponds to (xi)''.

By observing $(x)''$ and $(xi)''$, we have the following proposition:

Proposition 6.14. *For a Dynkin quiver Q of type C_n , let $\langle\langle\alpha, \beta\rangle\rangle$ be a $[Q]$ -pair such that $\deg_Q(\alpha, \beta) = 2$. Then there exists a unique $[Q]$ -pair $\underline{p}$ such that*

$$\mathrm{hd}_Q(\langle\langle\alpha, \beta\rangle\rangle) \prec_Q^b \underline{p} \prec_Q^b \langle\langle\alpha, \beta\rangle\rangle.$$

Proof of Theorem 6.11. By Theorem 6.9, we have

$$\sum_{k \in \mathbb{Z}_{\geq 0}} \mathfrak{o}_k^{B_n}(i, j) t^{k-1} = \sum_{k \in \mathbb{Z}_{\geq 0}} \mathfrak{o}_k^{C_n}(i, j) t^{k-1}$$

for any $1 \leq i, j \leq n$. Since, when $\max(i, j) = n$, we have $\max(\mathbf{d}_i, \mathbf{d}_j) = 2$ and

$$\sum_{s=1}^{\min(i, j)} (t^{|i-j|+2s-1} + t^{2n-i-j+2s-1}) = 2 \sum_{s=1}^i t^{n-i+2s-1},$$

our assertion follows from the definition of degree polynomial

$$\mathfrak{d}_{i,j}^\Delta(t) := \sum_{k \in \mathbb{Z}_{\geq 0}} \max(\mathbf{d}_i, \mathbf{d}_j) \mathfrak{o}_k^\Delta(i, j) t^{k-1}. \quad \square$$

6.3. F_4 -case. In this case, we have only finite many Dynkin quivers. Recall that $\mathbf{d}_1 = \mathbf{d}_2 = 2$ and $\mathbf{d}_3 = \mathbf{d}_4 = 1$. Then one can check the results in this subsections and we exhibit several examples instead of giving proofs.

For a Dynkin quiver $Q = (\mathbf{\Delta}_{F_4}, \xi)$ with

$$\begin{array}{c} \textcircled{4} \\ 1 \end{array} \longrightarrow \begin{array}{c} \textcircled{3} \\ 2 \end{array} \longrightarrow \begin{array}{c} \textcircled{2} \\ 3 \end{array} \longrightarrow \begin{array}{c} \textcircled{1} \\ 4 \end{array},$$

the AR-quiver Γ_Q can be described as follows:

$$(6.6) \quad \Gamma_Q^{F_4} = \begin{array}{cccccccccccccccc} (i \setminus p) & -9 & -8 & -7 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 \\ \hline 1 & & & & & & & & & & & & & & \\ 2 & & & (1, 0, -1, 0) & (0, 0, 1, 1) & (1, 0, 0, -1) & (0, 1, 0, 1) & (0, 0, 1, -1) & (0, 1, -1, 0) & & & & & \\ 3 & & & (1, -1, 0, 0) & (1, 0, 0, 1) & (1, 0, 1, 0) & (1, 1, 0, 0) & (0, 1, 1, 0) & (0, 1, 0, -1) & & & & & \\ 4 & & & (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}) & (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) & (1, 0, 0, 0) & (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) & (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}) & (0, 1, 0, 0) & & & & & \\ & & & (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) & (0, 0, 0, 1) & (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}) & (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}) & (0, 0, 1, 0) & (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) & & & & & \end{array}$$

Here we use notation the notation $(a, b, c, d) := a\varepsilon_1 + b\varepsilon_2 + c\varepsilon_3 + d\varepsilon_4$, $\varepsilon_i = \sqrt{2}\epsilon_i$ and

$$\alpha_1 = (0, 1, -1, 0), \quad \alpha_2 = (0, 0, 1, -1), \quad \alpha_3 = (0, 0, 0, 1) \quad \text{and} \quad \alpha_4 = (1/2, -1/2, -1/2, -1/2).$$

Theorem 6.15. *For a Dynkin quiver Q of type F_4 and any $[Q]$ -pair $\langle\langle\alpha, \beta\rangle\rangle \in (\Phi_{F_4}^+)^2$, $\mathrm{hd}_Q(\langle\langle\alpha, \beta\rangle\rangle)$ is well-defined. In particular, if $\alpha + \beta = \gamma$, then $\mathrm{hd}_Q(\langle\langle\alpha, \beta\rangle\rangle) = \gamma$.*

Remark 6.16. In F_4 -case, $\mathfrak{d}_{i,j}(t) + \delta_{i,j}t^{h-1}$ does not coincide with $\tilde{\mathfrak{d}}_{i,j}(t)$ any more. For instance,

$$\tilde{\mathfrak{d}}_{2,3}[9] = 4 \quad \text{while} \quad \mathfrak{d}_{2,3}[9] = 6.$$

Here the fact that $\mathfrak{d}_{2,3}[9] = 6$ can be computed by using Γ_Q in (6.6) as follows: Let $\alpha = (0, 1, 0, -1)$ and $\beta = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Then $\gamma = \alpha + \beta \in \Phi^+$ and

$$(6.7) \quad \gamma \prec_Q^b \langle (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}), (0, 0, 1, 0) \rangle \prec_Q^b \langle (0, 1, 0, 0), (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}) \rangle \prec_Q^b \langle \alpha, \beta \rangle.$$

6.4. G_2 -case. In this case, we have only finite many Dynkin quivers. Recall that $\mathbf{d}_1 = 1$ and $\mathbf{d}_2 = 3$. Then one can check the results in this subsections and we exhibit several examples instead of giving proofs.

For a Dynkin quiver $Q = (\mathbf{\Delta}_{G_2}, \xi)$ with

$$\begin{array}{c} 2 \\ \circ \\ 1 \end{array} \longrightarrow \begin{array}{c} 1 \\ \circ \\ 2 \end{array},$$

we have

$$\Gamma_Q^{G_2} = \begin{array}{ccccccc} (i \setminus p) & -3 & -2 & -1 & 0 & 1 & 2 \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{array}{c} (0, 1, -1) \\ \Downarrow \\ (1, 1, -2) \end{array} & \begin{array}{c} (1, 0, -1) \\ \nearrow \\ (2, -1, -1) \end{array} & \begin{array}{c} (1, -1, 0) \\ \nearrow \\ (1, -2, 1) \end{array} & & & \end{array}.$$

Here we use the orthonormal basis $\{\epsilon_i \mid 1 \leq i \leq 3\}$ of $\mathbb{R}^3$, the notation $(a, b, c) := a\epsilon_1 + b\epsilon_2 + c\epsilon_3$ and

$$\alpha_1 = (0, 1, -1) \quad \text{and} \quad \alpha_2 = (1, -2, 1).$$

Theorem 6.17. *For a Dynkin quiver Q of type G_2 and any $[Q]$ -pair $\langle \alpha, \beta \rangle \in (\Phi_{G_2}^+)^2$, $\text{hd}_Q(\langle \alpha, \beta \rangle)$ is well-defined. In particular, if $\alpha + \beta = \gamma$, then $\text{hd}_Q(\langle \alpha, \beta \rangle) = \gamma$.*

Remark 6.18. As F_4 -case, $\mathfrak{d}_{i,j}(t) + \delta_{i,j}t^{\mathbf{h}-1}$ does not coincide with $\tilde{\mathfrak{d}}_{i,j}(t)$ any more in G_2 -case either. For instance,

$$\tilde{\mathfrak{d}}_{1,1}[4] = 2 \quad \text{while} \quad \mathfrak{d}_{1,1}[4] = 1.$$

Note that, for each Dynkin quiver Q of type G_2 , there exist only one $[Q]$ -pair of positive roots $\langle \alpha, \beta \rangle$ such that $\alpha + \beta = \gamma \in \Phi^+$ and $\deg_Q(\langle \alpha, \beta \rangle) = 2$ ($i = 1, 2$). In that case, there exists a unique $[Q]$ -pair $\underline{p}$ such that

$$\gamma \prec_Q^b \underline{p} \prec_Q^b \langle \alpha, \beta \rangle.$$

7. QUIVER HECKE ALGEBRAS AND R-MATRICES

In this section, we briefly recall the definition of unipotent quantum coordinate ring and quiver Hecke algebra R associated to a finite Cartan datum $(\mathbf{C}, \mathbf{P}, \Pi, \mathbf{P}^\vee, \Pi^\vee)$. Then we review the definition, invariants and properties of affinizations of modules over quiver Hecke algebras which are introduced and investigated in [36, 31]. In the final section, we will see that the $\mathfrak{d}$ -invariants for pairs of cuspidal modules over R associated with Dynkin quivers coincide with $(\tilde{\mathfrak{d}}_{i,j}(t))_{i,j \in I}$.

7.1. Unipotent quantum coordinate ring. Let q be an indeterminate. For $k \in \mathbb{Z}_{\geq 1}$ and $i \in I$, we set

$$q_i = q^{d_i}, \quad [k]_i := \frac{q_i^k - q_i^{-k}}{q_i - q_i^{-1}} \quad \text{and} \quad [k]_i! := \prod_{s=1}^k [s]_i.$$

We denote by $U_q(\mathfrak{g})$ the quantum group associated to a finite Cartan datum $(\mathbf{C}, \mathbf{P}, \Pi, \mathbf{P}^\vee, \Pi^\vee)$, which is the associative algebra over $\mathbb{Q}(q)$ generated by e_i, f_i ($i \in I$) and q^h ($h \in \mathbf{P}^\vee$). We set $U_q^\pm(\mathfrak{g})$ be the subalgebra generated by e_i (resp. f_i) for $i \in I$. Note that $U_q(\mathfrak{g})$ admits the weight space decomposition:

$$U_q(\mathfrak{g}) = \bigoplus_{\beta \in \mathbf{Q}} U_q(\mathfrak{g})_\beta.$$

For any $i \in I$, there exists a unique $\mathbb{Q}(q)$ -linear endomorphisms e'_i of $U_q^-(\mathfrak{g})$ such that

$$e'_i(f_j) = \delta_{i,j}, \quad e'_i(xy) = e'_i(x) + q^{(\alpha_i, \beta)} x e'_i(y) \quad (x \in U_q^-(\mathfrak{g})_\beta, y \in U_q^-(\mathfrak{g})).$$

Then there exists a unique non-degenerate symmetric bilinear form $(\cdot, \cdot)_K$ on $U_q^-(\mathfrak{g})$ such that

$$(\mathbf{1}, \mathbf{1})_K = 1, \quad (f_i u, v)_K = (u, e'_i v)_K \text{ for } i \in I, u, v \in U_q^-(\mathfrak{g}).$$

We set $\mathbb{A} = \mathbb{Z}[q^{\pm 1}]$ and let $U_q^+(\mathfrak{g})_{\mathbb{A}}$ be the $\mathbb{A}$ -subalgebra of $U_q^+(\mathfrak{g})$ generated by $e_i^{(n)} := e_i^n / [n]_i!$ ($i \in I, n \in \mathbb{Z}_{>0}$).

Let Δ_n be the algebra homomorphism $U_q^+(\mathfrak{g}) \rightarrow U_q^+(\mathfrak{g}) \otimes U_q^+(\mathfrak{g})$ given by $\Delta_n(e_i) = e_i \otimes 1 + 1 \otimes e_i$, where the algebra structure on $U_q^+(\mathfrak{g}) \otimes U_q^+(\mathfrak{g})$ is defined by

$$(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) = q^{-(\text{wt}(x_2), \text{wt}(y_1))} (x_1 y_1 \otimes x_2 y_2).$$

Set

$$A_q(\mathbf{n}) = \bigoplus_{\beta \in \mathbf{Q}^-} A_q(\mathbf{n})_\beta \quad \text{where } A_q(\mathbf{n})_\beta := \text{Hom}_{\mathbb{Q}(q)}(U_q^+(\mathfrak{g})_{-\beta}, \mathbb{Q}(q)).$$

Then $A_q(\mathbf{n})$ is an algebra with the multiplication given by $(\psi \cdot \theta)(x) = \theta(x_{(1)})\psi(x_{(2)})$, when $\Delta_n(x) = x_{(1)} \otimes x_{(2)}$ in Sweedler's notation.

Let us denote by $A_{\mathbb{A}}(\mathbf{n})$ the $\mathbb{A}$ -submodule of $A_q(\mathbf{n})$ consisting of $\psi \in A_q(\mathbf{n})$ such that $\psi(U_q^+(\mathfrak{g})_{\mathbb{A}}) \subset \mathbb{A}$. Then it is an $\mathbb{A}$ -subalgebra of $A_q(\mathbf{n})$. Note that Lusztig [42, 43] and Kashiwara [26] have constructed a specific $\mathbb{A}$ -basis $\mathbf{B}^{\text{up}}$ of $A_{\mathbb{A}}(\mathbf{n})$.

Note that for a commutation class $\mathbf{c}$, the convex order $\prec_{\mathbf{c}}$ provides the *dual PBW-vectors*

$$\{r_{\mathbf{c}}(\beta) \mid \beta \in \Phi^+\}$$

of $A_q(\mathbf{n})$ which satisfies the following conditions.

Theorem 7.1 ([3]). *The dual PBW vector $r_{\mathbf{c}}(\alpha_i)$ ($i \in I$) is the element such that $(f_j, r_{\mathbf{c}}(\alpha_i))_K = \delta_{i,j}$. For a $\mathbf{c}$ -minimal $\mathbf{c}$ -pair $\langle\langle \alpha, \beta \rangle\rangle$ for $\gamma \in \Phi^+ \setminus \Pi$, we have*

$$(7.1) \quad r_{\mathbf{c}}(\alpha)r_{\mathbf{c}}(\beta) - q^{-(\alpha, \beta)} r_{\mathbf{c}}(\beta)r_{\mathbf{c}}(\alpha) = q^{-p_{\beta, \alpha}} (1 - q^{2(p_{\beta, \alpha} - (\alpha, \beta))}) r_{\mathbf{c}}(\gamma).$$

For a dominant weight $\lambda \in P^+$, let $V(\lambda)$ be the irreducible highest weight $U_q(\mathfrak{g})$ -module with highest weight vector u_λ of weight λ . Let $(\ , \)_\lambda$ be the non-degenerate symmetric bilinear form on $V(\lambda)$ such that $(u_\lambda, u_\lambda)_\lambda = 1$ and $(xu, v)_\lambda = (u, \varphi(x)v)_\lambda$ for $u, v \in V(\lambda)$ and $x \in U_q(\mathfrak{g})$, where φ is the algebra antiautomorphism on $U_q(\mathfrak{g})$ defined by $\varphi(e_i) = f_i$, $\varphi(f_i) = e_i$ and $\varphi(q^h) = q^h$. For each $\mu, \zeta \in W_{\mathfrak{g}}\lambda$, the *unipotent quantum minor* $D(\mu, \zeta)$ is an element in $A_q(\mathfrak{n})$ given by $D(\mu, \zeta)(x) = (xu_\mu, u_\zeta)_\lambda$ for $x \in U_q^+(\mathfrak{g})$, where u_μ and u_ζ are the extremal weight vectors in $V(\lambda)$ of weight μ and ζ , respectively. Then we have $D(\mu, \zeta) \in \mathbf{B}^{\text{up}} \sqcup \{0\}$. Note that $D(\mu, \zeta) \in \mathbf{B}^{\text{up}}$ if and only if $\mu \preceq \zeta$. Here $\mu \preceq \zeta$ if there are $w, w' \in W_{\mathfrak{g}}$ such that $w' \prec w$ and $\mu = w\lambda$, $\zeta = w'\lambda$.

For a reduced expression $\underline{w}_0 = s_{i_1}s_{i_2}\cdots s_{i_\ell}$ of the longest element $w_0 \in W_{\mathfrak{g}}$, define $\underline{w}_{\leq k} := s_{i_1}\cdots s_{i_k}$ and $\lambda_k := \underline{w}_{\leq k}\varpi_{i_k}$ for $1 \leq k \leq \ell$. Note that $\lambda_{k^-} = \underline{w}_{\leq k-1}\varpi_{i_k}$ if $k^- > 0$. Here

$$k^- := \max(\{1 \leq s < k \mid i_s = i_k\} \sqcup \{0\}).$$

For $0 \leq t \leq s \leq \ell$, we set

$$D_{\underline{w}_0}(s, t) := \begin{cases} D(\underline{w}_{\leq s}\varpi_{i_s}, \underline{w}_{\leq t}\varpi_{i_t}) & \text{if } 1 \leq t \leq s \leq \ell \text{ and } i_s = i_t, \\ D(\underline{w}_{\leq s}\varpi_{i_s}, \varpi_{i_s}) & \text{if } 0 = t < s \leq \ell, \\ 1 & \text{if } 0 = t = s. \end{cases}$$

Then $D_{\underline{w}_0}(s, t)$ belongs to $\mathbf{B}^{\text{up}}$ and

$$(7.2) \quad r_{[\underline{w}_0]}(\beta_s^{w_0}) = D_{\underline{w}_0}(s, s^-) \quad \text{for } 1 \leq s \leq \ell.$$

7.2. Quiver Hecke algebras. Let $\mathbf{k}$ be a field. For $i, j \in I$, we choose polynomials $\mathcal{Q}_{i,j}(u, v) \in \mathbf{k}[u, v]$ such that $\mathcal{Q}_{i,j}(u, v) = \mathcal{Q}_{j,i}(v, u)$, which is of the form

$$\mathcal{Q}_{i,j}(u, v) = \begin{cases} \sum_{p(\alpha_i, \alpha_i) + q(\alpha_j, \alpha_j) = -2(\alpha_i, \alpha_j)} t_{i,j;p,q} u^p v^q & \text{if } i \neq j, \\ 0 & \text{if } i = j, \end{cases}$$

where $t_{i,j;-c_{i,j},0} \in \mathbf{k}^\times$

For $\beta \in \mathbf{Q}^+$ with $\text{ht}(\beta) = m$, we set

$$I^\beta := \left\{ \nu = (\nu_1, \dots, \nu_m) \in I^m \mid \sum_{k=1}^m \alpha_{\nu_k} = \beta \right\},$$

on which the symmetric group $\mathfrak{S}_m = \langle r_k \mid k = 1, \dots, m-1 \rangle$ acts by place permutations.

Definition 7.2. For $\beta \in \mathbf{Q}^+$, the *quiver Hecke algebra* $R(\beta)$ associated with $\mathbf{C}$ and $(\mathcal{Q}_{i,j}(u, v))_{i,j \in I}$ is the $\mathbf{k}$ -algebra generated by

$$\{e(\nu) \mid \nu \in I^\beta\}, \{x_k \mid 1 \leq k \leq m\}, \{\tau_l \mid 1 \leq l < m\}$$

satisfying the following defining relations:

$$\begin{aligned} e(\nu)e(\nu') &= \delta_{\nu, \nu'}, \sum_{\nu \in I^\beta} e(\nu) = 1, \quad x_k e(\nu) = e(\nu) x_k, \quad x_k x_l = x_l x_k, \\ \tau_l e(\nu) &= e(r_l(\nu)) \tau_l, \quad \tau_k \tau_l = \tau_l \tau_k \quad \text{if } |k - l| > 1, \quad \tau_k^2 e(\nu) = \mathcal{Q}_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}) e(\nu), \end{aligned}$$

$$\begin{aligned}
(\tau_k x_l - x_{r_k(l)} \tau_k) e(\nu) &= \begin{cases} -e(\nu) & \text{if } l = k \text{ and } \nu_k = \nu_{k+1}, \\ e(\nu) & \text{if } l = k+1 \text{ and } \nu_k = \nu_{k+1}, \\ 0 & \text{otherwise,} \end{cases} \\
(\tau_{k+1} \tau_k \tau_{k+1} - \tau_k \tau_{k+1} \tau_k) e(\nu) &= \begin{cases} \overline{\mathcal{Q}}_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}, x_{k+2}) e(\nu) & \text{if } \nu_k = \nu_{k+2}, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

where

$$\overline{\mathcal{Q}}_{i,j}(u, v, w) := \frac{\mathcal{Q}_{i,j}(u, v) - \mathcal{Q}_{i,j}(w, v)}{u - w} \in \mathbf{k}[u, v, w].$$

The algebra $R(\beta)$ has the $\mathbb{Z}$ -grading defined by

$$\deg(e(\nu)) = 0, \quad \deg(x_k e(\eta)) = (\alpha_{\eta_k}, \alpha_{\eta_k}), \quad \deg(\tau_l e(\eta)) = -(\alpha_{\eta_l}, \alpha_{\eta_{l+1}}).$$

We denote by $R(\beta)\text{-gmod}$ the category of graded finite-dimensional $R(\beta)$ -modules with degree preserving homomorphisms.. We set $R\text{-gmod} := \bigoplus_{\beta \in \mathbf{Q}^+} R(\beta)\text{-gmod}$. The trivial $R(0)$ -module of degree 0 is denoted by $\mathbf{1}$. For simplicity, we write “a module” instead of “a graded module”. We define the grading shift functor q by $(qM)_k = M_{k-1}$ for a $\mathbb{Z}$ -graded module $M = \bigoplus_{k \in \mathbb{Z}} M_k$. For $M, N \in R(\beta)\text{-gmod}$, $\text{Hom}_{R(\beta)}(M, N)$ denotes the space of degree preserving module homomorphisms. We define

$$\text{Hom}_{R(\beta)}(M, N) := \bigoplus_{k \in \mathbb{Z}} \text{Hom}_{R(\beta)}(q^k M, N),$$

and set $\deg(f) := k$ for $f \in \text{Hom}_{R(\beta)}(q^k M, N)$. We sometimes write R for $R(\beta)$ in $\text{Hom}_{R(\beta)}(M, N)$ for simplicity.

For $M \in R(\beta)\text{-gmod}$, we set $M^* := \text{Hom}_{R(\beta)}(M, R(\beta))$ with the $R(\beta)$ -action given by

$$(r \cdot f)(u) := f(\psi(r)u), \quad \text{for } f \in M^*, r \in R(\beta) \text{ and } u \in M,$$

where ψ is the antiautomorphism of $R(\beta)$ which fixes the generators. We say that M is *self-dual* if $M \simeq M^*$. For an $R(\beta)$ -module M , we set $\text{wt}(M) := -\beta$.

We sometimes ignore grading shifts in the rest of this paper.

Let $e(\beta, \beta') := \sum_{\nu \in I^\beta, \nu' \in I^{\beta'}} e(\nu, \nu')$, where $e(\nu, \nu')$ is the idempotent corresponding to the concatenation $\nu * \nu'$ of ν and ν' . Then there is an injective ring homomorphism

$$R(\beta) \otimes R(\beta') \rightarrow e(\beta, \beta') R(\beta + \beta') e(\beta, \beta')$$

given by $e(\nu) \otimes e(\nu') \mapsto e(\nu, \nu')$, $x_k e(\beta) \otimes 1 \mapsto x_k e(\beta, \beta')$, $1 \otimes x_k e(\beta') \mapsto x_{k+\text{ht}(\beta)} e(\beta, \beta')$, $\tau_k e(\beta) \otimes 1 \mapsto \tau_k e(\beta, \beta')$ and $1 \otimes \tau_k e(\beta') \mapsto \tau_{k+\text{ht}(\beta)} e(\beta, \beta')$.

For $R(\beta)$ -modules M and N , we set

$$M \circ N := R(\beta + \beta') e(\beta, \beta') \otimes_{R(\beta) \otimes R(\beta')} (M \otimes N).$$

We denote by $M \nabla N$ the head of $M \circ N$ and by $M \Delta N$ the socle of $M \circ N$. We say that simple R -modules M and N *strongly commute* if $M \circ N$ is simple. A simple R -module L is *real* if L strongly commutes with itself. Note that if M and N strongly commute, then $M \circ N \simeq N \circ M$ up to a grading shift.

We denote by $K(R\text{-gmod})$ the Grothendieck ring of $R\text{-gmod}$.

Theorem 7.3 ([39, 40, 54]). *There exists an isomorphism of $\mathbb{A}$ -bialgebras*

$$(7.3) \quad \Omega: K(R\text{-gmod}) \xrightarrow{\sim} A_{\mathbb{A}}(\mathfrak{n}),$$

which preserves weights.

Proposition 7.4 ([29, Proposition 4.1]). *For $\varpi \in \mathbf{P}^+$ and $\mu, \zeta \in W\varpi$ with $\mu \preceq \zeta$, there exists a self-dual real simple R -module $\mathbf{M}(\mu, \zeta)$ such that*

$$\Omega([\mathbf{M}(\mu, \zeta)]) = D(\mu, \zeta).$$

We call $\mathbf{M}(\mu, \zeta)$ the *determinantal module* associated to $D(\mu, \zeta)$. Note that every determinantal module is real.

7.3. Affinization and R-matrices. In this subsection, we recall the notion of affinization and R-matrices for quiver Hecke algebras which were mainly investigated in [21, 24, 36].

For $\beta \in \mathbf{Q}^+$ with $\text{ht}(\beta) = m$ and $i \in I$, let

$$(7.4) \quad \mathfrak{p}_{i,\beta} = \sum_{\eta \in I^\beta} \left(\prod_{a \in [1,m]; \eta_a = i} x_a \right) e(\eta) \in R(\beta),$$

which belongs to the center $Z(R(\beta))$ of $R(\beta)$.

Definition 7.5. Let M be a simple $R(\beta)$ -module. An *affinization* of M with degree $t_{\widehat{\mathbf{M}}}$ is an $R(\beta)$ -module $\widehat{\mathbf{M}}$ with an endomorphism $z_{\widehat{\mathbf{M}}}$ of $\widehat{\mathbf{M}}$ with degree $t_{\widehat{\mathbf{M}}} \in \mathbb{Z}_{>0}$ and an isomorphism $\widehat{\mathbf{M}}/z_{\widehat{\mathbf{M}}}\widehat{\mathbf{M}} \simeq M$ such that

- (i) $\widehat{\mathbf{M}}$ is a finitely generated free module over the polynomial ring $\mathbf{k}[z_{\widehat{\mathbf{M}}}]$,
- (ii) $\mathfrak{p}_{i,\beta}\widehat{\mathbf{M}} \neq 0$ for all $i \in I$.

Note that every affinization is *essentially even*, i.e., $t_{\widehat{\mathbf{M}}} \in 2\mathbb{Z}_{>0}$ ([36, Proposition 2.5]). Thus, from now on, we assume that every affinization has an even degree.

Definition 7.6. We say that a simple R -module M is *affreal* if M is real and admits an affinization. If an affinization has degree $t \in 2\mathbb{Z}_{>0}$, we say that M is *t-affreal*.

Theorem 7.7 ([31, Theorem 3.26]). *For $\varpi \in \mathbf{P}^+$ and $\mu, \zeta \in W\varpi$ with $\mu \preceq \zeta$, the determinantal module $\mathbf{M}(\mu, \zeta)$ admits an affinization $\widehat{\mathbf{M}}(\mu, \zeta)$. When $\varpi = \varpi_i$, $\mathbf{M}(\mu, \zeta)$ admits an affinization of degree $(\alpha_i, \alpha_i) = 2\mathbf{d}_i$.*

Let $\beta \in \mathbf{Q}^+$ with $m = \text{ht}(\beta)$. For $k = 1, \dots, m-1$ and $\nu \in I^\beta$, the *intertwiner* φ_k is defined by

$$\varphi_k e(\nu) = \begin{cases} (\tau_k x_k - x_k \tau_k) e(\nu) = (x_{k+1} \tau_k - \tau_k x_{k+1}) e(\nu) & \text{if } \nu_k = \nu_{k+1}, \\ \tau_k e(\nu) & \text{otherwise.} \end{cases}$$

Lemma 7.8 ([21, Lemma 1.5]). *Let $\mathfrak{S}_m$ be the symmetric group of degree $m \in \mathbb{Z}_{>0}$.*

- (i) $\varphi_k^2 e(\nu) = (Q_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}) + \delta_{\nu_k, \nu_{k+1}}) e(\nu)$.
- (ii) $\{\varphi_k\}_{k=1, \dots, m-1}$ satisfies the braid relation.

- (iii) For a reduced expression $w = s_{i_1} \cdots s_{i_t} \in \mathfrak{S}_m$, we set $\varphi_w := \varphi_{i_1} \cdots \varphi_{i_t}$. Then φ_w does not depend on the choice of reduced expression of w .
- (iv) For $w \in \mathfrak{S}_m$ and $1 \leq k \leq m$, we have $\varphi_w x_k = x_{w(k)} \varphi_w$.
- (v) For $w \in \mathfrak{S}_m$ and $1 \leq k < m$, if $w(k+1) = w(k) + 1$, then $\varphi_w \tau_k = \tau_{w(k)} \varphi_w$.

For $m, n \in \mathbb{Z}_{\geq 0}$, we denote by $w[m, n]$ the element of $\mathfrak{S}_{m+n}$ defined by

$$w[m, n](k) := \begin{cases} k + n & \text{if } 1 \leq k \leq m, \\ k - m & \text{if } m < k \leq m + n. \end{cases}$$

Let $\beta, \gamma \in \mathbb{Q}^+$ and set $m := \text{ht}(\beta)$ and $n := \text{ht}(\gamma)$. For $M \in R(\beta)\text{-Mod}$ and $N \in R(\gamma)\text{-Mod}$, the $R(\beta) \otimes R(\gamma)$ -linear map $M \otimes N \rightarrow N \circ M$ defined by $u \otimes v \mapsto \varphi_{w[n, m]}(v \otimes u)$ can be extended to an $R(\beta + \gamma)$ -module homomorphism (up to a grading shift)

$$R_{M, N}: M \circ N \longrightarrow N \circ M.$$

Let $\widehat{M}$ be an affinization of a simple R -module M , and let N be a non-zero R -module. We define a homomorphism (up to a grading shift)

$$R_{\widehat{M}, N}^{\text{norm}} := z_{\widehat{M}}^{-s} R_{\widehat{M}, N}: \widehat{M} \circ N \longrightarrow N \circ \widehat{M},$$

where s is the largest integer such that $R_{\widehat{M}, N}(\widehat{M} \circ N) \subset z_{\widehat{M}}^s(N \circ \widehat{M})$. We define

$$\mathbf{r}_{M, N}^l: M \circ N \longrightarrow N \circ M$$

to be the homomorphism (up to a grading shift) induced from $R_{\widehat{M}, N}^{\text{norm}}$ by specializing at $z_{\widehat{M}} = 0$. By the definition, $\mathbf{r}_{M, N}^l$ never vanishes.

Similarly, for a simple module N admitting an affinization $\widehat{N}$, we can define

$$\mathbf{r}_{M, N}^r: M \circ N \longrightarrow N \circ M.$$

Proposition 7.9 ([22], [31, Proposition 3.13]). *Let M and N be simple modules. We assume that one of them is affreal. Then we have*

$$\text{Hom}_R(M \circ N, N \circ M) = \mathbf{k} \mathbf{r},$$

where $\mathbf{r} = \mathbf{r}_{M, N}^l$ or $\mathbf{r}_{M, N}^r$.

By the above proposition, we can write the homomorphism as $\mathbf{r}_{M, N}$, called the $\mathbf{R}$ -matrix, without superscript. Now we define

$$\begin{aligned} \Lambda(M, N) &:= \deg(\mathbf{r}_{M, N}), \\ \widetilde{\Lambda}(M, N) &:= \frac{1}{2}(\Lambda(M, N) + (\text{wt}(M), \text{wt}(N))), \\ \mathfrak{d}(M, N) &:= \frac{1}{2}(\Lambda(M, N) + \Lambda(N, M)). \end{aligned}$$

for simple modules M, N such that one of them admits an affinization.

Proposition 7.10 ([22, Theorem 3.2], [36, Proposition 2.10], [31, Proposition 3.13, Lemma 3.17]). *Let M and N be simple R -modules such that one of M and N is affreal. Then we have*

- (i) $M \circ N$ has simple socles and simple heads,
- (ii) $\text{Im}(\mathbf{r}_{M,N})$ is isomorphic to $M \nabla N$ and $N \Delta M$,
- (iii) $M \nabla N$ and $M \Delta N$ appear once in the composition series of $M \circ N$, respectively.

Now, let us collect properties of $\Lambda(M, N)$, $\tilde{\Lambda}(M, N)$ and $\mathfrak{d}(M, N)$ for simple R -modules M and N such that one of M and N is affreal. They are proved in [22, 24, 29, 31] or can be proved by using the arguments in those papers:

Lemma 7.11 ([31, Lemma 3.11]). *Let M and N be simple R -modules. Assume that one of them admit affinization of degree t . Then we have*

$$\tilde{\Lambda}(M, N), \mathfrak{d}(M, N) \in \frac{t}{2} \mathbb{Z}_{\geq 0}.$$

By this lemma, for t_1 -affreal M and t_2 -affreal N , we have

$$\tilde{\Lambda}(M, N), \mathfrak{d}(M, N) \in \frac{\text{lcm}(t_1, t_2)}{2} \mathbb{Z}_{\geq 0}.$$

Here $\text{lcm}(a, b)$ denotes the least common multiple of $a, b \in \mathbb{Z} \setminus \{0\}$.

Proposition 7.12 ([24, Proposition 3.2.17]). *Let M_i be t_i -affreal ($i = 1, 2$). Assume that*

$$\mathfrak{d}(M_1, M_2) = \frac{\max(t_1, t_2)}{2}.$$

Then we have an exact sequence

$$0 \rightarrow M_1 \Delta M_2 \rightarrow M_1 \circ M_2 \rightarrow M_1 \nabla M_2 \rightarrow 0.$$

Lemma 7.13 ([24, Lemma 3.1.4]). *Let M and N be self-dual simple modules. If one of them is affreal, then*

$$q^{\tilde{\Lambda}(M, N)} M \nabla N \text{ is a self-dual simple module.}$$

Proposition 7.14 ([22, Corollary 3.11]). *Let M_k be R -modules ($k = 1, 2, 3$) and assume that M_2 is simple. If $\varphi_1: L \rightarrow M_1 \circ M_2$ and $\varphi_2: M_2 \circ M_3 \rightarrow L'$ are non-zero homomorphisms, then the composition*

$$L \circ M_3 \xrightarrow{\varphi_1 \circ M_3} M_1 \circ M_2 \circ M_3 \xrightarrow{M_1 \circ \varphi_2} M_1 \circ L'$$

does not vanish. Similarly, if $\psi_1: L \rightarrow M_2 \circ M_3$ and $\psi_2: M_1 \circ M_2 \rightarrow L'$ are non-zero homomorphisms, then the composition

$$M_1 \circ L \xrightarrow{M_1 \circ \varphi_1} M_1 \circ M_2 \circ M_3 \xrightarrow{\varphi_2 \circ M_3} L' \circ M_3.$$

does not vanish.

The following proposition was incorrectly stated in [50, Proposition 5.11], and we present here its correct statement and its proof.

Proposition 7.15 ([50, Proposition 5.11]). *Let M_i and N_i be simple R -modules such that one of them is affreal and $M_i \circ N_i$ has composition length 2 for $i = 1, 2$. Let M and N be simple R -modules such that one of them is affreal. Let $\varphi_i: M_i \circ N_i \rightarrow M \circ N$ ($i = 1, 2$) be a non-zero homomorphisms. We assume that*

$$M_1 \Delta N_1 \simeq M_2 \Delta N_2 \quad \text{and} \quad M_1 \nabla N_1 \not\simeq M_2 \nabla N_2.$$

Then we have

$$M_1 \Delta N_1 \simeq M_2 \Delta N_2 \simeq M \Delta N \quad \text{and} \quad \varphi_i: M_i \circ N_i \rightarrow M \circ N.$$

Proof. If φ_i is not injective, then $\text{Im}(\varphi_i) \simeq M_i \nabla N_i \simeq M \Delta N$. If φ_i is injective, then $M_i \Delta N_i \simeq M \Delta N$.

- (i) If φ_1 and φ_2 are not injective, then $M_1 \nabla N_1 \simeq M_2 \nabla N_2$, which is a contradiction.
- (ii) Assume that one of φ_1 and φ_2 (say φ_1) is injective and the other is not injective. Then we have $M \Delta N \simeq M_1 \Delta N_1 \simeq M_2 \nabla N_2$. Hence $M_2 \nabla N_2 \simeq M_1 \Delta N_1 \simeq M_2 \Delta N_2$, which contradicts Proposition 7.10 (iii).
- (iii) Hence φ_1 and φ_2 are injective, and $M \Delta N \simeq M_1 \Delta N_1 \simeq M_2 \Delta N_2$. □

Definition 7.16 ([28, Definition 2.5]). Let $L_1, L_2, \dots, L_r$ be affreal modules. The sequence $(L_1, \dots, L_r)$ is called a *normal sequence* if the composition of the R -matrices

$$\begin{aligned} \mathbf{r}_{L_1, \dots, L_r} &:= \prod_{1 \leq i < j \leq r} \mathbf{r}_{L_i, L_j} \\ &= (\mathbf{r}_{L_{r-1}, L_r}) \circ \dots \circ (\mathbf{r}_{L_2, L_r} \circ \dots \circ \mathbf{r}_{L_2, L_3}) \circ (\mathbf{r}_{L_1, L_r} \circ \dots \circ \mathbf{r}_{L_1, L_2}) \\ &: q^{\sum_{1 \leq i < k \leq r} \Lambda(L_i, L_k)} L_1 \circ L_2 \circ \dots \circ L_r \longrightarrow L_r \circ \dots \circ L_2 \circ L_1. \end{aligned}$$

does not vanishes.

The following lemmas are proved in [28, 32] when $\mathfrak{g}$ is of symmetric type. However, by using the same arguments, we can prove when $\mathfrak{g}$ is of symmetrizable type.

Lemma 7.17 ([28, Lemma 2.6]). *Let $(L_1, \dots, L_r)$ be a normal sequence of affreal R -modules. Then $\text{Im}(\mathbf{r}_{L_1, \dots, L_r})$ is simple and it coincides with the head of $L_1 \circ \dots \circ L_r$ and also with the socle of $L_r \circ \dots \circ L_1$.*

Lemma 7.18 ([28, Lemma 2.8, Lemma 2.10]).

- (i) *A sequence $(L_1, \dots, L_r)$ of affreal R -modules is a normal sequence if and only if $(L_1, \dots, L_{r-1})$ is a normal sequence and*

$$\Lambda(\text{hd}(L_1, \dots, L_{r-1}), L_r) = \sum_{1 \leq j \leq r-1} \Lambda(L_j, L_r).$$

- (ii) *A sequence $(L_1, \dots, L_r)$ of affreal R -modules is a normal sequence if and only if $(L_2, \dots, L_r)$ is a normal sequence and*

$$\Lambda(L_1, \text{hd}(L_2, \dots, L_r)) = \sum_{2 \leq j \leq r} \Lambda(L_1, L_j).$$

Lemma 7.19 ([32, Lemma 2.23]). *Let L, M, N be affreal R -modules. Then $\mathfrak{d}(L, M \nabla N) = \mathfrak{d}(L, M) + \mathfrak{d}(L, N)$, if and only if (L, M, N) and (M, N, L) are normal sequences.*

8. CUSPIDAL MODULES AND t -QUANTIZED CARTAN MATRIX

Let $(\mathbf{C}, \mathbf{P}, \Pi, \mathbf{P}^\vee, \Pi^\vee)$ be an arbitrary finite Cartan datum. In this section, we first recall the *cuspidal modules* over quiver Hecke algebras categorifying the dual PBW vectors. Then we will prove that the $\mathfrak{d}$ -invariants among cuspidal modules associated to Dynkin quivers are encoded in $(\widehat{\mathfrak{d}}_{i,j}(t))_{i,j \in I}$, and $(\mathfrak{d}_{i,j}(t))_{i,j \in I}$ informs us of the information on the composition length of convolution product of pair of the cuspidal modules. To prove them, we will use the combinatorial properties and statistics of AR-quivers in a crucial way.

8.1. Cuspidal modules. Let R be the quiver Hecke algebra of type of $\mathbf{C}$. Then, in [3, 44, 37, 41], the following theorem is proved:

Theorem 8.1. *For any reduced expression $\underline{w}_0 = s_{i_1} \cdots s_{i_\ell}$ of $w_0 \in \mathbf{W}$, there exists a family of self-dual affreal simple modules*

$$\{S_{\underline{w}_0}(\beta) \mid \beta \in \Phi^+\}$$

satisfying the following properties:

(1) *For each $\beta \in \Phi^+$, we have*

$$\Omega([S_{\underline{w}_0}(\beta)]) = r_{[\underline{w}_0]}(\beta) \quad \text{and} \quad \text{wt}(S_{\underline{w}_0}(\beta)) = -\beta.$$

(2) *For any exponent $\underline{m} = \{m_\beta\}_{\beta \in \Phi^+} \in \mathbb{Z}_{\geq 0}^{\Phi^+}$, there exists a unique non-zero R -module homomorphism (up to grading shift)*

$$\begin{aligned} \vec{\mathcal{S}}_{\underline{w}_0}(\underline{m}) &:= S_{\underline{w}_0}(\beta_\ell)^{\circ m_{\beta_\ell}} \circ \cdots \circ S_{\underline{w}_0}(\beta_1)^{\circ m_{\beta_1}} \\ &\xrightarrow{\mathbf{r}_{\underline{m}} \leftarrow} \mathcal{S}_{\underline{w}_0}(\underline{m}) := S_{\underline{w}_0}(\beta_1)^{\circ m_{\beta_1}} \circ \cdots \circ S_{\underline{w}_0}(\beta_\ell)^{\circ m_{\beta_\ell}} \end{aligned}$$

where $\beta_k = \beta_k^{w_0}$. Moreover, $\text{Im}(\mathbf{r}_{\underline{m}})$ is simple and $\text{Im}(\mathbf{r}_{\underline{m}}) \simeq \text{soc}(\mathcal{S}_{\underline{w}_0}(\underline{m})) \simeq \text{hd}(\vec{\mathcal{S}}_{\underline{w}_0}(\underline{m}))$ up to a grading shift.

(3) *For any simple R -module M , there exists a unique exponent $\underline{m} \in \mathbb{Z}_{\geq 0}^{\Phi^+}$ such that*

$$M \simeq \text{hd}(\vec{\mathcal{S}}_{\underline{w}_0}(\underline{m})) \quad \text{up to a grading shift.}$$

Note that, for each $\underline{m} = \{m_1, \dots, m_\ell\}_{\beta \in \Phi^+} \in \mathbb{Z}_{\geq 0}^\ell$, $\text{hd}(q^{s_{\underline{m}}} \vec{\mathcal{S}}_{\underline{w}_0}(\underline{m}))$ is a self-dual simple R -module, where

$$s_{\underline{m}} := \sum_{k=1}^{\ell} d_{\beta_k^{w_0}} \frac{m_k(m_k - 1)}{2}.$$

We denote

$$S_{\underline{w}_0}(\underline{m}) := \text{hd}(q^{s_{\underline{m}}} \vec{\mathcal{S}}_{\underline{w}_0}(\underline{m})).$$

We call $S_{\underline{w}_0}(\beta)$ the *cuspidal module* associated to $\underline{w}_0$. It is known (see [50, Theorem 5.8]) that the set $\{S_{\underline{w}_0}(\beta) \mid \beta \in \Phi^+\}$ does depend only on the commutation class $[\underline{w}_0]$; i.e.,

$$S_{\underline{w}_0}(\beta) \simeq S_{\underline{w}'_0}(\beta) \quad \text{if } \underline{w}_0 \text{ and } \underline{w}'_0 \text{ are commutation equivalent.}$$

Thus, for any commutation class $\mathbf{c}$, $S_{\mathbf{c}}(\beta)$ for $\beta \in \Phi^+$ is well-defined.

Lemma 8.2 ([50, Proposition 5.7]). *Let $\underline{w}_0$ be a reduced expression of w_0 . If α and β are incomparable with respect to $\preceq_{[\underline{w}_0]}$, then $S_{[\underline{w}_0]}(\alpha)$ and $S_{[\underline{w}_0]}(\beta)$ commute.*

By this lemma, $S_{\underline{w}_0}(\underline{m})$, as well as $\overleftarrow{\mathcal{S}}_{\underline{w}_0}(\underline{m})$ and $\overrightarrow{\mathcal{S}}_{\underline{w}_0}(\underline{m})$, depends only on the commutation class $[\underline{w}_0]$. Hence, $S_{[\underline{w}_0]}(\underline{m})$, $\overleftarrow{\mathcal{S}}_{[\underline{w}_0]}(\underline{m})$ and $\overrightarrow{\mathcal{S}}_{[\underline{w}_0]}(\underline{m})$ are well-defined.

Proposition 8.3 ([3, 44, 50]). *Let $\mathbf{c}$ be a commutation class.*

(i) *For any $\mathbf{c}$ -minimal $\mathbf{c}$ -pair $\langle\langle \alpha, \beta \rangle\rangle$ for $\gamma \in \Phi^+$, there exist exact sequences of R -modules*

$$\begin{aligned} 0 \rightarrow S_{\mathbf{c}}(\gamma) \rightarrow S_{\mathbf{c}}(\beta) \circ S_{\mathbf{c}}(\alpha) \rightarrow S_{\mathbf{c}}(\beta) \nabla S_{\mathbf{c}}(\alpha) \rightarrow 0, \\ 0 \rightarrow S_{\mathbf{c}}(\beta) \nabla S_{\mathbf{c}}(\alpha) \rightarrow S_{\mathbf{c}}(\alpha) \circ S_{\mathbf{c}}(\beta) \rightarrow S_{\mathbf{c}}(\gamma) \rightarrow 0 \end{aligned}$$

and

$$[S_{\mathbf{c}}(\alpha)] \cdot [S_{\mathbf{c}}(\beta)] = [S_{\mathbf{c}}(\gamma)] + [S_{\mathbf{c}}(\alpha) \Delta S_{\mathbf{c}}(\beta)] \quad \text{in } K(R\text{-gmod})$$

up to grading shifts.

(ii) *For any exponent $\underline{m}$, we have*

$$(8.1) \quad [\overrightarrow{\mathcal{S}}_{\mathbf{c}}(\underline{m})] \in [S_{\mathbf{c}}(\underline{m})] + \sum_{\underline{m}' \prec_{\mathbf{c}}^b \underline{m}} \mathbb{Z}_{\geq 0} [S_{\mathbf{c}}(\underline{m}')].$$

up to grading shifts.

(iii) *If $\underline{m}$ is a $\mathbf{c}$ -simple sequence, then $\overrightarrow{\mathcal{S}}_{\mathbf{c}}(\underline{m})$ and $\overleftarrow{\mathcal{S}}_{\mathbf{c}}(\underline{m})$ are isomorphic up to a grading shift and they are simple.*

Proposition 8.4 ([3, Theorem 4.7]). *For any commutation class $\mathbf{c}$ and a $\mathbf{c}$ -pair $\langle\langle \alpha, \beta \rangle\rangle$, we have*

$$(8.2) \quad \widetilde{\Lambda}(S_{\mathbf{c}}(\beta), S_{\mathbf{c}}(\alpha)) = 0 \quad \text{and hence} \quad \Lambda(S_{\mathbf{c}}(\beta), S_{\mathbf{c}}(\alpha)) = -(\alpha, \beta).$$

In particular, if the $\mathbf{c}$ -pair $\langle\langle \alpha, \beta \rangle\rangle$ is $\mathbf{c}$ -minimal for a positive root $\gamma \in \Phi^+$, we have short exact sequences (including grading shifts)

$$(8.3) \quad \begin{aligned} 0 \rightarrow q^{p_{\beta, \alpha} - (\beta, \alpha)} S_{\mathbf{c}}(\gamma) \rightarrow S_{\mathbf{c}}(\beta) \circ S_{\mathbf{c}}(\alpha) \rightarrow S_{\mathbf{c}}(\langle\langle \alpha, \beta \rangle\rangle) \rightarrow 0, \\ 0 \rightarrow q^{-(\beta, \alpha)} S_{\mathbf{c}}(\langle\langle \alpha, \beta \rangle\rangle) \rightarrow S_{\mathbf{c}}(\alpha) \circ S_{\mathbf{c}}(\beta) \rightarrow q^{-p_{\beta, \alpha}} S_{\mathbf{c}} \rightarrow 0, \end{aligned}$$

which tells

$$\begin{aligned} \widetilde{\Lambda}(S_{\mathbf{c}}(\alpha), S_{\mathbf{c}}(\beta)) &= p_{\beta, \alpha}, \quad \Lambda(S_{\mathbf{c}}(\alpha), S_{\mathbf{c}}(\beta)) = 2p_{\beta, \alpha} - (\alpha, \beta), \quad \text{and} \\ \mathfrak{d}(S_{\mathbf{c}}(\alpha), S_{\mathbf{c}}(\beta)) &= p_{\beta, \alpha} - (\alpha, \beta). \end{aligned}$$

Proposition 8.5 ([50, Corollary 5.12]). *Let $\underline{m}^{(k)}$ ($k = 1, 2, 3$) be distinct sequences such that $\text{wt}_{\mathbf{c}}(\underline{m}_{k_1}^{(a)}) = \text{wt}_{\mathbf{c}}(\underline{m}^{(b)})$ ($a, b \in \{1, 2, 3\}$), $\deg_{\mathbf{c}}(\underline{m}^{(j)}) = 1$ and $\deg_{\mathbf{c}}(\underline{m}^{(3)}) > 1$ and $\underline{m}^{(j)} \prec_{\mathbf{c}}^{\mathbf{b}} \underline{m}^{(3)}$ ($j = 1, 2$). If there exist non-zero $R(\gamma)$ -homomorphisms (up to grading shift)*

$$\vec{\mathcal{S}}_{\mathbf{c}}(\underline{m}^{(j)}) \longrightarrow \vec{\mathcal{S}}_{\mathbf{c}}(\underline{m}^{(3)}) \quad \text{for } j = 1, 2,$$

and $\vec{\mathcal{S}}_{\mathbf{c}}(\underline{m}^{(k)})$ ($k = 1, 2, 3$) have the common simple head, then we have

$$\vec{\mathcal{S}}_{\mathbf{c}}(\underline{m}^{(j)}) \hookrightarrow \vec{\mathcal{S}}_{\mathbf{c}}(\underline{m}^{(3)}) \quad \text{for } j = 1, 2.$$

For a Q -datum $Q = (\Delta, \xi)$ and $(i, p) \in (\Gamma_Q)_0$, we set

$$S_Q(i; p) := S_{[Q]}(\beta) \quad \text{where } \phi_Q(i, p) = (\beta, 0).$$

The following is a main theorem of this paper:

Theorem 8.6. *Let $Q = (\Delta, \xi)$ be a Dynkin quiver of an arbitrary type. Then we have*

$$(8.4) \quad \mathfrak{d}(S_Q(i; p), S_Q(j; s)) = \widetilde{\mathfrak{d}}_{i,j}^{\Delta} [|p - s|] \quad \text{for any } (i, p), (j, s) \in (\Gamma_Q)_0.$$

Moreover, for a $[Q]$ -pair $\langle\langle \alpha, \beta \rangle\rangle$ with $\phi_Q(i, p) = (\alpha, 0)$ and $\phi_Q(j, s) = (\beta, 0)$, we have the following statements:

- (a) $S_Q(\alpha) \nabla S_Q(\beta) \simeq S_Q(\text{hd}_Q(\langle\langle \alpha, \beta \rangle\rangle))$ up to a grading shift.
- (b) If $\widetilde{\mathfrak{d}}_{i,j}^{\Delta} [|p - s|] = \max(\mathbf{d}_i, \mathbf{d}_j)$, we have $\ell(S_Q(\alpha) \circ S_Q(\beta)) = 2$.
- (c) If $\widetilde{\mathfrak{d}}_{i,j}^{\Delta} [|p - s|] > \max(\mathbf{d}_i, \mathbf{d}_j)$, we have $\ell(S_Q(\alpha) \circ S_Q(\beta)) > 2$.
- (d) *labelit: length1* If $\widetilde{\mathfrak{d}}_{i,j}^{\Delta} [|p - s|] = 0$, then $S_Q(\alpha) \circ S_Q(\beta)$ is simple.

Here, $\ell(M)$ denotes the composition length of M for $M \in R\text{-gmod}$.

Remark 8.7. In Theorem 8.6 (a), $S_Q(\text{hd}_Q(\langle\langle \alpha, \beta \rangle\rangle))$ is isomorphic to a convolution product of mutually commuting cuspidal modules, since $\text{hd}_Q(\langle\langle \alpha, \beta \rangle\rangle)$ is $[Q]$ -simple. For $\mathfrak{g}$ of simply-laced type, we refer the reader to [50, Theorem 5.18].

The simply-laced case of this theorem is obtained in [32, Theorem 4.12], [11, Theorem 1.2], [50]. In the next subsection, we will give a proof of this theorem in BCFG case.

Corollary 8.8. *Recall (4.13). For $\mathfrak{g}$ of type B_n , C_n or D_{n+1} and $(i, p), (j, s) \in (\Gamma_Q)_0$, the R -module $S_Q(i; p) \circ S_Q(j; s)$ is of composition length > 2 if and only if*

$$2 \leq i, j \leq n-1, \quad i+j \geq n+1, \quad \mathbf{h}+2-i-j \leq |p-s| \leq i+j \quad \text{and} \quad |p-s| \equiv_2 i+j.$$

The corollary below for ADE-case was also proved in [12, Lemma 9.12] (see also [32]).

Corollary 8.9. *For $\phi_Q(\alpha_a, 0) = (i, p)$ and $\phi_Q(\alpha_b, 0) = (j, s)$ with $a \neq b$, we have*

$$(8.5) \quad \widetilde{\mathfrak{d}}_{i^*,j}[\mathbf{h} - |p-s|] = \widetilde{\mathfrak{d}}_{i,j^*}[\mathbf{h} - |p-s|] = 0.$$

Proof. Note that

$$(8.6) \quad \begin{aligned} & \text{(A) } \widetilde{\mathfrak{d}}_{i,n}^{D_{n+1}}(t) = \widetilde{\mathfrak{d}}_{i,n+1}^{D_{n+1}}(t) \text{ for } i < n \text{ and} \\ & \text{(B) } \widetilde{\mathfrak{d}}_{i,j}^{C_n}(t) = \kappa_{i,j} \widetilde{\mathfrak{d}}_{i,j}^{B_n}(t), \text{ where } \kappa_{i,j} = \begin{cases} 1/2 & \text{if } i, j < n, \\ 1 & \text{if } \min(i, j) < \max(i, j) = n, \\ 2 & \text{if } i = j = n. \end{cases} \end{aligned}$$

(1) Let us assume that $\max(a, b) < n$ and $\triangle$ is of type B_n or C_n . In this case, we have $\max(i, j) < n$ since $\mathbf{d}_{\alpha_a} = \mathbf{d}_{\alpha_n} \neq \mathbf{d}_{\alpha_n}$. By regarding $\alpha_a \in \Phi_{B_n}^+$, we have $\alpha_a = \psi^{-1}(\alpha_a) \in \Phi_{D_{n+1}}^+$ and $\phi_Q^{-1}(\alpha_a, 0) = (i, p)$ for $i = \pi^{-1}(i)$ by Proposition 5.4. Similar observations hold for $\alpha_b = \psi^{-1}(\alpha_b)$. Since (8.5) holds for D_{n+1} -case, we have

$$\frac{1}{2} \widetilde{\mathfrak{d}}_{i,j}^{B_n}[\mathbf{h} - |p - s|] = \widetilde{\mathfrak{d}}_{i,j}^{C_n}[\mathbf{h} - |p - s|] = \widetilde{\mathfrak{d}}_{i,j}^{D_{n+1}}[\mathbf{h} - |p - s|] = 0.$$

Here the second equation follows from Theorem 3.7. Thus the assertion for this case follows.

(2) Let us assume that $\max(a, b) = n$ and $\triangle$ is of type B_n or C_n . Without loss of generality, we set $\alpha_a = \alpha_n$. Note that n is a sink or source in Q and $i = n$. By regarding $\alpha_n \in \Phi_{B_n}^+$, we have $\{\alpha_n, \alpha_{n+1}\} = \psi^{-1}(\alpha_n) \subset \Phi_{D_{n+1}}^+$ and $\{\phi_Q^{-1}(\alpha_n, 0), \phi_Q^{-1}(\alpha_{n+1}, 0)\} = \{(n, p), (n+1, p)\}$. Then, as in (1)-case, we have

$$\widetilde{\mathfrak{d}}_{n,j}^{B_n}[\mathbf{h} - |p - s|] = \widetilde{\mathfrak{d}}_{n,j}^{C_n}[\mathbf{h} - |p - s|] = \widetilde{\mathfrak{d}}_{n,j}^{D_{n+1}}[\mathbf{h} - |p - s|] + \widetilde{\mathfrak{d}}_{n+1,j}^{D_{n+1}}[\mathbf{h} - |p - s|] = 0.$$

(3) The cases for F_4 and G_2 can be directly checked by considering all Dynkin quivers of those types. $\square$

8.2. BCFG cases. Throughout this subsection, we consider the convolution products of $[Q]$ -cuspidal modules, when Q is a Dynkin quiver of type BCFG.

We set $\phi_Q(i, p) = (\alpha, 0)$ and $\phi_Q(j, s) = (\beta, 0)$.

Proposition 8.10. *Let Q be any Dynkin quiver of types B_n and C_n . For a Dynkin quiver Q , let $\langle\langle \alpha, \beta \rangle\rangle$ be a $[Q]$ -minimal $[Q]$ -pair for $\alpha + \beta = \gamma \in \Phi^+$. Then we have*

$$\mathfrak{d}(S_Q(\alpha), S_Q(\beta)) = \max(\mathbf{d}_i, \mathbf{d}_j) = \widetilde{\mathfrak{d}}_{i,j}^\triangle[|p - s|].$$

Proof. The second equality follows from Theorem 6.6.

(1) Consider the case when $\mathbf{d}_\gamma = 2$, $\mathbf{d}_\alpha = \mathbf{d}_\beta = 1$ and $\mathfrak{g}$ is of type B_n . Then we have $\text{res}^{[Q]}(\alpha) = \text{res}^{[Q]}(\beta) = n$ and $\text{res}^{[Q]}(\gamma) < n$. Hence $\gamma = \varepsilon_a + \varepsilon_b$ and $\{\alpha, \beta\} = \{\varepsilon_a, \varepsilon_b\}$ for $1 \leq a \neq b \leq n$. Thus we have

$$\mathfrak{d}(S_Q(\alpha), S_Q(\beta)) = 1 - (\varepsilon_a, \varepsilon_b) = 1 = \max(\mathbf{d}_\alpha, \mathbf{d}_\beta).$$

(2) For the remained cases of type B_n or C_n , we need to show that

$$\mathfrak{d}(S_Q(\alpha), S_Q(\beta)) = -(\alpha, \beta) = \max(\mathbf{d}_\alpha, \mathbf{d}_\beta).$$

Then we have

- (a) $\{\alpha, \beta\} = \{\varepsilon_a - \varepsilon_b, \varepsilon_b \pm \varepsilon_c\}$ or $\{\varepsilon_a - \varepsilon_b, \varepsilon_b\}$ if $\mathfrak{g}$ is of type B_n ,
- (b) $\{\alpha, \beta\} = \{\varepsilon_a - \varepsilon_b, \varepsilon_b \pm \varepsilon_c\}$, $\{\varepsilon_a - \varepsilon_b, 2\varepsilon_b\}$ or $\{\varepsilon_a + \varepsilon_b, \varepsilon_a - \varepsilon_b\}$ if $\mathfrak{g}$ is of type C_n ,

$$-(\alpha, \beta) = \max(\mathbf{d}_i, \mathbf{d}_j),$$

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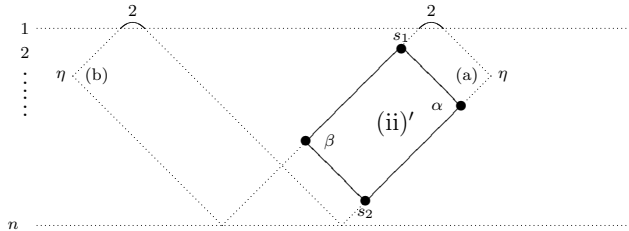
8.2.1. B_n case. Let Q be *any* Dynkin quiver of type B_n . Under the assumption that

it suffices to consider the cases (ii)', (iv)', (v)', (vi)', (vii)' in (6.2), and (x)', (xi)' in (6.3). Recall the values (α, β) and $p_{\beta, \alpha}$ are recorded in Table 3 and (6.4), and do not depend on the choice of Q .

$$\phi_Q(i, p) = (\alpha, 0) \quad \text{and} \quad \phi_Q(j, s) = (\beta, 0).$$

Consider the case (ii)'. By Proposition 4.8 and the fact that ψ is additive, there exists $\eta \in \Phi^+$ such that

- (a) the $[Q]$ -pair $\langle\eta, s_1\rangle$ is $[Q]$ -minimal for α and of the form (i)', or
(b) the $[Q]$ -pair $\langle\alpha, \eta\rangle$ is a $[Q]$ -minimal for s_1 and of the form (iii)'.


$$q^{p_{s_1, \eta} - (s_1, \eta)} S_Q(\alpha) \circ S_Q(\beta) \hookrightarrow S_Q(s_1) \circ S_Q(\eta) \circ S_Q(\beta).$$
$$S_Q(s_1) \circ S_Q(\eta) \circ S_Q(\beta) \twoheadrightarrow q^{-p_{\beta, \eta}} S_Q(s_1) \circ S_Q(s_2)$$
$$(8.7) \quad q^{p_{s_1, \eta} - (s_1, \eta)} S_Q(\alpha) \circ S_Q(\beta) \hookrightarrow S_Q(s_1) \circ S_Q(\eta) \circ S_Q(\beta) \twoheadrightarrow q^{-p_{\beta, \eta}} S_Q(s_1) \circ S_Q(s_2)$$
$$S_Q(\alpha) \circ S_Q(\beta) \twoheadrightarrow q^{-p_{\beta, \eta} - p_{s_1, \eta} + (s_1, \eta)} S_Q(s_1) \circ S_Q(s_2),$$
$$\tilde{\Lambda}(S_Q(\alpha), S_Q(\beta)) = p_{\beta, \eta} + p_{s_1, \eta} - (s_1, \eta),$$

by Lemma 7.13. Thus (8.2) tells that

$$\mathfrak{d}(S_Q(\alpha), S_Q(\beta)) = p_{\beta, \eta} + p_{s_1, \eta} - (s_1, \eta) - (\alpha, \beta).$$

Since $p_{\beta, \eta} = p_{s_1, \eta} = 0$, $(s_1, \eta) = -2$ and $(\alpha, \beta) = 0$, we have

$$\mathfrak{d}(S_Q(\alpha), S_Q(\beta)) = 2 = \max(\mathbf{d}_\alpha, \mathbf{d}_\beta) = \mathfrak{d}_{i,j}[p - s].$$

Now let us assume (b). Similarly, we have a homogeneous R -module homomorphism,

$$q^{p_{\eta, \alpha} - (\eta, \alpha)} S_Q(s_2) \circ S_Q(s_1) \hookrightarrow S_Q(s_2) \circ S_Q(\eta) \circ S_Q(\alpha)$$

Note that the $[Q]$ -pair $\langle\langle s_2, \eta \rangle\rangle$ is $[Q]$ -minimal for β and of the form (iii)'. Thus we have a non-zero composition of R -module homomorphism,

$$(8.8) \quad q^{p_{\eta, \alpha} - (\eta, \alpha)} S_Q(s_2) \circ S_Q(s_1) \hookrightarrow S_Q(s_2) \circ S_Q(\eta) \circ S_Q(\alpha) \twoheadrightarrow q^{-p_{\eta, s_2}} S_Q(\beta) \circ S_Q(\alpha),$$

which tells $S_Q(\beta) \Delta S_Q(\alpha) \simeq S_Q(s_1) \circ S_Q(s_2)$ up to a grading shift and

$$\mathfrak{d}(S_Q(\alpha), S_Q(\beta)) = p_{\eta, \alpha} + p_{\eta, s_2} - (s_2, s_1) - (\eta, \alpha) = 2 = \max(\mathbf{d}_\alpha, \mathbf{d}_\beta) = \mathfrak{d}_{i,j}[p - s].$$

From (8.7) and (8.8), we have a non-zero R -module monomorphism (up to grading shifts)

$$S_Q(s_2) \circ S_Q(s_1) \hookrightarrow S_Q(\beta) \circ S_Q(\alpha).$$

We can apply the similar argument to the length 1 chains of $\prec_Q^b$ -sequence, which are of the form $\underline{m} \prec_Q^b \underline{m}'$ and depicted in the cases (iv)', (vii)', (x)' and (xi)', for obtaining non-zero R -module homomorphisms (up to grading shifts)

$$(8.9) \quad \overleftarrow{\mathcal{S}}_{[w_0]}(\underline{m}) \rightarrow \overleftarrow{\mathcal{S}}_{[w_0]}(\underline{m}').$$

More precisely, those length 1-chains are listed below:

$$\left\{ \begin{array}{ll} \langle\langle s_1, s_2 \rangle\rangle \prec_Q^b \langle\langle \alpha, \beta \rangle\rangle & \text{(iv)' and (vii)' cases,} \\ \langle\langle \eta_1, \eta_2 \rangle\rangle, \langle\langle \tau_1, \zeta_1 \rangle\rangle \prec_Q^b \langle\langle \alpha, \beta \rangle\rangle & \text{(x)' case,} \\ \langle\langle \eta_1, \eta_2 \rangle\rangle \prec_Q^b \langle\langle \alpha, \beta \rangle\rangle, \{s_1, \tau_1, \zeta_1\}_{\beta \in \Phi^+} \prec_Q^b \langle\langle \alpha, \beta \rangle\rangle, & \text{(xi)' case,} \\ \langle\langle s_1, s_2 \rangle\rangle \prec_Q^b \langle\langle \eta_1, \eta_2 \rangle\rangle, \langle\langle s_1, s_2 \rangle\rangle \prec_Q^b \{s_1, \tau_1, \zeta_1\}_{\beta \in \Phi^+}, & \text{(xi)' case continued.} \end{array} \right.$$

In cases (iv)' and (vii)', one can prove

$$S_Q(\alpha) \nabla S_Q(\beta) \simeq S_Q(s_1) \circ S_Q(s_2) \quad (\text{up to grading shifts})$$

and

$$\mathfrak{d}(S_Q(\alpha), S_Q(\beta)) = 2$$

as in (ii)'-case. Moreover, by (8.1) together with Proposition 7.10, we have

$$[S_Q(\alpha) \circ S_Q(\beta)] = [S_Q(\alpha) \Delta S_Q(\beta)] + [S_Q(s_2) \circ S_Q(s_1)],$$

for the cases (ii)', (iv)' and (vii)'. Thus the assertions in Theorem 8.6 hold for those cases.

Example 8.11. Using Γ_Q in Example 5.11 and the $[Q]$ -pairs corresponding to (iv)' and (vii)' in Example 6.8, we can compute $\mathfrak{d}(S_Q(\alpha), S_Q(\beta))$ as follows:

- (1) Since $p_{\langle 1, -2 \rangle, \langle 2, -4 \rangle} = 0$, $p_{\langle 1, -2 \rangle, \langle 2, -3 \rangle} = 0$, $(\langle 1, -2 \rangle, \langle 2, -4 \rangle) = -2$ and $(\langle 2, -3 \rangle, \langle 1, -4 \rangle) = 0$, we have

$$\mathfrak{d}(S_Q(\langle 2, -3 \rangle), S_Q(\langle 1, -4 \rangle)) = 2.$$

- (2) Since $p_{\langle 1 \rangle, \langle 2 \rangle} = 1$, $p_{\langle 3 \rangle, \langle 2 \rangle} = 1$, $(\langle 1 \rangle, \langle 2 \rangle) = 0$ and $(\langle 2 \rangle, \langle 3 \rangle) = 0$, we have

$$\mathfrak{d}(S_Q(\langle 1, 2 \rangle), S_Q(\langle 3 \rangle)) = 2.$$

Now let us consider the case $(x)'$. In this case, we have non-zero R -module homomorphisms (up to grading shifts)

$$(8.10) \quad S_Q(\eta_2) \circ S_Q(\eta_1) \rightarrow S_Q(\beta) \circ S_Q(\alpha) \quad \text{and} \quad S_Q(\zeta_1) \circ S_Q(\tau_1) \rightarrow S_Q(\beta) \circ S_Q(\alpha)$$

by (8.9) and hence the homomorphisms are injective, and

$$(8.11) \quad S_Q(\alpha) \nabla S_Q(\beta) \simeq S_Q(\text{hd}_Q(\alpha, \beta)) \simeq S_Q(\gamma) \simeq S_Q(\eta_1) \nabla S_Q(\eta_2) \simeq S_Q(\tau_1) \nabla S_Q(\zeta_1)$$

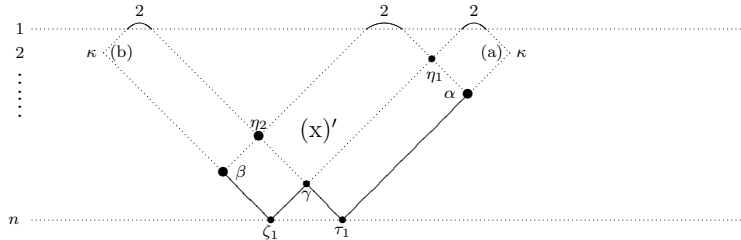
(up to grading shifts) by Proposition 7.15 and Proposition 8.5. Moreover, by Proposition 8.3 and Proposition 7.15, we can conclude that

$$\ell(S_Q(\alpha) \circ S_Q(\beta)) > 2.$$

Similarly, we can conclude the same result for case $(xi)'$.

Let us compute $\mathfrak{d}(S_Q(\alpha), S_Q(\beta))$ in case $(x)'$: Since (α, η_1) is sectional, there exists $\zeta \in \Phi^+$ such that

- (a) the $[Q]$ -pair $\langle\langle \kappa, \eta_1 \rangle\rangle$ is $[Q]$ -minimal for α and of the form (i)', or
- (b) the $[Q]$ -pair $\langle\langle \alpha, \kappa \rangle\rangle$ is $[Q]$ -minimal for η_1 and of the form (iii)'.



Assume (a). Then as in the previous cases, we have a homogeneous R -module homomorphism

$$q^{p_{\eta_1, \kappa} - (\eta_1, \kappa)} S_Q(\alpha) \circ S_Q(\beta) \hookrightarrow S_Q(\eta_1) \circ S_Q(\kappa) \circ S_Q(\beta).$$

Since the $[Q]$ -pair $\langle\langle \kappa, \beta \rangle\rangle$ is $[Q]$ -minimal for η_2 and of the form (iii)', we have a non-zero composition of R -module homomorphisms

$$q^{p_{\eta_1, \kappa} - (\eta_1, \kappa)} S_Q(\alpha) \circ S_Q(\beta) \hookrightarrow S_Q(\eta_1) \circ S_Q(\kappa) \circ S_Q(\beta) \twoheadrightarrow q^{-p_{\beta, \kappa}} S_Q(\eta_1) \circ S_Q(\eta_2).$$

Note that the homomorphism is surjective by Proposition 8.5 and (8.10). Hence we can obtain a non-zero R -module homomorphism

$$q^{p_{\eta_1, \kappa} - (\eta_1, \kappa)} S_Q(\alpha) \circ S_Q(\beta) \twoheadrightarrow q^{-p_{\beta, \kappa} - p_{\eta_2, \eta_1}} S_Q(\gamma),$$

since $S_Q(\eta_1) \nabla S_Q(\eta_2) \simeq q^{-p_{\eta_2, \eta_1}} S_Q(\gamma)$. Thus we have

$$\tilde{\Lambda}(S_Q(\alpha), S_Q(\beta)) = p_{\beta, \kappa} + p_{\eta_2, \eta_1} + p_{\eta_1, \kappa} - (\eta_1, \kappa)$$

and

$$\mathfrak{d}(S_Q(\alpha), S_Q(\beta)) = p_{\beta, \kappa} + p_{\eta_2, \eta_1} + p_{\eta_1, \kappa} - (\eta_1, \kappa) - (\alpha, \beta).$$

Since $p_{\beta, \kappa} = p_{\eta_2, \eta_1} = p_{\eta_1, \kappa} = 0$ and $(\eta_1, \kappa) = (\alpha, \beta) = -2$, we have

$$(8.12) \quad \mathfrak{d}(S_Q(\alpha), S_Q(\beta)) = 4 = \deg_Q(\alpha, \beta) \times \max(\mathbf{d}_\alpha, \mathbf{d}_\beta) = \mathfrak{d}_{i,j}[p - s].$$

Similarly, we have the same results for $\tilde{\Lambda}(S_Q(\alpha), S_Q(\beta))$ and $\mathfrak{d}(S_Q(\alpha), S_Q(\beta))$ in (b) case.

Let us compute $\mathfrak{d}(S_Q(\alpha), S_Q(\beta))$ in case (xi)': As in case (x)', we have

- (a) the $[Q]$ -pair $\langle\langle \zeta, \eta_1 \rangle\rangle$ is $[Q]$ -minimal for α and of the form (i)', or
- (b) the $[Q]$ -pair $\langle\langle \alpha, \zeta \rangle\rangle$ is $[Q]$ -minimal for η_1 and of the form (iii)'.

Assume (a). As in case (x)', we have a non-zero composition of R -module homomorphisms

$$q^{p_{\eta_1, \zeta} - (\eta_1, \zeta)} S_Q(\alpha) \circ S_Q(\beta) \hookrightarrow S_Q(\eta_1) \circ S_Q(\zeta) \circ S_Q(\beta) \twoheadrightarrow q^{-p_{\beta, \zeta}} S_Q(\eta_1) \circ S_Q(\eta_2).$$

Since the $[Q]$ -pair $\langle\langle \eta_1, \eta_2 \rangle\rangle$ corresponds to case (ii)', (8.11) tells that we can obtain a non-zero R -module homomorphism

$$q^{p_{\eta_1, \zeta} - (\eta_1, \zeta)} S_Q(\alpha) \circ S_Q(\beta) \twoheadrightarrow q^{-p_{\beta, \zeta} - \tilde{\Lambda}(S_Q(\eta_1), S_Q(\eta_2))} S_Q(s_1) \circ S_Q(s_2),$$

which is surjective. Thus we have

$$\tilde{\Lambda}(S_Q(\alpha), S_Q(\beta)) = p_{\beta, \zeta} + \tilde{\Lambda}(S_Q(\eta_1), S_Q(\eta_2)) + p_{\eta_1, \zeta} - (\eta_1, \zeta)$$

and hence

$$\mathfrak{d}(S_Q(\alpha), S_Q(\beta)) = p_{\beta, \zeta} + \tilde{\Lambda}(S_Q(\eta_1), S_Q(\eta_2)) + p_{\eta_1, \zeta} - (\eta_1, \zeta) - (\alpha, \beta).$$

Since the $[Q]$ -pair $\langle\langle \eta_1, \eta_2 \rangle\rangle$ corresponds to case (ii)', we have

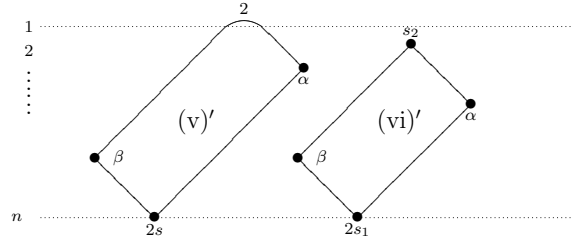
$$\tilde{\Lambda}(S_Q(\eta_1), S_Q(\eta_2)) = \mathfrak{d}(S_Q(\eta_1), S_Q(\eta_2)) + (\eta_1, \eta_2) = 2.$$

By Table 3, and the facts that $p_{\beta, \zeta} = p_{\eta_1, \zeta} = 0$, $(\eta_1, \zeta) = -2$ and $(\alpha, \beta) = 0$, we have

$$\mathfrak{d}(S_Q(\alpha), S_Q(\beta)) = 4 = \deg_Q(\alpha, \beta) \times \max(\mathbf{d}_\alpha, \mathbf{d}_\beta) = \mathfrak{d}_{i,j}[p - s].$$

Thus the assertions in Theorem 8.6 hold for cases (x)' and (xi)'.

Let us consider the cases (v)' and (vi)', in which contain $m_i = 2$ in the head $\underline{m}$ of $\langle\langle \alpha, \beta \rangle\rangle$.



Consider (v)' case. As in the previous cases, there exists $\eta \in \Phi^+$ such that

- (a) the $[Q]$ -pair $\langle\langle \eta, s \rangle\rangle$ is $[Q]$ -minimal for α and of the form (ix)' or
- (b) the $[Q]$ -pair $\langle\langle s, \eta \rangle\rangle$ is $[Q]$ -minimal for β and of the form (ix)'.

Assume (a). Then we have a homogeneous R -module monomorphism

$$q^{p_{s,\eta}-(s,\eta)} S_Q(\alpha) \circ S_Q(\beta) \hookrightarrow S_Q(s) \circ S_Q(\eta) \circ S_Q(\beta).$$

Note that $\langle\langle \eta, \beta \rangle\rangle$ is a $[Q]$ -minimal $[Q]$ -pair for s and of the form (viii)'. Thus the composition

$$q^{p_{s,\eta}-(s,\eta)} S_Q(\alpha) \circ S_Q(\beta) \hookrightarrow S_Q(s) \circ S_Q(\eta) \circ S_Q(\beta) \twoheadrightarrow q^{-p_{\beta,\eta}} S_Q(s)^{\circ 2}$$

does not vanish by Proposition 7.14 and hence $S_Q(\alpha) \nabla S_Q(\beta) \simeq S_Q(s)^{\circ 2}$ up to a grading shift. Note that $S_Q(s)^{\circ 2}$ is not self-dual simple, while $S_Q(2s) := q S_Q(s)^{\circ 2}$ is self-dual simple. Thus we have

$$S_Q(\alpha) \circ S_Q(\beta) \twoheadrightarrow q^{-p_{\beta,\eta}-p_{s,\eta}+(s,\eta)-1} S_Q(2s)$$

which tells

$$\tilde{\Lambda}(S_Q(\alpha), S_Q(\beta)) = p_{\beta,\eta} + p_{s,\eta} - (s, \eta) + 1$$

and hence

$$\mathfrak{d}(S_Q(\alpha), S_Q(\beta)) = p_{\beta,\eta} + p_{s,\eta} - (s, \eta) - (\alpha, \beta) + 1.$$

Since $p_{\beta,\eta} = 1$ and $p_{s,\eta} = (s, \eta) = (\alpha, \beta) = 0$, we have

$$(8.13) \quad \mathfrak{d}(S_Q(\alpha), S_Q(\beta)) = 2 = \max(\mathbf{d}_\alpha, \mathbf{d}_\beta) = \mathfrak{d}_{i,j}[p - s].$$

By Proposition 7.12, $\ell(S_Q(\alpha) \circ S_Q(\beta)) = 2$. We can apply the similar arguments to (b).

Consider (vi)' case. As in (v)'-case, there exists $\eta \in \Phi^+$ such that

- (a) the $[Q]$ -pair $\langle\langle \eta, s_2 \rangle\rangle$ is $[Q]$ -minimal for α and of the form (ix)', or
- (b) the $[Q]$ -pair $\langle\langle s_2, \eta \rangle\rangle$ is $[Q]$ -minimal for β and of the form (ix)'.

Assume (a). Then we have

$$q^{p_{s_2,\eta}-(s_2,\eta)} S_Q(\alpha) \circ S_Q(\beta) \hookrightarrow S_Q(s_2) \circ S_Q(\eta) \circ S_Q(\beta).$$

Then the $[Q]$ -pair $\langle\langle \eta, \beta \rangle\rangle$ corresponds to (vii)'. Thus we have a non-zero composition of homogeneous R -module homomorphisms

$$\begin{aligned} q^{p_{s_2,\eta}-(s_2,\eta)} S_Q(\alpha) \circ S_Q(\beta) &\hookrightarrow S_Q(s_2) \circ S_Q(\eta) \circ S_Q(\beta) \\ &\twoheadrightarrow q^{-\tilde{\Lambda}(S_Q(\eta), S_Q(\beta))} S_Q(s_2) \circ S_Q(s_2) \circ S_Q(s_3) \simeq q^{-\tilde{\Lambda}(S_Q(\eta), S_Q(\beta))-1} S_Q(2s_2) \circ S_Q(s_3), \end{aligned}$$

and hence $S_Q(\alpha) \circ S_Q(\beta) \simeq S_Q(2s_2) \circ S_Q(s_3)$ up to a grading shift. Furthermore, we have

$$S_Q(\alpha) \circ S_Q(\beta) \twoheadrightarrow q^{-p_{s_2,\eta}+(s_2,\eta)-\tilde{\Lambda}(S_Q(\eta), S_Q(\beta))-1} S_Q(2s_2) \circ S_Q(s_3)$$

which tells

$$\tilde{\Lambda}(S_Q(\alpha), S_Q(\beta)) = p_{s_2,\eta} - (s_2, \eta) + \tilde{\Lambda}(S_Q(\eta), S_Q(\beta)) + 1$$

and hence

$$\mathfrak{d}(S_Q(\alpha), S_Q(\beta)) = p_{s_2,\eta} + \tilde{\Lambda}(S_Q(\eta), S_Q(\beta)) - (s_2, \eta) - (\alpha, \beta) + 1.$$

Since the $[Q]$ -pair $\langle\langle \eta, \beta \rangle\rangle$ corresponds to (vii)', we have

$$\tilde{\Lambda}(S_Q(\eta), S_Q(\beta)) = \tilde{\Lambda}(S_Q(\eta), S_Q(\beta)) + (\eta, \beta) = 2 + 0 = 2.$$

Then the facts that $p_{s_2, \eta} = 1$, $(s_2, \eta) = 0$ and $(\alpha, \beta) = 2$ tell that we have

$$(8.14) \quad \mathfrak{d}(S_Q(\alpha), S_Q(\beta)) = 2 = \max(\mathbf{d}_\alpha, \mathbf{d}_\beta) = \mathfrak{d}_{i,j}[p - s].$$

We can apply the similar arguments to (b). Thus the assertions in Theorem 8.6 hold for cases (v)' and (vi)'.

8.2.2. C_n case. Now we shall consider the C_n -cases. Comparing with B_n -case, there is no C_n -analogue of Proposition 6.10; i.e., for a $[Q]$ -pair $\underline{p} = \langle\langle \alpha, \beta \rangle\rangle$ with $\deg_Q(\alpha, \beta) = 2$, there exists a unique sequence $\underline{m}$ such that

$$\mathrm{hd}_Q(\underline{p}) \prec_Q^b \underline{m} \prec_Q^b \underline{p}.$$

Thus we can not apply Proposition 7.15 for the cases (x)'' and (xi)''. For the cases except (x)'' and (xi)'', we can apply the same strategies of B_n type to show Theorem 8.6.

Let us consider the (x)''-case. Before we consider the general (x)''-case, let us observe the AR-quiver Γ_Q of type C_3 described in Example 4.5. In this case, $\langle\langle \alpha, \beta \rangle\rangle := \langle\langle \langle 1, -3 \rangle, \langle 2, 3 \rangle \rangle\rangle$ is the unique $[Q]$ -pair corresponding to (x)''-case.

Now we want to show

$$\mathfrak{d}(S_Q(\langle 1, -3 \rangle), S_Q(\langle 2, 3 \rangle)) = 2.$$

Since $[Q]$ -pair $\langle\langle \langle 2, -3 \rangle, \langle 3, 3 \rangle \rangle\rangle$ is $[Q]$ -minimal for $\langle 2, 3 \rangle$, we have

$$S_Q(\langle 2, 3 \rangle) \simeq S_Q(\langle 2, -3 \rangle) \nabla S_Q(\langle 3, 3 \rangle) \quad (\text{up to grading shift}).$$

Let $L = S_Q(\langle 1, -3 \rangle)$, $M = S_Q(\langle 2, -3 \rangle)$ and $N = S_Q(\langle 3, 3 \rangle)$. Since L and M commute, the sequence (L, M, N) is a normal sequence. On the other hand, we can compute

$$\Lambda(M, L) = -1 \quad \Lambda(N, L) = 2 \quad \text{and} \quad \Lambda(M \nabla N, L) = 1$$

since

$$\langle 1, -3 \rangle \prec_Q \langle 2, -3 \rangle, \quad \langle 1, -3 \rangle \prec_Q \langle 3, 3 \rangle \quad \text{and} \quad \langle 1, -3 \rangle \prec_Q \langle 2, 3 \rangle.$$

Then Lemma 7.18 implies (M, N, L) is normal and hence

$$\mathfrak{d}(L, M \nabla N) = \mathfrak{d}(L, M) + \mathfrak{d}(L, N) = 0 + 2 = 2,$$

by Lemma 7.19. Here we use the fact that $\mathfrak{d}(L, N) = 2$ followed by Proposition 8.10. Moreover, we have

$$\begin{aligned} L \nabla (M \nabla N) &\simeq \mathrm{hd}(L \circ M \circ N) \simeq \mathrm{hd}(M \circ L \circ N) \simeq \mathrm{hd}(M \circ (L \nabla N)) \\ &\simeq S_Q(\langle 2, -3 \rangle) \nabla S_Q(\langle 1, 3 \rangle) \simeq S_Q(\langle 1, 2 \rangle) \end{aligned}$$

as we desired. Since there exists a non-zero R -module homomorphism

$$S_Q(\langle 1, 3 \rangle) \circ S_Q(\langle 2, -3 \rangle) \rightarrow S_Q(\langle 2, 3 \rangle) \circ S_Q(1, -3),$$

$\ell(S_Q(\langle 1, 3 \rangle) \circ S_Q(\langle 2, -3 \rangle)) = 2$ and they have common simple socle, the R -module homomorphism is injective. Thus we have

$$\ell(S_Q(\langle 2, 3 \rangle) \circ S_Q(1, -3)) > 2.$$

Thus we can check the assertions in Theorem 8.6 hold for the $[Q]$ -pair.

Let us consider the general (x)''-case. Then there exists $\kappa \in \Phi^+$ such that

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Assume (a). Then we have

Now we shall prove the following claims:

(I) Note that

$$\Lambda(S_Q(\beta), S_Q(\kappa) \nabla S_Q(\eta_1)) = \Lambda(S_Q(\beta), S_Q(\kappa)) + \Lambda(S_Q(\beta), S_Q(\eta_1))$$

(II) We have

Here we use the facts

that follow from Proposition 8.10 and (8.15). Thus we have

which implies that the sequence $(S_Q(\kappa), S_Q(\eta_1), S_Q(\beta))$ is normal.

$$\mathfrak{d}(S_Q(\alpha), S_Q(\beta)) = \mathfrak{d}(S_Q(\kappa) \nabla S_Q(\eta_1), S_Q(\beta)) = \mathfrak{d}(S_Q(\kappa), S_Q(\beta)) + \mathfrak{d}(S_Q(\eta_1), S_Q(\beta)) = 2,$$

as we desired. Moreover, we have

$$\begin{aligned} S_Q(\alpha) \nabla S_Q(\beta) &\simeq \text{hd}(S_Q(\kappa) \circ S_Q(\eta_1) \circ S_Q(\beta)) \\ &\simeq S_Q(\kappa) \nabla (S_Q(\eta_1) \nabla S_Q(\beta)) \simeq S_Q(\kappa) \nabla S_Q(\tau_2) \\ &\simeq S_Q(\gamma), \end{aligned}$$

up to grading shifts. Finally we have

$$\ell(S_Q(\alpha) \circ S_Q(\beta)) > 2,$$

as in the particular case of type C_3 . For the case (b) and (xi)'', we can apply the similar argument. Thus the assertions in Theorem 8.6 hold for cases (x)'' and (xi)''.

8.2.3. G_2 case. Note that there exist only 2-reduced expressions of w_0 of type G_2 , which are all adapted to the two Dynkin quivers (up to constant). We shall only consider Q and Γ_Q in Section 6.4. Then we have

$$\begin{aligned} \beta_1 = \alpha_2 &= (1, -2, 1), & \beta_2 = \alpha_1 + \alpha_2 &= (1, -1, 0), & \beta_3 = 3\alpha_1 + \alpha_2 &= (2, -1, -1), \\ \beta_4 = 2\alpha_1 + \alpha_2 &= (1, 0, -1), & \beta_5 = 3\alpha_1 + \alpha_2 &= (1, 1, -2), & \beta_6 = \alpha_1 &= (0, 1, -1). \end{aligned}$$

Note that non $[Q]$ -simple $[Q]$ -pairs are given as follows:

$$(8.16) \quad \begin{aligned} \beta_3 \prec_Q^b \langle\langle \beta_2, \beta_4 \rangle\rangle, \quad \beta_5 \prec_Q^b \langle\langle \beta_4, \beta_6 \rangle\rangle, \quad \beta_4 \prec_Q^b \langle\langle \beta_2, \beta_6 \rangle\rangle, \\ 2\beta_2 \prec_Q^b \langle\langle \beta_1, \beta_4 \rangle\rangle, \quad 2\beta_4 \prec_Q^b \langle\langle \beta_3, \beta_6 \rangle\rangle, \quad 2\beta_3 \prec_Q^b \langle\langle \beta_2, \beta_5 \rangle\rangle, \end{aligned}$$

and

$$(8.17) \quad \text{(i) } 3\beta_2 \prec_Q^b \langle\langle \beta_1, \beta_3 \rangle\rangle, \quad \text{(ii) } 3\beta_4 \prec_Q^b \langle\langle \beta_3, \beta_5 \rangle\rangle, \quad \text{(iii) } \beta_3 \prec_Q^b \langle\langle \beta_2, \beta_4 \rangle\rangle \prec_Q^b \langle\langle \beta_5, \beta_1 \rangle\rangle.$$

The cases in (8.16) can be proved as in B_n or C_n -cases. So let us consider the cases in (8.17).

At first, we shall deal with (iii) in (8.17). Take $L = S_Q(\beta_1)$, $M = S_Q(\beta_4)$, $N = S_Q(\beta_6)$ and hence $M \nabla N \simeq S_Q(\beta_5)$ and $L \nabla M \simeq S_Q(\beta_2)^{\circ 2}$ up to grading shifts. Since we know $\mathfrak{d}(S_Q(\beta_2), N) = 2$, $\mathfrak{d}(L, N) = 3$ and $\mathfrak{d}(M, N) = 1$ by Proposition 8.4, we know

$$\begin{aligned} 2 \times 3 &= 2\Lambda(S_Q(\beta_2), N) = \Lambda(L \nabla M, N) \\ &= \Lambda(L, N) + \Lambda(M, N) = 3 + 3, \end{aligned}$$

which implies that the sequence (L, M, N) is normal by Lemma 7.18. By the convex order $\prec_Q$, the sequence (M, N, L) is also normal. Thus

$$\mathfrak{d}(L, M \nabla N) = \mathfrak{d}(L, M) + \mathfrak{d}(L, N) = 6,$$

by Lemma 7.19. Furthermore, we have

$$\begin{aligned} L \nabla (M \nabla N) &\simeq (L \nabla M) \nabla N \simeq S_Q(\beta_2)^{\circ 2} \nabla N \\ &\simeq S_Q(\beta_2) \nabla (S_Q(\beta_2) \nabla N) \simeq S_Q(\beta_2) \nabla S_Q(\beta_4) \simeq S_Q(\beta_3). \end{aligned}$$

Since we can construct a non-zero homomorphism

$$S_Q(\beta_4) \circ S_Q(\beta_2) \rightarrow S_Q(\beta_5) \circ S_Q(\beta_1)$$

and they have common simple socle, we have

$$S_Q(\beta_4) \circ S_Q(\beta_2) \hookrightarrow S_Q(\beta_5) \circ S_Q(\beta_1)$$

and hence

$$\ell(S_Q(\beta_5) \circ S_Q(\beta_1)) > 2.$$

Now let us consider (i) in (8.17). Since $\langle\langle \beta_2, \beta_4 \rangle\rangle$ is a minimal $[Q]$ -pair for β_5 , we have a non-zero composition of R -module homomorphisms

$$S_Q(\beta_1) \circ S_Q(\beta_3) \hookrightarrow S_Q(\beta_1) \circ S_Q(\beta_4) \circ S_Q(\beta_2) \twoheadrightarrow S_Q(\beta_2)^{\circ 3},$$

which implies

$$S_Q(\beta_1) \circ S_Q(\beta_3) \simeq S_Q(\beta_2)^{\circ 3}.$$

The assertion (ii) in (8.17) can be proved in the same way.

8.2.4. F_4 case. In F_4 -case, we can prove Theorem 8.6 by using techniques we have used in the previous subsections to each $[Q]$ -pair $\langle\langle \alpha, \beta \rangle\rangle$. In this subsection, we shall present several cases, instead of considering all cases. Let us use Γ_Q in (6.6) and assume that Theorem 8.6 holds for $[Q]$ -pairs $\langle\langle \theta, \mu \rangle\rangle$ with $\deg_Q(\theta, \mu) \leq 1$.

(1) Let us compute $\mathfrak{d}(S_Q(\alpha'), S_Q(\beta'))$ when $\alpha' = (0, 1, 0, 0)$ and $\beta' = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$. In this case, we have

$$(a) \quad \phi_Q(3, 2) = (\alpha', 0), \quad \phi_Q(4, -5) = (\beta', 0), \quad \tilde{\mathfrak{d}}_{3,4}[7] = 2,$$

(b) there exists a unique $[Q]$ -pair $\underline{p} = \langle\langle (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}), (0, 0, 1, 0) \rangle\rangle = \langle\langle \alpha^{(1)}, \beta^{(1)} \rangle\rangle$ such that

$$\gamma := \alpha' + \beta' \prec_Q^b \underline{p} \prec_Q^b \langle\langle \alpha', \beta' \rangle\rangle.$$

Note that $\langle\langle \beta^{(1)}, \eta := (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) \rangle\rangle$ is a minimal $[Q]$ -pair for β' and $\langle\langle \alpha', \beta^{(1)} \rangle\rangle$ is a minimal $[Q]$ -pair for $\zeta = (0, 1, 1, 0)$. Setting $L = S_Q(\alpha')$, $M = S_Q(\beta^{(1)})$, $N = S_Q(\eta)$ and $T = S_Q(\zeta)$, we have $\mathfrak{d}(S_Q(\alpha'), S_Q(\beta')) = \mathfrak{d}(L, M \nabla N)$ and $L \nabla M \simeq T$. Since $\alpha' \prec_Q \beta^{(1)} \prec_Q \beta \prec_Q \eta$, the sequence (M, N, L) is normal.

On the other hand, we have

$$\Lambda(L \nabla M, N) = \Lambda(T, N) = 2 \mathfrak{d}(T, N) + (\zeta, \eta) = 2 - 2 = 2$$

and

$$\Lambda(L, N) = 2\mathfrak{d}(L, N) + (\alpha', \eta) = 2 \times 1 - 1 = 1, \quad \Lambda(M, N) = 2\mathfrak{d}(M, N) + (\beta^{(1)}, \eta) = 2 \times 1 - 1 = 1.$$

Thus the sequence (L, M, N) is also normal. Thus we have

$$\mathfrak{d}(L, M \nabla N) = \mathfrak{d}(L, M) + \mathfrak{d}(L, N) = 1 + 1 = 2.$$

Finally, we have

$$L \nabla (M \nabla N) \simeq (L \nabla M) \nabla N \simeq T \nabla N \simeq S_Q(\gamma)$$

and hence $\ell(S_Q(\alpha') \circ S_Q(\beta')) > 2$.

(2) Now let us keep the notations in (1) and compute $\mathfrak{d}(S_Q(\alpha), S_Q(\beta))$ when $\alpha = (0, 1, 0, -1)$ and $\beta = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ in (6.7). As we have seen in (6.7), we have

$$\gamma = \alpha + \beta \prec_Q^b \langle\langle \alpha^{(1)}, \beta^{(1)} \rangle\rangle \prec_Q^b \langle\langle \alpha', \beta' \rangle\rangle \prec_Q^b \langle\langle \alpha, \beta \rangle\rangle.$$

However, we have another chain given as follows:

$$\gamma = \alpha + \beta \prec_Q^b \langle\langle \alpha^{(1)}, \beta^{(1)} \rangle\rangle \prec_Q^b \langle\langle \alpha'', \beta'' \rangle\rangle \prec_Q^b \langle\langle \alpha, \beta \rangle\rangle,$$

where $\alpha'' = (0, 0, 1, -1)$ and $\beta'' = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$. As in (1), we can prove that $\mathfrak{d}(J, K) = 2$ and $J \nabla K \simeq S_Q(\gamma)$, where $J := S_Q(\alpha'')$ and $K := S_Q(\beta'')$. Let $\kappa = (0, 1, -1, 0)$, and set $P := S_Q(\kappa)$, $R = S_Q(\alpha)$, $X = S_Q(\beta')$ and $U = S_Q(\beta)$. Then $\mathfrak{d}(R, U) = \mathfrak{d}(P \nabla J, U)$, since $\langle\langle \alpha'', \kappa \rangle\rangle$ is a minimal $[Q]$ -pair of α . Note that $J \nabla U \simeq M \circ X \simeq X \circ M$. Since

$$\kappa \prec_Q \alpha'' \prec_Q \beta^{(1)} \prec_Q \beta' \prec_Q \beta,$$

the sequence (J, U, P) is normal.

On the other hand, we have

$$\begin{aligned} \Lambda(P, J \nabla U) &= \Lambda(P, X \circ M) = \Lambda(P, X) + \Lambda(P, M) \\ &= (2\mathfrak{d}(P, X) + (\kappa, \beta')) + (2\mathfrak{d}(P, M) + (\kappa, \beta^{(1)})) = (2 \times 2 - 2) + (2 \times 2 - 2) = 4 \end{aligned}$$

and

$$\begin{aligned} \Lambda(P, J) &= 2\mathfrak{d}(P, J) + (\kappa, \alpha'') = 2 \times 2 - 2 = 2, \\ \Lambda(P, U) &= 2\mathfrak{d}(P, U) + (\kappa, \beta) = 2 \times 2 - 2 = 2. \end{aligned}$$

Thus the sequence (P, J, U) is also normal. Hence

$$\mathfrak{d}(P \nabla J, U) = \mathfrak{d}(P, U) + \mathfrak{d}(J, U) = 2 + 2 = 4 = \tilde{\mathfrak{d}}_{1,3}[9].$$

Consequently, we have

$$\begin{aligned} (P \nabla J) \nabla U &\simeq P \nabla (J \nabla U) \simeq P \nabla (X \circ M) \simeq (P \nabla X) \nabla M \\ &\simeq L \nabla M \simeq S_Q(\gamma). \end{aligned}$$

Thus we have injective R -module homomorphisms

$$S_Q(\beta'') \circ S_Q(\alpha'') \hookrightarrow S_Q(\beta) \circ S_Q(\alpha) \quad \text{and} \quad S_Q(\beta') \circ S_Q(\alpha') \hookrightarrow S_Q(\beta) \circ S_Q(\alpha),$$

since they have the common socle $S_Q(\gamma)$, which appears once in their composition series. Thus we have $\ell(S_Q(\alpha) \circ S_Q(\beta)) > 2$.

8.3. Ending Remark. We can also apply the techniques in the previous subsections to prove (8.4) for ADE -types. Also the result for $\deg_Q(\alpha, \beta) = 1$ can be written in the form of (7.1).

Example 8.12. For a reduced expression $s_1 s_2 s_3 s_1 s_2 s_3 s_1 s_2 s_3$ of w_0 of type C_3 , adapted to Q in Example 4.5, the fact that

$$S_Q(\langle 1, 1 \rangle) \nabla S_Q(\langle 2, 2 \rangle) \simeq S_Q(\langle 1, 2 \rangle)^{\circ 2} \quad (\text{up to a grading shift})$$

corresponds to

$$r_{[Q]}(\langle 1, 1 \rangle) r_{[Q]}(\langle 2, 2 \rangle) - r_{[Q]}(\langle 2, 2 \rangle) r_{[Q]}(\langle 1, 1 \rangle) = (q^{-1} - q^3) r_{[Q]}(\langle 1, 2 \rangle)^2.$$

APPENDIX A. $\widetilde{\mathfrak{d}}_{i,j}(t)$ FOR E_7 AND E_8

A.1. E_7 . Here is the list of $\widetilde{\mathfrak{d}}_{i,j}(t)$ for E_7 .

$$\begin{aligned}
\widetilde{\mathfrak{d}}_{1,1}(t) &= t^1 + t^7 + t^{11} + t^{17}, & \widetilde{\mathfrak{d}}_{1,2}(t) &= t^4 + t^8 + t^{10} + t^{14}, \\
\widetilde{\mathfrak{d}}_{1,3}(t) &= t^2 + t^6 + t^8 + t^{10} + t^{12} + t^{16}, & \widetilde{\mathfrak{d}}_{1,4}(t) &= t^3 + t^5 + t^7 + 2t^9 + t^{11} + t^{13} + t^{15}, \\
\widetilde{\mathfrak{d}}_{1,6}(t) &= t^5 + t^7 + t^{11} + t^{13}, & \widetilde{\mathfrak{d}}_{1,7}(t) &= t^6 + t^{12}, \\
\widetilde{\mathfrak{d}}_{2,2}(t) &= t^1 + t^5 + t^7 + t^9 + t^{11} + t^{13} + t^{17}, & \widetilde{\mathfrak{d}}_{2,3}(t) &= \widetilde{\mathfrak{d}}_{1,4}(t) \\
\widetilde{\mathfrak{d}}_{2,4}(t) &= t^2 + t^4 + 2t^6 + 2t^8 + 2t^{10} + 2t^{12} + t^{14} + t^{16}, \\
\widetilde{\mathfrak{d}}_{2,5}(t) &= t^3 + t^5 + 2t^7 + t^9 + 2t^{11} + t^{13} + t^{15}, & \widetilde{\mathfrak{d}}_{2,7}(t) &= t^5 + t^9 + t^{13}, \\
\widetilde{\mathfrak{d}}_{2,6}(t) &= t^4 + t^6 + t^8 + t^{10} + t^{12} + t^{14}, \\
\widetilde{\mathfrak{d}}_{3,3}(t) &= t^1 + t^3 + t^5 + 2t^7 + 2t^9 + 2t^{11} + t^{13} + t^{15} + t^{17}, \\
\widetilde{\mathfrak{d}}_{3,4}(t) &= t^2 + 2t^4 + 2t^6 + 3t^8 + 3t^{10} + 2t^{12} + 2t^{14} + t^{16}, \\
\widetilde{\mathfrak{d}}_{3,5}(t) &= t^3 + 2t^5 + 2t^7 + 2t^9 + 2t^{11} + 2t^{13} + t^{15}, \\
\widetilde{\mathfrak{d}}_{3,6}(t) &= t^4 + 2t^6 + t^8 + t^{10} + 2t^{12} + t^{14}, & \widetilde{\mathfrak{d}}_{3,7}(t) &= t^5 + t^7 + t^{11} + t^{13}, \\
\widetilde{\mathfrak{d}}_{4,4}(t) &= t^1 + 2t^3 + 3t^5 + 4t^7 + 4t^9 + 4t^{11} + 3t^{13} + 2t^{15} + t^{17}, \\
\widetilde{\mathfrak{d}}_{4,5}(t) &= t^2 + 2t^4 + 3t^6 + 3t^8 + 3t^{10} + 3t^{12} + 2t^{14} + t^{16}, \\
\widetilde{\mathfrak{d}}_{4,6}(t) &= \widetilde{\mathfrak{d}}_{3,5}(t), & \widetilde{\mathfrak{d}}_{4,7}(t) &= t^4 + t^6 + t^8 + t^{10} + t^{12} + t^{14}, \\
\widetilde{\mathfrak{d}}_{5,5}(t) &= t^1 + t^3 + 2t^5 + 2t^7 + 3t^9 + 2t^{11} + 2t^{13} + t^{15} + t^{17}, \\
\widetilde{\mathfrak{d}}_{5,6}(t) &= t^2 + t^4 + t^6 + 2t^8 + 2t^{10} + t^{12} + t^{14} + t^{16}, & \widetilde{\mathfrak{d}}_{5,7}(t) &= t^3 + t^7 + t^{11} + t^{15}, \\
\widetilde{\mathfrak{d}}_{6,6}(t) &= t^1 + t^3 + t^7 + 2t^9 + t^{11} + t^{15} + t^{17}, & \widetilde{\mathfrak{d}}_{6,7}(t) &= t^2 + t^8 + t^{10} + t^{16}, \\
\widetilde{\mathfrak{d}}_{7,7}(t) &= t^1 + t^9 + t^{17} & \text{and} & \widetilde{\mathfrak{d}}_{i,j}(t) = \widetilde{\mathfrak{d}}_{j,i}(t).
\end{aligned}$$

A.2. E_8 . Here is the list of $\widetilde{\mathfrak{d}}_{i,j}(t)$ for E_8 .

$$\begin{aligned}
\widetilde{\mathfrak{d}}_{1,1}(t) &= t^1 + t^7 + t^{11} + t^{13} + t^{17} + t^{19} + t^{23} + t^{29}, \\
\widetilde{\mathfrak{d}}_{1,2}(t) &= t^4 + t^8 + t^{10} + t^{12} + t^{14} + t^{16} + t^{18} + t^{20} + t^{22} + t^{26}, \\
\widetilde{\mathfrak{d}}_{1,3}(t) &= t^2 + t^6 + t^8 + t^{10} + 2t^{12} + t^{14} + t^{16} + 2t^{18} + t^{20} + t^{22} + t^{24} + t^{28}, \\
\widetilde{\mathfrak{d}}_{1,4}(t) &= t^3 + t^5 + t^7 + 2t^9 + 2t^{11} + 2t^{13} + 2t^{15} + 2t^{17} + 2t^{19} + 2t^{21} + t^{23} + t^{25} + t^{27}, \\
\widetilde{\mathfrak{d}}_{1,5}(t) &= t^4 + t^6 + t^8 + 2t^{10} + t^{12} + 2t^{14} + 2t^{16} + t^{18} + 2t^{20} + t^{22} + t^{24} + t^{26}, \\
\widetilde{\mathfrak{d}}_{1,6}(t) &= t^5 + t^7 + t^9 + t^{11} + t^{13} + 2t^{15} + t^{17} + t^{19} + t^{21} + t^{23} + t^{25}, \\
\widetilde{\mathfrak{d}}_{1,7}(t) &= t^6 + t^8 + t^{12} + t^{14} + t^{16} + t^{18} + t^{22} + t^{24}, \\
\widetilde{\mathfrak{d}}_{1,8}(t) &= t^7 + t^{13} + t^{17} + t^{23},
\end{aligned}$$

$$\begin{aligned}
\widetilde{\mathfrak{d}}_{2,2}(t) &= t^1 + t^5 + t^7 + t^9 + 2t^{11} + t^{13} + 2t^{15} + t^{17} + 2t^{19} + t^{21} + t^{23} + t^{25} + t^{29}, \\
\widetilde{\mathfrak{d}}_{2,3}(t) &= t^3 + t^5 + t^7 + 2(t^9 + t^{11} + t^{13} + t^{15} + t^{17} + t^{19} + t^{21}) + t^{23} + t^{25} + t^{27}, \\
\widetilde{\mathfrak{d}}_{2,4}(t) &= t^2 + t^4 + 2t^6 + 2t^8 + 3(t^{10} + t^{12} + t^{14} + t^{16} + t^{18} + t^{20}) + 2t^{22} + 2t^{24} + t^{26} + t^{28}, \\
\widetilde{\mathfrak{d}}_{2,5}(t) &= t^3 + t^5 + 2t^7 + 2t^9 + 2t^{11} + 3t^{13} + 3t^{15} + 3t^{17} + 2t^{19} + 2t^{21} + 2t^{23} + t^{25} + t^{27}, \\
\widetilde{\mathfrak{d}}_{2,6}(t) &= t^4 + t^6 + 2t^8 + t^{10} + 2t^{12} + 2t^{14} + 2t^{16} + 2t^{18} + t^{20} + 2t^{22} + t^{24} + t^{26}, \\
\widetilde{\mathfrak{d}}_{2,7}(t) &= \widetilde{\mathfrak{d}}_{1,6}(t), \quad \widetilde{\mathfrak{d}}_{2,8}(t) = t^6 + t^{10} + t^{14} + t^{16} + t^{20} + t^{24}, \\
\widetilde{\mathfrak{d}}_{3,3}(t) &= t^1 + t^3 + t^5 + 2t^7 + 2t^9 + 3t^{11} + 3t^{13} + 2t^{15} + 3t^{17} + 3t^{19} + 2t^{21} + 2t^{23} \\
&\quad + t^{25} + t^{27} + t^{29}, \\
\widetilde{\mathfrak{d}}_{3,4}(t) &= t^2 + 2(t^4 + t^6) + 3t^8 + 4(t^{10} + t^{12} + t^{14} + t^{16} + t^{18} + t^{20}) + 3t^{22} + 2(t^{24} + t^{26}) + t^{28}, \\
\widetilde{\mathfrak{d}}_{3,5}(t) &= t^3 + 2t^5 + 2t^7 + 3t^9 + 3t^{11} + 3t^{13} + 4t^{15} + 3t^{17} + 3t^{19} + 3t^{21} + 2t^{23} + 2t^{25} + t^{27}, \\
\widetilde{\mathfrak{d}}_{3,6}(t) &= t^4 + 2t^6 + 2t^8 + 2t^{10} + 2t^{12} + 3t^{14} + 3t^{16} + 2t^{18} + 2t^{20} + 2t^{22} + 2t^{24} + t^{26}, \\
\widetilde{\mathfrak{d}}_{3,7}(t) &= t^5 + 2t^7 + t^9 + t^{11} + 2t^{13} + 2t^{15} + 2t^{17} + t^{19} + t^{21} + 2t^{23} + t^{25}, \\
\widetilde{\mathfrak{d}}_{3,8}(t) &= t^6 + t^8 + t^{12} + t^{14} + t^{16} + t^{18} + t^{22} + t^{24}, \\
\widetilde{\mathfrak{d}}_{4,4}(t) &= t^1 + 2t^3 + 3t^5 + 4t^7 + 5t^9 + 6t^{11} + 6t^{13} + 6t^{15} + 6t^{17} + 6t^{19} + 5t^{21} \\
&\quad + 4t^{23} + 3t^{25} + 2t^{27} + t^{29}, \\
\widetilde{\mathfrak{d}}_{4,5}(t) &= t^2 + 2t^4 + 3t^6 + 4t^8 + 4t^{10} + 5t^{12} + 5t^{14} + 5t^{16} + 5t^{18} + 4t^{20} + 4t^{22} \\
&\quad + 3t^{24} + 2t^{26} + t^{28}, \\
\widetilde{\mathfrak{d}}_{4,6}(t) &= t^3 + 2t^5 + 3t^7 + 3t^9 + 3t^{11} + 4t^{13} + 4t^{15} + 4t^{17} + 3t^{19} + 3t^{21} + 3t^{23} + 2t^{25} + t^{27}, \\
\widetilde{\mathfrak{d}}_{4,7}(t) &= \widetilde{\mathfrak{d}}_{3,6}(t), \quad \widetilde{\mathfrak{d}}_{4,8}(t) = t^5 + t^7 + t^9 + t^{11} + t^{13} + 2t^{15} + t^{17} + t^{19} + t^{21} + t^{23} + t^{25}, \\
\widetilde{\mathfrak{d}}_{5,5}(t) &= t^1 + t^3 + 2t^5 + 3t^7 + 3t^9 + 4t^{11} + 4t^{13} + 4t^{15} + 4t^{17} + 4t^{19} + 3t^{21} \\
&\quad + 3t^{23} + 2t^{25} + t^{27} + t^{29}, \\
\widetilde{\mathfrak{d}}_{5,6}(t) &= t^2 + t^4 + 2t^6 + 2t^8 + 3t^{10} + 3t^{12} + 3t^{14} + 3t^{16} + 3t^{18} + 3t^{20} + 2t^{22} \\
&\quad + 2t^{24} + t^{26} + t^{28}, \\
\widetilde{\mathfrak{d}}_{5,7}(t) &= t^3 + t^5 + t^7 + 2t^9 + 2t^{11} + 2t^{13} + 2t^{15} + 2t^{17} + 2t^{19} + 2t^{21} + t^{23} + t^{25} + t^{27}, \\
\widetilde{\mathfrak{d}}_{5,8}(t) &= t^4 + t^8 + t^{10} + t^{12} + t^{14} + t^{16} + t^{18} + t^{20} + t^{22} + t^{26}, \\
\widetilde{\mathfrak{d}}_{6,6}(t) &= t^1 + t^3 + t^5 + t^7 + 2t^9 + 3t^{11} + 2t^{13} + 2t^{15} + 2t^{17} + 3t^{19} + 2t^{21} \\
&\quad + t^{23} + t^{25} + t^{27} + t^{29}, \\
\widetilde{\mathfrak{d}}_{6,7}(t) &= t^2 + t^4 + t^8 + 2t^{10} + 2t^{12} + t^{14} + t^{16} + 2t^{18} + 2t^{20} + t^{22} + t^{26} + t^{28}, \\
\widetilde{\mathfrak{d}}_{6,8}(t) &= t^3 + t^9 + t^{11} + t^{13} + t^{17} + t^{19} + t^{21} + t^{27}, \\
\widetilde{\mathfrak{d}}_{7,7}(t) &= t^1 + t^3 + t^9 + 2t^{11} + t^{13} + t^{17} + 2t^{19} + t^{21} + t^{27} + t^{29},
\end{aligned}$$

$$\begin{aligned}\widetilde{\mathfrak{d}}_{7,8}(t) &= t^2 + t^{10} + t^{12} + t^{18} + t^{20} + t^{28}, \\ \widetilde{\mathfrak{d}}_{8,8}(t) &= t^1 + t^{11} + t^{19} + t^{29} \quad \text{and} \quad \widetilde{\mathfrak{d}}_{i,j}(t) = \widetilde{\mathfrak{d}}_{j,i}(t).\end{aligned}$$

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