

On the duality property of Blaschke products and its application

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Abstract

We study geometric properties of finite Blaschke products. For a Blaschke product B of degree d , the interior curve and the exterior curve are defined. In this paper, we explain the existence of duality-like geometrical property lies between the interior curve and the exterior curve. Using this property, we construct some examples of Blaschke products whose interior curves consist of two ellipses.

1 Geometry of Blaschke products

1.1 Blaschke Products

A *Blaschke product* of degree d is a rational function defined by

$$B(z) = e^{i\theta} \prod_{k=1}^d \frac{z - a_k}{1 - \overline{a_k}z} \quad (a_k \in \mathbb{D}, \theta \in \mathbb{R}).$$

In the case that $\theta = 0$ and $B(0) = 0$, B is called *canonical*.

Note that we only need to consider a canonical Blaschke product for the following discussions. Moreover, we remark that there are d distinct preimages $z_1, \dots, z_d$ of $\lambda \in \partial\mathbb{D}$ by B because the derivative B' has no zeros on $\partial\mathbb{D}$ (for instance, see [Mas13]).

1.2 The interior curves and exterior curves

For a Blaschke product B of degree d and $\lambda \in \partial\mathbb{D}$, let ℓ_λ be the set of lines joining each distinct two preimages in $B^{-1}(\lambda)$. Then, the envelope of the family of lines $\{\ell_\lambda\}_\lambda$ called the *interior curve* associated with B .

While, for a canonical Blaschke product B of degree d and $\lambda \in \partial\mathbb{D}$, let L_λ be the set of d lines tangent to $\partial\mathbb{D}$ at the d preimages of λ . Then, the trace of the intersection points of each two elements in L_λ as λ ranges over the unit circle called the *exterior curve* associated with B .

For a canonical Blaschke product of degree d , the exterior curve is an algebraic curve of degree at most $d - 1$ ([Fuj17]).

Example 1

For a canonical Blaschke product $B(z) = z \frac{z - a}{1 - \overline{a}z} \frac{z - b}{1 - \overline{b}z}$ of degree 3, the interior curve is the ellipse (see [DGM02])

$$|z - a| + |z - b| = |1 - \overline{a}b|, \quad (1)$$

and the exterior curve is the non-degenerate conic (see [Fuj17])

$$\overline{a}bz^2 + (-|ab|^2 + |a + b|^2 - 1)z\overline{z} + ab\overline{z}^2 - 2(\overline{a} + \overline{b})z - 2(a + b)\overline{z} + 4 = 0. \quad (2)$$

Example 2

For a canonical Blaschke product $B(z) = z \frac{z-a}{1-\bar{a}z} \frac{z-b}{1-\bar{b}z} \frac{z-c}{1-\bar{c}z}$ of degree 4, the interior curve is defined by the equation S of degree 6. We describe the defining equation S in Appendix A below. The exterior curve is written as follows (see [Fuj17])

$$\begin{aligned} & \bar{\sigma}_3 z^3 + (\sigma_1 \bar{\sigma}_2 - \sigma_2 \bar{\sigma}_3 - \bar{\sigma}_1) z^2 \bar{z} - (\sigma_1 - \sigma_2 \bar{\sigma}_1 + \sigma_3 \bar{\sigma}_2) z \bar{z}^2 + \sigma_3 \bar{z}^3 \\ & - 2\bar{\sigma}_2 z^2 - (2\sigma_1 \bar{\sigma}_1 - 2\sigma_3 \bar{\sigma}_3 - 4) z \bar{z} - 2\sigma_2 \bar{z}^2 + 4\bar{\sigma}_1 z + 4\sigma_1 \bar{z} - 8 = 0, \end{aligned}$$

where σ_k are the elementary symmetric polynomials on three variables a, b, c of degree k ($k = 1, 2, 3$), i.e. $\sigma_1 = a + b + c$, $\sigma_2 = ab + bc + ca$, and $\sigma_3 = abc$.

1.3 Duality-like property

There exists a duality-like relationship between the interior and exterior curves.

Theorem 1 ([Fuj18])

Let B be a canonical Blaschke product of degree d , and E_B^* the dual curve of the homogenized exterior curve E_B . Then the interior curve is given by

$$I_B : u_B^*(-z) = 0,$$

where $u_B^*(z) = 0$ is a defining equation of the affine part of E_B^* .

Equivalently, the converse also holds.

Corollary 2

Let B be a canonical Blaschke product of degree d , and I_B^* be the dual curve of the homogenized interior curve I_B . Then the exterior curve is given by

$$E_B : v_B^*(-z) = 0,$$

where $v_B^*(z) = 0$ is a defining equation of the affine part of I_B^* .

In general, the defining equation of the interior curve is hard to calculate, even using an algebraic computation system. On the other hand, the defining equation of the exterior curve is relatively simple, as seen in Example 2 above. Theorem 1 allows us to get the defining equation of the interior curve via the exterior curve. In the next section, we will show some examples as an application of this theorem.

2 Examples

Here, we construct Blaschke products of degree 5 whose interior curve consists of two ellipses. The defining equation of the exterior curve is an algebraic curve of degree four. We need to find an example of Blaschke product whose exterior curve can be resolved into two conics because the dual curve of a conic is a conic.

Example 3

Let

$$B_A(z) = z \frac{z^2 - a}{1 - az^2} \frac{z^2 - b}{1 - bz^2} \quad (0 < a, b < 1),$$

where a, b satisfy $a^3 b^3 - 2a^2 b^2 - (b^2 + a^2) + 3ab = 0$. Then the exterior curve is given by

$$\begin{aligned} E_{B_A} : & \left(a(b+1)^2 x^2 + a(b-1)^2 y^2 - 4b \right) \\ & \times \left((a^2 b^3 - ab^2 + 2b^2 + 3b - a)x^2 + (a^2 b^3 - ab^2 - 2b^2 + 3b - a)y^2 - 4b \right) = 0, \end{aligned}$$

and the interior curve consists of two ellipses

$$I_{B_A} : \left(\frac{4b}{a(b+1)^2}x^2 + \frac{4b}{a(b-1)^2}y^2 - 1 \right) \left(\frac{4a}{b(a+1)^2}x^2 + \frac{4a}{b(a-1)^2}y^2 - 1 \right) = 0,$$

where we set $z = x + iy$. Their foci are $\pm\sqrt{a}$ (the first factor) and $\pm\sqrt{b}$ (the second factor).

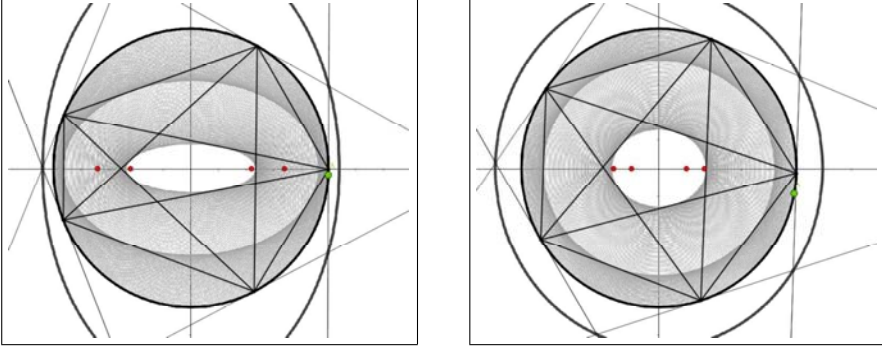


Figure 1: The interior curve I_{B_A} for $a = 0.4801 \dots$, $b = 0.2$ (left) and $a = 0.04$, $b = 0.1043 \dots$ (right). The interior curve consists of two ellipses, one is inscribed in a family of pentagons and the other is inscribed in a family of pentagrams.

The interior curve of a finite Blaschke product is closely related to the numerical range of a matrix (see [GMR18], [DGSV18], for example). In fact, the interior curve in the above example corresponds to the elliptic domain that appears in the following result of Chien and Nakazato ([CN17, Theorem 3.2]).

Theorem: Let A be a 4×4 unitary bordering matrix with real eigenvalues $\pm\sqrt{a}$, $\pm\sqrt{b}$ for some $0 < a \neq b < 1$. Then $F_A(x, y, z) := \det(x \cdot \operatorname{Re}(A) + y \cdot \operatorname{Im}(A) + z \cdot I_4)$ is a product of two quadratic forms if and only if $a^3b^3 - 2a^2b^2 - a^2 + 3ab - b^2 = 0$, where $\operatorname{Re}(A) = \frac{A+A^*}{2}$, $\operatorname{Im}(A) = \frac{A-A^*}{2i}$. In this case, the higher rank numerical range $\Lambda_k(A)$ is an elliptical disc.

Here we construct some more examples of Blaschke products of degree 5 whose interior curve consists of two ellipses.

Example 4

Let

$$B_B(z) = z \left(\frac{z-a}{1-az} \right)^2 \left(\frac{z-b}{1-bz} \right)^2 \quad (0 < a, b < 1),$$

where a, b satisfy $a^2b^3 - 2a^2b^2 - (a^2 + b^2) + 3ab = 0$. Then the exterior curve is given by

$$E_{B_B} : \left((a(b^2-1)^2 - 4b^3)x^2 + 8b^2x + a(b^2-1)^2y^2 - 4b \right) \\ \times \left(((a^2-1)b - 4a^3)x^2 + 8a^2x + (a^2-1)by^2 - 4a \right) = 0,$$

and the interior curve consists of two circles (see Figures 2, 3, and 4).

$$I_{B_B} : \left(4a(x-a)^2 + 4ay^2 - (a^2-1)^2b \right) \left(4b(x-b)^2 + 4by^2 - a(b^2-1)^2 \right) = 0,$$

where we set $z = x + iy$. Their centers coincide with non-zero zeros of B_B , i.e. a (the first factor) and b (the second factor).

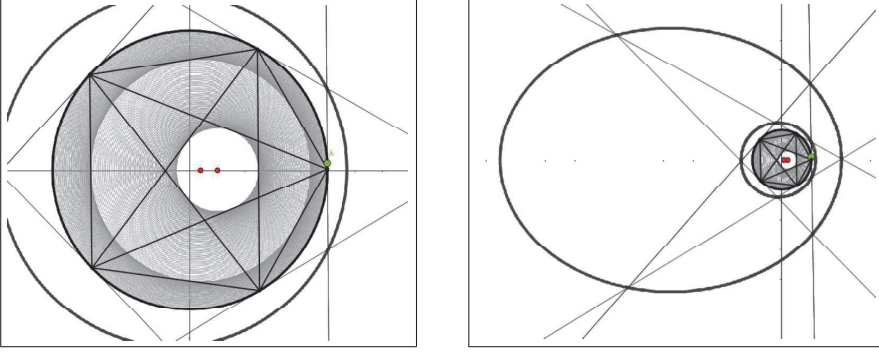


Figure 2: The interior and exterior curves of B_B for $a = 0.07746 \dots$, $b = \frac{1}{5}$. In this case, the exterior and interior curves consist of two ellipses and two circles, respectively.

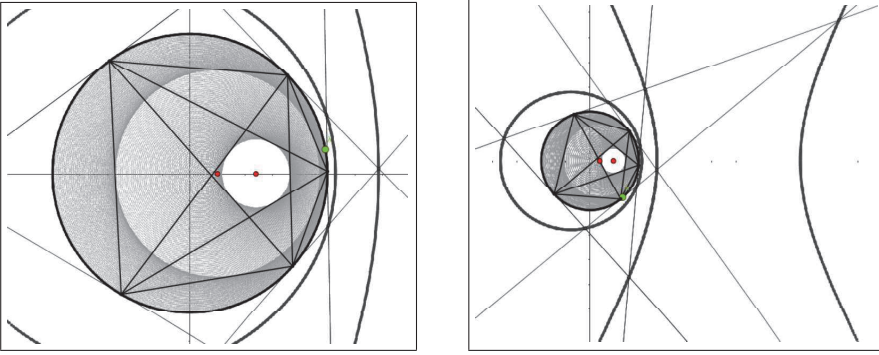


Figure 3: The interior and exterior curves of B_B for $a = 0.48012 \dots$, $b = \frac{1}{5}$. In this case, the exterior curve consists of an ellipse and a hyperbolic curve.

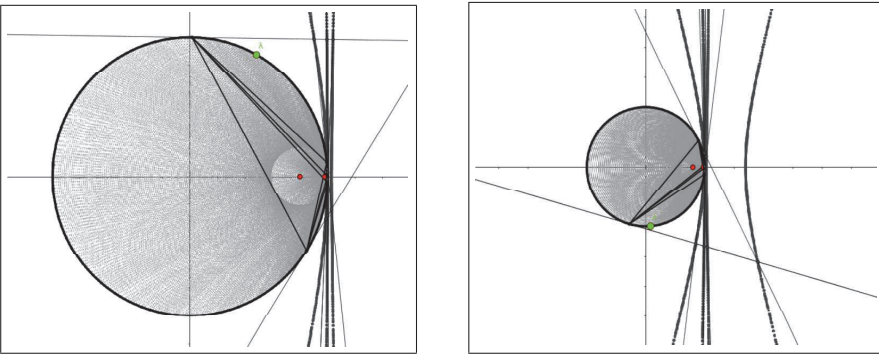


Figure 4: The interior and exterior curves of B_B for $a = 0.98697541 \dots$, $b = \frac{4}{5}$. In this case, the exterior curve consists of two hyperbolic curves.

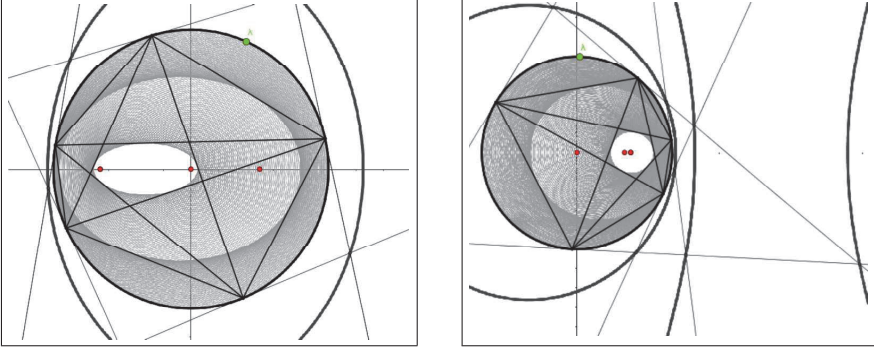


Figure 5: The interior and exterior curves of B_C . The left figure indicates the case of $P = 0$ (I_P for $a = 0.5$, $b = -0.660442 \dots$). The right figure indicates the case of $Q = 0$ (I_Q for $a = 0.5$, $b = 0.5653036 \dots$).

Example 5

Let

$$B_C(z) = z^2 \frac{z-a}{1-az} \left(\frac{z-b}{1-bz} \right)^2 \quad (-1 < a, b < 1),$$

where a, b satisfy

$$P(a, b) = a^2 + (b^3 - b)a - b^2 = 0 \quad \text{or} \quad Q(a, b) = (b^3 - 3b)a^3 - 2(b^2 - 2)a^2 + (3b^3 - b)a - 2b^2 = 0.$$

Then, the exterior curve for $P = 0$ and $Q = 0$ are given by

$$\begin{aligned} E_{P(a,b)} : & \left((a^3 + 3ba^2 - a + b)x^2 - 4a(a+b)x + (a^2 - 1)(a-b)y^2 + 4a \right) \\ & \times \left((b^2 - 1)(a+b)x^2 - 4b^2x + (b^2 - 1)(a+b)y^2 + 4b \right) = 0 \end{aligned}$$

and

$$\begin{aligned} E_{Q(a,b)} : & \left(2((b^2 + 1)a + b^3 - b)x^2 - 8abx - 2(b^2 - 1)(a-b)y^2 + 4a \right) \\ & \times \left((ba^2 + (2b^2 - 2)a - b)x^2 - 4abx + (ba^2 + (2b^2 - 2)a - b)y^2 + 4b \right) = 0 \end{aligned}$$

respectively. Here, we assume that a, b satisfy $E_{P(a,b)} \cap \partial\mathbb{D} = \emptyset$ or $E_{Q(a,b)} \cap \partial\mathbb{D} = \emptyset$. The interior curve for $P = 0$ is given by

$$\begin{aligned} I_{P(a,b)} : & \left(a(a^2 - 1)(2x - (a+b))^2 + 4a(ab - 1)y^2 + (a^2 - 1)(a-b)(ab - 1) \right) \\ & \times \left(b(b^2 - 1)(a+b)(2x - b)^2 + 4b((b^2 - 1)a - b)y^2 + (b^2 - 1)(a+b)((b^2 - 1)a - b) \right) = 0, \end{aligned}$$

if $E_{P(a,b)} \cap \partial\mathbb{D} = \emptyset$. Similarly, the interior curve for $Q = 0$ is given by

$$\begin{aligned} I_{Q(a,b)} : & \left(b(ba^2 + 2(b^2 - 1)a - b)(2x - a)^2 + 4b(2(b^2 - 1)a - b)y^2 \right. \\ & \left. + (2(b^2 - 1)a - b)(ba^2 + 2(b^2 - 1)a - b) \right) \\ & \times \left(2a(x - b)^2 + 2ay^2 - (b^2 - 1)(a - b) \right) = 0, \end{aligned}$$

if $E_{Q(a,b)} \cap \partial\mathbb{D} = \emptyset$ (see Figure 5).

The zeros of each examples B_A , B_B , and B_C are on a line passing through the origin (the zero points are placed on the real axis by suitable rotation). The following gives an example of Blaschke product whose zeros are not collinear.

Example 6

Let

$$B_D(z) = z \frac{(z-a)(z-\bar{a})(z-b)(z-\bar{b})}{(1-\bar{a}z)(1-az)(1-\bar{b}z)(1-bz)},$$

where $a \approx -0.44096 + 0.37267i$, $b \approx -0.27103 + 0.65310i$. Then, the exterior curve consists of two conics

$$E_D : \left(\frac{1}{3}z^2 + \frac{1}{3}\bar{z}^2 + vz + v\bar{z} + 4 \right) \left(\frac{1}{2}z^2 - \frac{1}{4}(3v\bar{v} + 2)z\bar{z} + \frac{1}{2}\bar{z}^2 + \bar{v}z + \bar{v}\bar{z} + 4 \right) = 0,$$

where $v = \frac{\sqrt{28}}{3}$ and $\bar{v}$ is the unique positive root of $\bar{v}^2 + 2v\bar{v} - 5 = 0$. Therefore, the interior curve is also written as two conics as follows.

$$\begin{aligned} I_D : & \left((9v^2 - 48)z^2 - 18v^2z\bar{z} + (9v^2 - 48)\bar{z}^2 - 24vz - 24v\bar{z} - 16 \right) \\ & \times \left(4(\bar{v}^2 - 8)z^2 - 8(6\bar{v}v + \bar{v}^2 + 4)z\bar{z} + 4(\bar{v}^2 - 8)\bar{z}^2 - 12(2\bar{v} + v\bar{v}^2)(z + \bar{z}) \right. \\ & \left. + 9\bar{v}^2v^2 + 12\bar{v}v - 12 \right) = 0. \end{aligned}$$

Since these two conics are included in the unit disk, they are necessarily two ellipses.

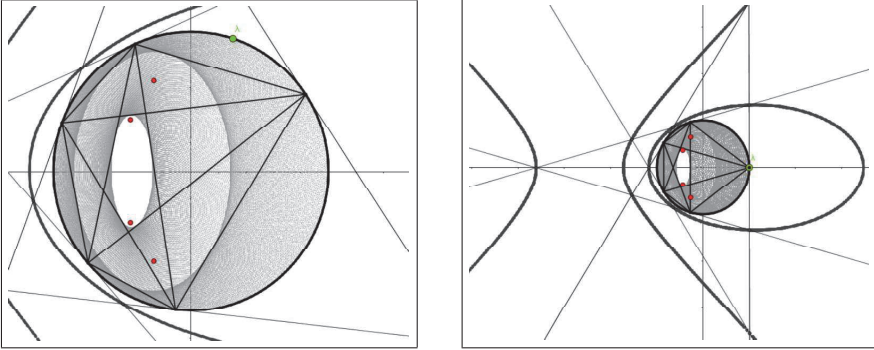


Figure 6: The interior and exterior curves for B_D . In this case, the exterior curve E_D consists of an ellipse and a hyperbola. So, the interior curve I_D consists of two ellipses.

Acknowledgements

This work was partially supported by JSPS KAKENHI Grant Number JP19K03531.

A The interior curve for a Blaschke product of degree 4

For a canonical Blaschke product

$$B(z) = z \frac{z-a}{1-\bar{a}z} \frac{z-b}{1-\bar{b}z} \frac{z-c}{1-\bar{c}z}$$

of degree 4, let σ_k be the elementary symmetric polynomials on three variables a, b, c of degree k ($k = 1, 2, 3$), i.e.

$$\sigma_1 = a + b + c, \quad \sigma_2 = ab + bc + ca, \quad \text{and} \quad \sigma_3 = abc.$$

Then, the defining equation of the interior curve is given as follows.

$$\begin{aligned} S : & (-4\bar{\sigma}_3\sigma_1^3 + \bar{\sigma}_2^2\sigma_1^2 + 18\bar{\sigma}_3\sigma_2\sigma_1 - 4\bar{\sigma}_2^3 - 27\bar{\sigma}_3^2)z^6 + (((2\bar{\sigma}_2\sigma_1^3 - 6\bar{\sigma}_3\sigma_1^2 - 8\bar{\sigma}_2^2\sigma_1 + 36\bar{\sigma}_3\sigma_2)\sigma_1 + \\ & (-4\bar{\sigma}_3\sigma_1^3 + 18\bar{\sigma}_3\sigma_2\sigma_1 - 54\bar{\sigma}_3^2)\sigma_2 + (2\bar{\sigma}_3\sigma_2\sigma_1^2 + 18\bar{\sigma}_3^2\sigma_1 - 12\bar{\sigma}_3\sigma_2^2)\sigma_3 - 4\bar{\sigma}_1^4 + 22\bar{\sigma}_2\sigma_1^2 - 18\bar{\sigma}_3\sigma_1 - \\ & 24\bar{\sigma}_2^2)\bar{z} + (-12\bar{\sigma}_3\sigma_1^3 + 4\bar{\sigma}_2^2\sigma_1^2 + 54\bar{\sigma}_3\sigma_2\sigma_1 - 16\bar{\sigma}_2^3 - 54\bar{\sigma}_3^2)\sigma_1 + (-2\bar{\sigma}_3\sigma_2\sigma_1^2 - 18\bar{\sigma}_3^2\sigma_1 + 12\bar{\sigma}_3\sigma_2^2)\sigma_2 + \\ & (12\bar{\sigma}_3^2\sigma_1^2 - 2\bar{\sigma}_3\sigma_2^2\sigma_1 - 18\bar{\sigma}_3^2\sigma_2)\sigma_3 - 2\bar{\sigma}_2\sigma_1^3 + 6\bar{\sigma}_3\sigma_1^2 + 8\bar{\sigma}_2^2\sigma_1 - 36\bar{\sigma}_3\sigma_2)z^5 + (((\bar{\sigma}_1^4 - 2\bar{\sigma}_2\sigma_1^2 + 6\bar{\sigma}_3\sigma_1 - \\ & 8\bar{\sigma}_2^2)\sigma_1^2 + ((-6\bar{\sigma}_3\sigma_1^2 + 36\bar{\sigma}_3\sigma_2)\sigma_2 + (-2\bar{\sigma}_3\sigma_1^3 + 2\bar{\sigma}_3\sigma_2\sigma_1 - 18\bar{\sigma}_3^2)\sigma_3 - 6\bar{\sigma}_1^3 + 22\bar{\sigma}_2\sigma_1 + 18\bar{\sigma}_3)\sigma_1 - 27\bar{\sigma}_3^2\sigma_2^2 + \\ & (18\bar{\sigma}_3^2\sigma_1\sigma_3 - 4\bar{\sigma}_1^4 + 20\bar{\sigma}_2\sigma_1^2 - 54\bar{\sigma}_3\sigma_1 - 12\bar{\sigma}_2^2)\sigma_2 + (\bar{\sigma}_3^2\sigma_1^2 - 12\bar{\sigma}_3^2\sigma_2)\sigma_3^2 + (4\bar{\sigma}_2\sigma_1^3 + 22\bar{\sigma}_3\sigma_1^2 - 18\bar{\sigma}_2^2\sigma_1 + \\ & 6\bar{\sigma}_3\sigma_2)\sigma_3 + 13\bar{\sigma}_1^2 - 48\bar{\sigma}_2)\bar{z}^2 + ((6\bar{\sigma}_2\sigma_1^3 - 12\bar{\sigma}_3\sigma_1^2 - 24\bar{\sigma}_2^2\sigma_1 + 72\bar{\sigma}_3\sigma_2)\sigma_1^2 + ((-10\bar{\sigma}_3\sigma_1^3 + 52\bar{\sigma}_3\sigma_2\sigma_1 - \\ & 90\bar{\sigma}_3^2)\sigma_2 + (-2\bar{\sigma}_3\sigma_2\sigma_1^2 + 30\bar{\sigma}_3^2\sigma_1 - 16\bar{\sigma}_3\sigma_2^2)\sigma_3 - 10\bar{\sigma}_1^4 + 52\bar{\sigma}_2\sigma_1^2 + 6\bar{\sigma}_3\sigma_1 - 56\bar{\sigma}_2^2)\sigma_1 - 18\bar{\sigma}_3^2\sigma_1\sigma_2^2 + \\ & ((10\bar{\sigma}_3^2\sigma_1^2 + 6\bar{\sigma}_3^2\sigma_2)\sigma_3 - 4\bar{\sigma}_2\sigma_1^3 - 28\bar{\sigma}_3\sigma_1^2 + 20\bar{\sigma}_2^2\sigma_1 - 24\bar{\sigma}_3\sigma_2)\sigma_2 + (-4\bar{\sigma}_3^2\sigma_2\sigma_1 - 18\bar{\sigma}_3^3)\sigma_3^2 + (10\bar{\sigma}_3\sigma_1^3 + \\ & 4\bar{\sigma}_2^2\sigma_1^2 + 26\bar{\sigma}_3\sigma_2\sigma_1 - 24\bar{\sigma}_2^3 - 72\bar{\sigma}_3^2)\sigma_3 + 2\bar{\sigma}_1^3 - 4\bar{\sigma}_2\sigma_1 - 72\bar{\sigma}_3)\bar{z} + (-12\bar{\sigma}_3\sigma_1^3 + 6\bar{\sigma}_2^2\sigma_1^2 + 54\bar{\sigma}_3\sigma_2\sigma_1 - \\ & 24\bar{\sigma}_2^3 - 27\bar{\sigma}_3^2)\sigma_1^2 + ((-6\bar{\sigma}_3\sigma_2\sigma_1^2 - 36\bar{\sigma}_3^2\sigma_1 + 36\bar{\sigma}_3\sigma_2^2)\sigma_2 + (24\bar{\sigma}_3^2\sigma_1^2 - 6\bar{\sigma}_3\sigma_2^2\sigma_1 - 36\bar{\sigma}_3^2\sigma_2)\sigma_3 - 6\bar{\sigma}_2\sigma_1^3 + \\ & 24\bar{\sigma}_3\sigma_1^2 + 22\bar{\sigma}_2^2\sigma_1 - 90\bar{\sigma}_3\sigma_2)\sigma_1 + (\bar{\sigma}_3^2\sigma_1^2 - 12\bar{\sigma}_3^2\sigma_2)\sigma_2^2 + ((4\bar{\sigma}_3^2\sigma_2\sigma_1 + 18\bar{\sigma}_3^3)\sigma_3 - 10\bar{\sigma}_3\sigma_1^3 + 2\bar{\sigma}_2^2\sigma_1^2 + \\ & 20\bar{\sigma}_3\sigma_2\sigma_1 - 4\bar{\sigma}_2^3 - 18\bar{\sigma}_3^2)\sigma_2 + (-12\bar{\sigma}_3^2\sigma_1 + \bar{\sigma}_3^2\sigma_2^2)\sigma_3^2 + (16\bar{\sigma}_3\sigma_2\sigma_1^2 + (-2\bar{\sigma}_2^3 - 30\bar{\sigma}_3^2)\sigma_1 - 14\bar{\sigma}_3\sigma_2^2)\sigma_3 + \\ & \bar{\sigma}_1^4 - 4\bar{\sigma}_2\sigma_1^2 - 12\bar{\sigma}_3\sigma_1 + 4\bar{\sigma}_2^2)z^4 + (((2\bar{\sigma}_1^3 - 8\bar{\sigma}_2\sigma_1 + 4\bar{\sigma}_3)\sigma_1^3 + (6\bar{\sigma}_3\sigma_1\sigma_2 + (-8\bar{\sigma}_3\sigma_1^2 + 20\bar{\sigma}_3\sigma_2)\sigma_3 - \\ & 4\bar{\sigma}_1^2 + 4\bar{\sigma}_2)\sigma_1^2 + ((-18\bar{\sigma}_3^2\sigma_3 - 8\bar{\sigma}_1^3 + 38\bar{\sigma}_2\sigma_1)\sigma_2 + 10\bar{\sigma}_3^2\sigma_1\sigma_3^2 + (6\bar{\sigma}_2\sigma_1^2 + 4\bar{\sigma}_3\sigma_1 - 36\bar{\sigma}_2^2)\sigma_3 + 22\bar{\sigma}_1)\sigma_1 - \\ & 36\bar{\sigma}_3\sigma_1\sigma_2^2 + ((20\bar{\sigma}_3\sigma_1^2 + 30\bar{\sigma}_3\sigma_2)\sigma_3 + 4\bar{\sigma}_1^2 - 48\bar{\sigma}_2)\sigma_2 - 4\bar{\sigma}_3^3\sigma_3^3 + (-18\bar{\sigma}_3\sigma_2\sigma_1 - 24\bar{\sigma}_3^2)\sigma_3^2 + (4\bar{\sigma}_1^3 + 6\bar{\sigma}_3)\sigma_3 - \\ & 32)\bar{z}^3 + ((2\bar{\sigma}_1^4 - 4\bar{\sigma}_2\sigma_1^2 + 6\bar{\sigma}_3\sigma_1 - 16\bar{\sigma}_2^2)\sigma_1^3 + ((-4\bar{\sigma}_3\sigma_1^2 + 52\bar{\sigma}_3\sigma_2)\sigma_2 + (-6\bar{\sigma}_3\sigma_1^3 + 6\bar{\sigma}_3\sigma_2\sigma_1 - 6\bar{\sigma}_3^2)\sigma_3 - \\ & 8\bar{\sigma}_1^3 + 14\bar{\sigma}_2\sigma_1 + 60\bar{\sigma}_3)\sigma_1^2 + (-36\bar{\sigma}_3^2\sigma_2^2 + (10\bar{\sigma}_3^2\sigma_1\sigma_3 - 8\bar{\sigma}_1^4 + 40\bar{\sigma}_2\sigma_1^2 - 46\bar{\sigma}_3\sigma_1 + 4\bar{\sigma}_2^2)\sigma_2 + (6\bar{\sigma}_3^2\sigma_1^2 - \\ & 2\bar{\sigma}_3^2\sigma_2)\sigma_3^2 + (8\bar{\sigma}_2\sigma_1^3 + 26\bar{\sigma}_3\sigma_1^2 - 50\bar{\sigma}_2^2\sigma_1 + 34\bar{\sigma}_3\sigma_2)\sigma_3 + 46\bar{\sigma}_1^2 - 68\bar{\sigma}_2)\sigma_1 + (-32\bar{\sigma}_3\sigma_1^2 - 12\bar{\sigma}_3\sigma_2)\sigma_2^2 + \\ & (-6\bar{\sigma}_3^3\sigma_3^3 + (8\bar{\sigma}_3\sigma_1^3 + 72\bar{\sigma}_3\sigma_2\sigma_1 - 78\bar{\sigma}_3^2)\sigma_3 - 16\bar{\sigma}_1^3 + 24\bar{\sigma}_2\sigma_1 - 132\bar{\sigma}_3)\sigma_2 - 2\bar{\sigma}_3^3\sigma_1\sigma_3^3 + (-8\bar{\sigma}_3\sigma_2\sigma_1^2 - \\ & 18\bar{\sigma}_3^2\sigma_1 - 30\bar{\sigma}_3\sigma_2^2)\sigma_3^2 + (44\bar{\sigma}_2\sigma_1^2 + 6\bar{\sigma}_3\sigma_1 - 96\bar{\sigma}_2^2)\sigma_3 - 40\bar{\sigma}_1)\bar{z}^2 + ((6\bar{\sigma}_2\sigma_1^3 - 6\bar{\sigma}_3\sigma_1^2 - 24\bar{\sigma}_2^2\sigma_1 + 36\bar{\sigma}_3\sigma_2)\sigma_1^3 + \\ & ((-8\bar{\sigma}_3\sigma_1^3 + 50\bar{\sigma}_3\sigma_2\sigma_1 - 36\bar{\sigma}_3^2)\sigma_2 + (-10\bar{\sigma}_3\sigma_2\sigma_1^2 + 12\bar{\sigma}_3^2\sigma_1 + 4\bar{\sigma}_3\sigma_2^2)\sigma_3 - 8\bar{\sigma}_1^4 + 34\bar{\sigma}_2\sigma_1^2 + 54\bar{\sigma}_3\sigma_1 - \\ & 44\bar{\sigma}_2^2)\sigma_1^2 + (-26\bar{\sigma}_3^2\sigma_1\sigma_2^2 + ((16\bar{\sigma}_3^2\sigma_1^2 - 10\bar{\sigma}_3^2\sigma_2)\sigma_3 - 4\bar{\sigma}_2\sigma_1^3 - 48\bar{\sigma}_3\sigma_1^2 + 34\bar{\sigma}_2^2\sigma_1 + 28\bar{\sigma}_3\sigma_2)\sigma_2 + (2\bar{\sigma}_3^2\sigma_2\sigma_1 - \\ & 6\bar{\sigma}_3^3)\sigma_3^2 + (16\bar{\sigma}_3\sigma_1^3 + 4\bar{\sigma}_2^2\sigma_1^2 + 52\bar{\sigma}_3\sigma_2\sigma_1 - 52\bar{\sigma}_2^3 - 78\bar{\sigma}_3^2)\sigma_3 + 14\bar{\sigma}_1^3 - 6\bar{\sigma}_2\sigma_1 - 132\bar{\sigma}_3)\sigma_1 + (6\bar{\sigma}_3^3\sigma_3 - \\ & 8\bar{\sigma}_3\sigma_1^3 - 4\bar{\sigma}_3\sigma_2\sigma_1 - 60\bar{\sigma}_3^2)\sigma_2^2 + (-8\bar{\sigma}_3^3\sigma_1\sigma_3^2 + (20\bar{\sigma}_3\sigma_2\sigma_1^2 + 4\bar{\sigma}_3^2\sigma_1 + 22\bar{\sigma}_3\sigma_2^2)\sigma_3 - 8\bar{\sigma}_1^4 + 20\bar{\sigma}_2\sigma_1^2 - 32\bar{\sigma}_3\sigma_1 - \\ & 16\bar{\sigma}_2^2)\sigma_2 + 2\bar{\sigma}_3^3\sigma_2\sigma_3^3 + (-8\bar{\sigma}_3^2\sigma_1^2 - 12\bar{\sigma}_3\sigma_2^2\sigma_1 - 42\bar{\sigma}_3^2\sigma_2)\sigma_3^2 + (8\bar{\sigma}_2\sigma_1^3 + 10\bar{\sigma}_3\sigma_1^2 - 138\bar{\sigma}_3\sigma_2)\sigma_3 - 20\bar{\sigma}_1^2 + \\ & 16\bar{\sigma}_2)\bar{z} + (-4\bar{\sigma}_3\sigma_1^3 + 4\bar{\sigma}_2^2\sigma_1^2 + 18\bar{\sigma}_3\sigma_2\sigma_1 - 16\bar{\sigma}_2^3)\sigma_1^3 + ((-6\bar{\sigma}_3\sigma_2\sigma_1^2 - 18\bar{\sigma}_3^2\sigma_1 + 36\bar{\sigma}_3\sigma_2^2)\sigma_2 + (12\bar{\sigma}_3^2\sigma_1^2 - \\ & 6\bar{\sigma}_3\sigma_2^2\sigma_1 - 18\bar{\sigma}_3^2\sigma_2)\sigma_3 - 6\bar{\sigma}_2\sigma_1^3 + 30\bar{\sigma}_3\sigma_1^2 + 18\bar{\sigma}_2^2\sigma_1 - 72\bar{\sigma}_3\sigma_2)\sigma_1^2 + ((2\bar{\sigma}_3^2\sigma_1^2 - 24\bar{\sigma}_3^2\sigma_2)\sigma_2^2 + ((8\bar{\sigma}_3^2\sigma_2\sigma_1 + \\ & 18\bar{\sigma}_3^3)\sigma_3 - 20\bar{\sigma}_3\sigma_1^3 + 6\bar{\sigma}_2^2\sigma_1^2 + 44\bar{\sigma}_3\sigma_2\sigma_1 - 12\bar{\sigma}_2^3)\sigma_2 + (-12\bar{\sigma}_3^3\sigma_1 + 2\bar{\sigma}_3^2\sigma_2^2)\sigma_3^2 + (32\bar{\sigma}_3\sigma_2\sigma_1^2 + (-6\bar{\sigma}_2^3 - \\ & 54\bar{\sigma}_3^2)\sigma_1 - 26\bar{\sigma}_3\sigma_2^2)\sigma_3 + 2\bar{\sigma}_1^4 - 4\bar{\sigma}_2\sigma_1^2 - 42\bar{\sigma}_3\sigma_1 + 12\bar{\sigma}_2^2)\sigma_1 + 4\bar{\sigma}_3^3\sigma_3^3 + (-2\bar{\sigma}_3^3\sigma_1\sigma_3 - 4\bar{\sigma}_3\sigma_2\sigma_1^2 - 28\bar{\sigma}_3^2\sigma_1 + \\ & 12\bar{\sigma}_3\sigma_2^2)\sigma_2^2 + (-2\bar{\sigma}_3^3\sigma_2\sigma_3^2 + (20\bar{\sigma}_3^2\sigma_1^2 + 2\bar{\sigma}_3\sigma_2^2\sigma_1 - 8\bar{\sigma}_3^2\sigma_2)\sigma_3 - 4\bar{\sigma}_2\sigma_1^3 + 16\bar{\sigma}_3\sigma_1^2 + 8\bar{\sigma}_2^2\sigma_1 - 44\bar{\sigma}_3\sigma_2)\sigma_2 + \\ & 4\bar{\sigma}_3^4\sigma_3^4 + (-26\bar{\sigma}_3^2\sigma_2\sigma_1^2 + 2\bar{\sigma}_3\sigma_2^3 + 24\bar{\sigma}_3^3)\sigma_3^2 + (-2\bar{\sigma}_3\sigma_1^3 + 4\bar{\sigma}_2^2\sigma_1^2 - 20\bar{\sigma}_3\sigma_2\sigma_1 - 6\bar{\sigma}_3^2)\sigma_3 - 8\bar{\sigma}_2\sigma_1 + 32\bar{\sigma}_3)z^3 + \\ & (((\bar{\sigma}_1^2 - 4\bar{\sigma}_2)\sigma_1^4 + (4\bar{\sigma}_3\sigma_2 - 2\bar{\sigma}_3\sigma_1\sigma_3 - 6\bar{\sigma}_1)\sigma_1^3 + ((-2\bar{\sigma}_1^2 + 20\bar{\sigma}_2)\sigma_2 + \bar{\sigma}_3^2\sigma_3^2 + (-6\bar{\sigma}_2\sigma_1 + 22\bar{\sigma}_3)\sigma_3 + 13)\sigma_1^2 + \\ & (-18\bar{\sigma}_3\sigma_2^2 + (2\bar{\sigma}_3\sigma_1\sigma_3 + 22\bar{\sigma}_1)\sigma_2 + 18\bar{\sigma}_3\sigma_2\sigma_3^2 + (6\bar{\sigma}_1^2 - 54\bar{\sigma}_2)\sigma_3)\sigma_1 + (-8\bar{\sigma}_1^2 - 12\bar{\sigma}_2)\sigma_2^2 + (-12\bar{\sigma}_3^2\sigma_3^2 + \\ & (36\bar{\sigma}_2\sigma_1 + 6\bar{\sigma}_3)\sigma_3 - 48)\sigma_2 + (-18\bar{\sigma}_3\sigma_1 - 27\bar{\sigma}_2^2)\sigma_2^2 + 18\bar{\sigma}_1\sigma_3)\bar{z}^4 + ((2\bar{\sigma}_1^3 - 8\bar{\sigma}_2\sigma_1)\sigma_1^4 + (8\bar{\sigma}_3\sigma_1\sigma_2 + (-6\bar{\sigma}_3\sigma_1^2 + \\ & 8\bar{\sigma}_3\sigma_2)\sigma_3 - 8\bar{\sigma}_1^2 - 16\bar{\sigma}_2)\sigma_1^3 + ((-8\bar{\sigma}_3^2\sigma_3 - 4\bar{\sigma}_1^3 + 40\bar{\sigma}_2\sigma_1 + 44\bar{\sigma}_3)\sigma_2 + 6\bar{\sigma}_3^2\sigma_1\sigma_3^2 + (-4\bar{\sigma}_2\sigma_1^2 + 26\bar{\sigma}_3\sigma_1 - \\ & 32\bar{\sigma}_2^2)\sigma_3 + 46\bar{\sigma}_1)\sigma_1^2 + (-50\bar{\sigma}_3\sigma_1\sigma_2^2 + ((6\bar{\sigma}_3\sigma_1^2 + 72\bar{\sigma}_3\sigma_2)\sigma_3 + 14\bar{\sigma}_1^2 + 24\bar{\sigma}_2)\sigma_2 - 2\bar{\sigma}_3^3\sigma_3^3 + (10\bar{\sigma}_3\sigma_2\sigma_1 - \\ & 18\bar{\sigma}_3^2)\sigma_3^2 + (6\bar{\sigma}_1^3 - 46\bar{\sigma}_2\sigma_1 + 6\bar{\sigma}_3)\sigma_3 - 40)\sigma_1 + (-30\bar{\sigma}_3^2\sigma_3 - 16\bar{\sigma}_1^3 + 4\bar{\sigma}_2\sigma_1 - 96\bar{\sigma}_3)\sigma_2^2 + (-2\bar{\sigma}_3^2\sigma_1\sigma_3^2 + (52\bar{\sigma}_2\sigma_1^2 + \\ & 34\bar{\sigma}_3\sigma_1 - 12\bar{\sigma}_2^2)\sigma_3 - 68\bar{\sigma}_1)\sigma_2 - 6\bar{\sigma}_3^2\sigma_2\sigma_3^2 + (-6\bar{\sigma}_3\sigma_1^2 - 36\bar{\sigma}_2^2\sigma_1 - 78\bar{\sigma}_3\sigma_2)\sigma_3^2 + (60\bar{\sigma}_1^2 - 132\bar{\sigma}_2)\sigma_3)\bar{z}^3 \end{aligned}$$

$$\begin{aligned}
& + ((\bar{\sigma}_1^4 - 2\bar{\sigma}_2\bar{\sigma}_1^2 - 8\bar{\sigma}_2^2)\sigma_1^4 + ((2\bar{\sigma}_3\bar{\sigma}_1^2 + 16\bar{\sigma}_3\bar{\sigma}_2)\sigma_2 + (-4\bar{\sigma}_3\bar{\sigma}_1^3 + 4\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1)\sigma_3 - 4\bar{\sigma}_1^3 - 20\bar{\sigma}_2\bar{\sigma}_1 + 30\bar{\sigma}_3)\sigma_1^3 + \\
& (-8\bar{\sigma}_3^2\sigma_2^2 + (-4\bar{\sigma}_3^2\bar{\sigma}_1\sigma_3 - 2\bar{\sigma}_1^4 + 24\bar{\sigma}_2\bar{\sigma}_1^2 + 32\bar{\sigma}_3\bar{\sigma}_1 + 8\bar{\sigma}_2^2)\sigma_2 + (6\bar{\sigma}_3^2\bar{\sigma}_1^2 - 2\bar{\sigma}_3^2\bar{\sigma}_2)\sigma_3^2 + (2\bar{\sigma}_2\bar{\sigma}_1^3 + 16\bar{\sigma}_3\bar{\sigma}_1^2 - \\
& 44\bar{\sigma}_2^2\bar{\sigma}_1 + 28\bar{\sigma}_3\bar{\sigma}_2)\sigma_3 + 54\bar{\sigma}_1^2 - 26\bar{\sigma}_2^2)\sigma_1^2 + ((-44\bar{\sigma}_3\bar{\sigma}_1^2 + 22\bar{\sigma}_3\bar{\sigma}_2)\sigma_2^2 + (2\bar{\sigma}_3^3\sigma_3^2 + (4\bar{\sigma}_3\bar{\sigma}_1^3 + 78\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1 - \\
& 40\bar{\sigma}_3^2)\sigma_3 - 20\bar{\sigma}_1^3 + 76\bar{\sigma}_2\bar{\sigma}_1 - 70\bar{\sigma}_3)\sigma_2 - 4\bar{\sigma}_3^3\bar{\sigma}_1\sigma_3^2 + (-4\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1^2 - 20\bar{\sigma}_3^2\bar{\sigma}_1 + 6\bar{\sigma}_3\bar{\sigma}_2^2)\sigma_3^2 + (32\bar{\sigma}_2\bar{\sigma}_1^2 + \\
& 34\bar{\sigma}_3\bar{\sigma}_1 - 96\bar{\sigma}_2^2)\sigma_3 - 64\bar{\sigma}_1)\sigma_1 - 30\bar{\sigma}_3^2\sigma_3^2 + (6\bar{\sigma}_3^2\bar{\sigma}_1\sigma_3 - 8\bar{\sigma}_1^4 + 8\bar{\sigma}_2\bar{\sigma}_1^2 - 96\bar{\sigma}_3\bar{\sigma}_1 - 8\bar{\sigma}_2^2)\sigma_2^2 + ((-2\bar{\sigma}_3^2\bar{\sigma}_1^2 - \\
& 26\bar{\sigma}_3^2\bar{\sigma}_2)\sigma_3^2 + (16\bar{\sigma}_2\bar{\sigma}_1^3 + 28\bar{\sigma}_3\bar{\sigma}_1^2 + 22\bar{\sigma}_2^2\bar{\sigma}_1 - 20\bar{\sigma}_3\bar{\sigma}_2)\sigma_3 - 26\bar{\sigma}_1^2 - 8\bar{\sigma}_2)\sigma_2 + \bar{\sigma}_3^4\sigma_3^4 + (2\bar{\sigma}_3^2\bar{\sigma}_2\bar{\sigma}_1 + 8\bar{\sigma}_3^3)\sigma_3^3 + \\
& (-8\bar{\sigma}_2^2\bar{\sigma}_1^2 - 40\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1 - 30\bar{\sigma}_3^2 - 84\bar{\sigma}_3^2)\sigma_3^2 + (30\bar{\sigma}_1^3 - 70\bar{\sigma}_2\bar{\sigma}_1 - 184\bar{\sigma}_3)\sigma_3 + 16)\bar{z}^2 + ((2\bar{\sigma}_2\bar{\sigma}_1^3 - 8\bar{\sigma}_2^2\bar{\sigma}_1)\sigma_1^4 + \\
& ((-2\bar{\sigma}_3\bar{\sigma}_1^3 + 16\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1)\sigma_2 + (-6\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1^2 + 8\bar{\sigma}_3\bar{\sigma}_2^2)\sigma_3 - 2\bar{\sigma}_1^4 + 30\bar{\sigma}_3\bar{\sigma}_1 - 16\bar{\sigma}_2^2)\sigma_1^3 + (-8\bar{\sigma}_3^2\bar{\sigma}_1\sigma_2^2 + ((6\bar{\sigma}_3^2\bar{\sigma}_1^2 - \\
& 16\bar{\sigma}_3^2\bar{\sigma}_2)\sigma_3 + 4\bar{\sigma}_2\bar{\sigma}_1^3 - 16\bar{\sigma}_3\bar{\sigma}_1^2 + 8\bar{\sigma}_2^2\bar{\sigma}_1 + 60\bar{\sigma}_3\bar{\sigma}_2)\sigma_2 + 6\bar{\sigma}_3^2\bar{\sigma}_2\bar{\sigma}_1\sigma_3^2 + (6\bar{\sigma}_3\bar{\sigma}_1^3 - 4\bar{\sigma}_2^2\bar{\sigma}_1^2 + 34\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1 - \\
& 32\bar{\sigma}_2^2 - 30\bar{\sigma}_3^2)\sigma_3 + 16\bar{\sigma}_1^3 + 14\bar{\sigma}_2\bar{\sigma}_1 - 96\bar{\sigma}_3)\sigma_1^2 + ((8\bar{\sigma}_3^3\sigma_3 - 12\bar{\sigma}_3\bar{\sigma}_1^3 + 8\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1 - 44\bar{\sigma}_3^2)\sigma_2^2 + (-6\bar{\sigma}_3^3\bar{\sigma}_1\sigma_2^2 + \\
& (16\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1^2 - 2\bar{\sigma}_3^3\bar{\sigma}_1 + 40\bar{\sigma}_3\bar{\sigma}_2^2)\sigma_3 - 12\bar{\sigma}_1^4 + 32\bar{\sigma}_2\bar{\sigma}_1^2 + 14\bar{\sigma}_3\bar{\sigma}_1 - 16\bar{\sigma}_2^2)\sigma_2 - 2\bar{\sigma}_3^3\bar{\sigma}_2\sigma_3^2 + (-6\bar{\sigma}_3^3\bar{\sigma}_1^2 - \\
& 4\bar{\sigma}_3\bar{\sigma}_2^2\bar{\sigma}_1 - 34\bar{\sigma}_3^2\bar{\sigma}_2)\sigma_3^2 + (12\bar{\sigma}_2\bar{\sigma}_1^3 + 26\bar{\sigma}_3\bar{\sigma}_1^2 - 10\bar{\sigma}_2^2\bar{\sigma}_1 - 142\bar{\sigma}_3\bar{\sigma}_2)\sigma_3 - 38\bar{\sigma}_1^2 + 16\bar{\sigma}_2)\sigma_1 - 16\bar{\sigma}_3^2\bar{\sigma}_1\sigma_2^2 + \\
& ((12\bar{\sigma}_3^2\bar{\sigma}_1^2 - 8\bar{\sigma}_3^2\bar{\sigma}_2)\sigma_3 - 8\bar{\sigma}_2\bar{\sigma}_1^3 - 32\bar{\sigma}_3\bar{\sigma}_1^2 + 20\bar{\sigma}_2^2\bar{\sigma}_1 - 40\bar{\sigma}_3\bar{\sigma}_2)\sigma_2^2 + (2\bar{\sigma}_3^4\sigma_3^4 + (-20\bar{\sigma}_3^2\bar{\sigma}_2\bar{\sigma}_1 + 18\bar{\sigma}_3^3)\sigma_3^2 + \\
& (12\bar{\sigma}_3\bar{\sigma}_1^3 + 16\bar{\sigma}_2^2\bar{\sigma}_1^2 + 20\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1 - 20\bar{\sigma}_2^2 - 6\bar{\sigma}_3^2)\sigma_3 + 8\bar{\sigma}_1^3 - 36\bar{\sigma}_2\bar{\sigma}_1 + 40\bar{\sigma}_3)\sigma_2 + (2\bar{\sigma}_3^3\bar{\sigma}_1 + 8\bar{\sigma}_3^2\bar{\sigma}_2^2)\sigma_3^3 + \\
& (-12\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1^2 + (-8\bar{\sigma}_2^2 - 42\bar{\sigma}_3^2)\bar{\sigma}_1 - 10\bar{\sigma}_3\bar{\sigma}_2^2)\sigma_3^2 + (-138\bar{\sigma}_3\bar{\sigma}_1 + 20\bar{\sigma}_2^2)\sigma_3 + 16\bar{\sigma}_1)\bar{z} + (\bar{\sigma}_2^2\bar{\sigma}_1^2 - 4\bar{\sigma}_2^3)\sigma_1^4 + \\
& ((-2\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1^2 + 12\bar{\sigma}_3\bar{\sigma}_2^2)\sigma_2 - 2\bar{\sigma}_3\bar{\sigma}_2^2\bar{\sigma}_1\sigma_3 - 2\bar{\sigma}_2\bar{\sigma}_1^3 + 12\bar{\sigma}_3\bar{\sigma}_1^2 + 2\bar{\sigma}_2^2\bar{\sigma}_1 - 18\bar{\sigma}_3\bar{\sigma}_2)\sigma_1^3 + ((\bar{\sigma}_3^2\bar{\sigma}_1^2 - 12\bar{\sigma}_3^2\bar{\sigma}_2)\sigma_2^2 + \\
& (4\bar{\sigma}_3^2\bar{\sigma}_2\bar{\sigma}_1\sigma_3 - 10\bar{\sigma}_3\bar{\sigma}_1^3 + 6\bar{\sigma}_2^2\bar{\sigma}_1^2 + 28\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1 - 12\bar{\sigma}_2^3 + 18\bar{\sigma}_3^2)\sigma_2 + \bar{\sigma}_3^3\bar{\sigma}_2^2\sigma_3^2 + (16\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1^2 + (-6\bar{\sigma}_2^3 - \\
& 24\bar{\sigma}_3^2)\bar{\sigma}_1 - 10\bar{\sigma}_3\bar{\sigma}_2^2)\sigma_3 + \bar{\sigma}_1^4 + 4\bar{\sigma}_2\bar{\sigma}_1^2 - 42\bar{\sigma}_3\bar{\sigma}_1 + 13\bar{\sigma}_2^2)\sigma_1^2 + (4\bar{\sigma}_3^3\sigma_3^3 + (-2\bar{\sigma}_3^3\bar{\sigma}_1\sigma_3 - 8\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1^2 - 30\bar{\sigma}_3^2\bar{\sigma}_1 + \\
& 24\bar{\sigma}_3\bar{\sigma}_2^2)\sigma_2^2 + (-2\bar{\sigma}_3^3\bar{\sigma}_2\sigma_3^2 + (20\bar{\sigma}_3^2\bar{\sigma}_1^2 + 4\bar{\sigma}_3\bar{\sigma}_2^2\bar{\sigma}_1 - 12\bar{\sigma}_3^2\bar{\sigma}_2)\sigma_3 - 8\bar{\sigma}_2\bar{\sigma}_1^3 + 36\bar{\sigma}_3\bar{\sigma}_1^2 + 14\bar{\sigma}_2^2\bar{\sigma}_1 - 70\bar{\sigma}_3\bar{\sigma}_2)\sigma_2 + \\
& (-26\bar{\sigma}_3^2\bar{\sigma}_2\bar{\sigma}_1 + 4\bar{\sigma}_3\bar{\sigma}_2^2 + 12\bar{\sigma}_3^3)\sigma_3^2 + (-2\bar{\sigma}_3\bar{\sigma}_1^3 + 8\bar{\sigma}_2^2\bar{\sigma}_1^2 - 30\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1 - 2\bar{\sigma}_2^3 - 6\bar{\sigma}_3^3)\sigma_3 - 2\bar{\sigma}_1^3 - 16\bar{\sigma}_2\bar{\sigma}_1 + \\
& 48\bar{\sigma}_3)\sigma_1 + (2\bar{\sigma}_3^2\bar{\sigma}_1^2 - 12\bar{\sigma}_3^2\bar{\sigma}_2)\sigma_2^2 + (\bar{\sigma}_3^4\sigma_3^4 + (2\bar{\sigma}_3^2\bar{\sigma}_2\bar{\sigma}_1 + 22\bar{\sigma}_3^3)\sigma_3 - 8\bar{\sigma}_3\bar{\sigma}_1^3 + \bar{\sigma}_2^2\bar{\sigma}_1^2 + 2\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1 + 13\bar{\sigma}_3^2)\sigma_2^2 + \\
& ((-10\bar{\sigma}_3^3\bar{\sigma}_1 - 4\bar{\sigma}_3^2\bar{\sigma}_2^2)\sigma_3^2 + (24\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1^2 + (-2\bar{\sigma}_2^2 - 40\bar{\sigma}_3^2)\bar{\sigma}_1 - 14\bar{\sigma}_3\bar{\sigma}_2^2)\sigma_3 + 2\bar{\sigma}_1^4 - 4\bar{\sigma}_2\bar{\sigma}_1^2 - 4\bar{\sigma}_3\bar{\sigma}_1)\sigma_2 + \\
& 12\bar{\sigma}_3^3\bar{\sigma}_2\sigma_3^2 + (\bar{\sigma}_3^2\bar{\sigma}_1^2 - 16\bar{\sigma}_3\bar{\sigma}_2^2\bar{\sigma}_1 + \bar{\sigma}_2^4 + 30\bar{\sigma}_3^2\bar{\sigma}_2)\sigma_3^2 + (-2\bar{\sigma}_2\bar{\sigma}_1^3 - 14\bar{\sigma}_3\bar{\sigma}_1^2 + 4\bar{\sigma}_2^2\bar{\sigma}_1 + 12\bar{\sigma}_3\bar{\sigma}_2)\sigma_3 + 4\bar{\sigma}_1^2)\bar{z}^2 + \\
& ((-4\bar{\sigma}_1^4 + (2\bar{\sigma}_1\sigma_2 - 4\bar{\sigma}_2\bar{\sigma}_3)\sigma_1^3 + ((2\bar{\sigma}_3\sigma_3 + 22)\sigma_2 - 6\bar{\sigma}_1\sigma_3)\sigma_1^2 + (-8\bar{\sigma}_1\sigma_2^2 + 18\bar{\sigma}_2\sigma_3\sigma_2 + 18\bar{\sigma}_3\sigma_3^2 - 18\sigma_3)\sigma_1 + \\
& (-12\bar{\sigma}_3\sigma_3 - 24)\sigma_2^2 + 36\bar{\sigma}_1\sigma_3\sigma_2 - 54\bar{\sigma}_2\sigma_3^2)\bar{z}^5 + (-10\bar{\sigma}_1\sigma_1^4 + ((6\bar{\sigma}_1^2 - 4\bar{\sigma}_2)\sigma_2 + (-10\bar{\sigma}_2\bar{\sigma}_1 + 10\bar{\sigma}_3)\sigma_3 + 2)\sigma_1^3 + \\
& (4\bar{\sigma}_3\sigma_2^2 + (-2\bar{\sigma}_3\bar{\sigma}_1\sigma_3 + 52\bar{\sigma}_1)\sigma_2 + 10\bar{\sigma}_3\bar{\sigma}_2\sigma_3^2 + (-12\bar{\sigma}_1^2 - 28\bar{\sigma}_2)\sigma_3)\sigma_1^2 + ((-24\bar{\sigma}_1^2 + 20\bar{\sigma}_2)\sigma_2^2 + (-4\bar{\sigma}_3^2\sigma_3^2 + \\
& (52\bar{\sigma}_2\bar{\sigma}_1 + 26\bar{\sigma}_3)\sigma_3 - 4)\sigma_2 + (30\bar{\sigma}_3\bar{\sigma}_1 - 18\bar{\sigma}_2^2)\sigma_3^2 + 6\bar{\sigma}_1\sigma_3)\sigma_1 - 24\bar{\sigma}_3\sigma_3^2 + (-16\bar{\sigma}_3\bar{\sigma}_1\sigma_3 - 56\bar{\sigma}_1)\sigma_2^2 + (6\bar{\sigma}_3\bar{\sigma}_2\sigma_3^2 + \\
& (72\bar{\sigma}_1^2 - 24\bar{\sigma}_2)\sigma_3)\sigma_2 - 18\bar{\sigma}_3^2\sigma_3^2 + (-90\bar{\sigma}_2\bar{\sigma}_1 - 72\bar{\sigma}_3)\sigma_3^2 - 72\sigma_3)\bar{z}^4 + ((-8\bar{\sigma}_1^2 - 8\bar{\sigma}_2)\sigma_1^4 + ((6\bar{\sigma}_1^3 - 4\bar{\sigma}_2\bar{\sigma}_1 + \\
& 8\bar{\sigma}_3)\sigma_2 + (-8\bar{\sigma}_2\bar{\sigma}_1^2 + 16\bar{\sigma}_3\bar{\sigma}_1 - 8\bar{\sigma}_2^2)\sigma_3 + 14\bar{\sigma}_1)\sigma_1^3 + (4\bar{\sigma}_3\bar{\sigma}_1\sigma_2^2 + ((-10\bar{\sigma}_3\bar{\sigma}_1^2 + 20\bar{\sigma}_3\bar{\sigma}_2)\sigma_3 + 34\bar{\sigma}_1^2 + 20\bar{\sigma}_2)\sigma_2 + \\
& (16\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1 - 8\bar{\sigma}_3^2)\sigma_3^2 + (-6\bar{\sigma}_1^3 - 48\bar{\sigma}_2\bar{\sigma}_1 + 10\bar{\sigma}_3)\sigma_3 - 20)\sigma_1^2 + ((-12\bar{\sigma}_3^2\sigma_3 - 24\bar{\sigma}_3^3 + 34\bar{\sigma}_2\bar{\sigma}_1)\sigma_2^2 + (2\bar{\sigma}_3^2\bar{\sigma}_1\sigma_3^2 + \\
& (50\bar{\sigma}_2\bar{\sigma}_1^2 + 52\bar{\sigma}_3\bar{\sigma}_1 - 4\bar{\sigma}_2^2)\sigma_3 - 6\bar{\sigma}_1)\sigma_2 - 8\bar{\sigma}_3^2\bar{\sigma}_2\sigma_3^2 + (12\bar{\sigma}_3\bar{\sigma}_1^2 - 26\bar{\sigma}_2^2\bar{\sigma}_1 + 4\bar{\sigma}_3\bar{\sigma}_2)\sigma_3^2 + (54\bar{\sigma}_1^2 - 32\bar{\sigma}_2)\sigma_3)\sigma_1 - \\
& 52\bar{\sigma}_3\bar{\sigma}_1\sigma_2^2 + ((4\bar{\sigma}_3\bar{\sigma}_1^2 + 22\bar{\sigma}_3\bar{\sigma}_2)\sigma_3 - 44\bar{\sigma}_1^2 - 16\bar{\sigma}_2)\sigma_2^2 + (2\bar{\sigma}_3^3\sigma_3^3 + (-10\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1 - 42\bar{\sigma}_3^2)\sigma_3^2 + (36\bar{\sigma}_1^3 + \\
& 28\bar{\sigma}_2\bar{\sigma}_1 - 138\bar{\sigma}_3)\sigma_3 - 4)\sigma_2 + (-6\bar{\sigma}_3^2\bar{\sigma}_1 + 6\bar{\sigma}_3\bar{\sigma}_2^2)\sigma_3^2 + (-36\bar{\sigma}_2\bar{\sigma}_1^2 - 78\bar{\sigma}_3\bar{\sigma}_1 - 60\bar{\sigma}_2^2)\sigma_3^2 - 132\bar{\sigma}_1\sigma_3)\bar{z}^3 + \\
& ((-2\bar{\sigma}_1^3 - 12\bar{\sigma}_2\bar{\sigma}_1)\sigma_1^4 + ((2\bar{\sigma}_1^4 + 4\bar{\sigma}_2\bar{\sigma}_1^2 + 12\bar{\sigma}_3\bar{\sigma}_1 - 8\bar{\sigma}_2^2)\sigma_2 + (-2\bar{\sigma}_2\bar{\sigma}_1^3 + 6\bar{\sigma}_3\bar{\sigma}_1^2 - 12\bar{\sigma}_2^2\bar{\sigma}_1 + 12\bar{\sigma}_3\bar{\sigma}_2)\sigma_3 + \\
& 16\bar{\sigma}_1^2 + 8\bar{\sigma}_2)\sigma_1^3 + ((-4\bar{\sigma}_3\bar{\sigma}_1^2 + 16\bar{\sigma}_3\bar{\sigma}_2)\sigma_2^2 + ((-6\bar{\sigma}_3\bar{\sigma}_1^3 + 16\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1 - 12\bar{\sigma}_3^2)\sigma_3 + 32\bar{\sigma}_2\bar{\sigma}_1)\sigma_2 + (6\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1^2 - \\
& 6\bar{\sigma}_3^2\bar{\sigma}_1 + 12\bar{\sigma}_3\bar{\sigma}_2^2)\sigma_3^2 + (-16\bar{\sigma}_2\bar{\sigma}_1^2 + 26\bar{\sigma}_3\bar{\sigma}_1 - 32\bar{\sigma}_2^2)\sigma_3 - 38\bar{\sigma}_1)\sigma_1^2 + (-8\bar{\sigma}_3^2\sigma_3^2 + (-4\bar{\sigma}_3^2\bar{\sigma}_1\sigma_3 - 8\bar{\sigma}_1^4 + \\
& 8\bar{\sigma}_2\bar{\sigma}_1^2 - 10\bar{\sigma}_3\bar{\sigma}_1 + 20\bar{\sigma}_2^2)\sigma_2^2 + ((6\bar{\sigma}_3^2\bar{\sigma}_1^2 - 20\bar{\sigma}_3^2\bar{\sigma}_2)\sigma_3^2 + (16\bar{\sigma}_2\bar{\sigma}_1^3 + 34\bar{\sigma}_3\bar{\sigma}_1^2 + 8\bar{\sigma}_2^2\bar{\sigma}_1 + 20\bar{\sigma}_3\bar{\sigma}_2)\sigma_3 + 14\bar{\sigma}_1^2 - \\
& 36\bar{\sigma}_2)\sigma_2 + (-6\bar{\sigma}_3^2\bar{\sigma}_2\bar{\sigma}_1 + 2\bar{\sigma}_3^3)\sigma_3^2 + (-8\bar{\sigma}_2^2\bar{\sigma}_1^2 - 2\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1 - 16\bar{\sigma}_2^2 - 42\bar{\sigma}_3^2)\sigma_3^2 + (30\bar{\sigma}_1^3 + 14\bar{\sigma}_2\bar{\sigma}_1 - 138\bar{\sigma}_3)\sigma_3 + \\
& 16)\sigma_1 + (-32\bar{\sigma}_3\bar{\sigma}_1^2 - 20\bar{\sigma}_3\bar{\sigma}_2)\sigma_2^2 + (8\bar{\sigma}_3^3\sigma_3^3 + (8\bar{\sigma}_3\bar{\sigma}_1^3 + 40\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1 - 10\bar{\sigma}_3^2)\sigma_3 - 16\bar{\sigma}_1^3 - 16\bar{\sigma}_2\bar{\sigma}_1 + 20\bar{\sigma}_3)\sigma_2^2 + \\
& (-2\bar{\sigma}_3^3\bar{\sigma}_1\sigma_3^2 + (-16\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1^2 - 34\bar{\sigma}_3^2\bar{\sigma}_1 - 8\bar{\sigma}_3\bar{\sigma}_2^2)\sigma_3^2 + (60\bar{\sigma}_2\bar{\sigma}_1^2 - 142\bar{\sigma}_3\bar{\sigma}_1 - 40\bar{\sigma}_2^2)\sigma_3 + 16\bar{\sigma}_1)\sigma_2 + 2\bar{\sigma}_3^3\bar{\sigma}_2\sigma_3^2 + \\
& (8\bar{\sigma}_3\bar{\sigma}_2^2\bar{\sigma}_1 + 18\bar{\sigma}_3^2\bar{\sigma}_2)\sigma_3^2 + (-30\bar{\sigma}_3\bar{\sigma}_1^2 - 44\bar{\sigma}_2^2\bar{\sigma}_1 - 6\bar{\sigma}_3\bar{\sigma}_2)\sigma_3^2 + (-96\bar{\sigma}_1^2 + 40\bar{\sigma}_2)\sigma_3)\bar{z}^2 + ((-4\bar{\sigma}_2\bar{\sigma}_1^2 - 4\bar{\sigma}_2^2)\sigma_1^4 + \\
& ((4\bar{\sigma}_2\bar{\sigma}_1^3 + 4\bar{\sigma}_3\bar{\sigma}_1^2 - 6\bar{\sigma}_2^2\bar{\sigma}_1 + 8\bar{\sigma}_3\bar{\sigma}_2)\sigma_2 + (-4\bar{\sigma}_2^2\bar{\sigma}_1^2 + 8\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1 - 4\bar{\sigma}_2^3)\sigma_3 + 4\bar{\sigma}_1^3 + 18\bar{\sigma}_2\bar{\sigma}_1 - 24\bar{\sigma}_3)\sigma_1^3 + \\
& ((-4\bar{\sigma}_3\bar{\sigma}_1^3 + 12\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1 - 4\bar{\sigma}_3^2)\sigma_2^2 + ((-4\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1^2 - 8\bar{\sigma}_3^2\bar{\sigma}_1 + 18\bar{\sigma}_3\bar{\sigma}_2^2)\sigma_3 - 4\bar{\sigma}_1^4 + 8\bar{\sigma}_2\bar{\sigma}_1^2 + 26\bar{\sigma}_3\bar{\sigma}_1 - 2\bar{\sigma}_2^2)\sigma_2 + \\
& (8\bar{\sigma}_3\bar{\sigma}_2^2\bar{\sigma}_1 - 4\bar{\sigma}_3^2\bar{\sigma}_2)\sigma_3^2 + (4\bar{\sigma}_2\bar{\sigma}_1^3 + 10\bar{\sigma}_3\bar{\sigma}_1^2 - 18\bar{\sigma}_2^2\bar{\sigma}_1 - 18\bar{\sigma}_3\bar{\sigma}_2)\sigma_3 - 24\bar{\sigma}_1^2 + 4\bar{\sigma}_2)\sigma_1^2 + ((-6\bar{\sigma}_3^2\bar{\sigma}_1\sigma_2^2 + \\
& ((8\bar{\sigma}_3^2\bar{\sigma}_1^2 - 24\bar{\sigma}_3^2\bar{\sigma}_2)\sigma_3 - 6\bar{\sigma}_2\bar{\sigma}_1^3 - 18\bar{\sigma}_3\bar{\sigma}_1^2 + 22\bar{\sigma}_2^2\bar{\sigma}_1)\sigma_2^2 + ((-4\bar{\sigma}_3^2\bar{\sigma}_2\bar{\sigma}_1 + 4\bar{\sigma}_3^3)\sigma_3^2 + (8\bar{\sigma}_3\bar{\sigma}_1^3 + 12\bar{\sigma}_2^2\bar{\sigma}_1^2 + \\
& 52\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1 - 22\bar{\sigma}_2^2 - 26\bar{\sigma}_3^2)\sigma_3 + 18\bar{\sigma}_1^3 - 42\bar{\sigma}_2\bar{\sigma}_1 + 4\bar{\sigma}_3)\sigma_2 - 4\bar{\sigma}_3^2\bar{\sigma}_2^2\sigma_3^2 + (-8\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1^2 + (-6\bar{\sigma}_2^2 - 32\bar{\sigma}_3^2)\bar{\sigma}_1 - \\
& 14\bar{\sigma}_3\bar{\sigma}_2^2)\sigma_3^2 + (26\bar{\sigma}_2\bar{\sigma}_1^2 - 128\bar{\sigma}_3\bar{\sigma}_1 + 18\bar{\sigma}_2^2)\sigma_3 + 16\bar{\sigma}_1)\sigma_1 + (10\bar{\sigma}_3^3\sigma_3 - 4\bar{\sigma}_3\bar{\sigma}_1^3 - 22\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1 + 2\bar{\sigma}_3^2)\sigma_2^2 + \\
& (-4\bar{\sigma}_3^3\bar{\sigma}_1\sigma_3^2 + (18\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1^2 - 14\bar{\sigma}_3^2\bar{\sigma}_1 + 22\bar{\sigma}_3\bar{\sigma}_2^2)\sigma_3 - 4\bar{\sigma}_1^4 - 2\bar{\sigma}_2\bar{\sigma}_1^2 + 18\bar{\sigma}_3\bar{\sigma}_1)\sigma_2^2 + (4\bar{\sigma}_3^3\bar{\sigma}_2\sigma_3^2 + (-4\bar{\sigma}_3^2\bar{\sigma}_1^2 - \\
& 24\bar{\sigma}_3\bar{\sigma}_2^2\bar{\sigma}_1 + 4\bar{\sigma}_3^2\bar{\sigma}_2)\sigma_3^2 + (8\bar{\sigma}_2\bar{\sigma}_1^3 - 18\bar{\sigma}_3\bar{\sigma}_1^2 - 98\bar{\sigma}_3\bar{\sigma}_2)\sigma_3 + 4\bar{\sigma}_1^2)\sigma_2 + (4\bar{\sigma}_3^2\bar{\sigma}_2\bar{\sigma}_1 + 10\bar{\sigma}_3\bar{\sigma}_2^2 + 18\bar{\sigma}_3^3)\sigma_3^2 + \\
& (-4\bar{\sigma}_2^2\bar{\sigma}_1^2 - 26\bar{\sigma}_3\bar{\sigma}_2\bar{\sigma}_1 + 2\bar{\sigma}_2^3 + 72\bar{\sigma}_3^2)\sigma_3^2 + (-24\bar{\sigma}_1^3 + 4\bar{\sigma}_2\bar{\sigma}_1 + 72\bar{\sigma}_3)\sigma_3)\bar{z}
\end{aligned}$$

$$\begin{aligned}
& -2\overline{\sigma_2^2}\overline{\sigma_1^4} + ((2\overline{\sigma_2^2}\overline{\sigma_1^2} + 4\overline{\sigma_3}\overline{\sigma_2}\overline{\sigma_1} - 4\overline{\sigma_2^3})\sigma_2 + (-2\overline{\sigma_2^3}\overline{\sigma_1} + 2\overline{\sigma_3}\overline{\sigma_2^2})\sigma_3 + 4\overline{\sigma_2}\overline{\sigma_1^2} - 12\overline{\sigma_3}\overline{\sigma_1} + 6\overline{\sigma_2^2})\sigma_1^3 + \\
& ((-4\overline{\sigma_3}\overline{\sigma_2}\overline{\sigma_1^2} - 2\overline{\sigma_3^2}\overline{\sigma_1} + 12\overline{\sigma_3}\overline{\sigma_2^2})\sigma_2^2 + ((2\overline{\sigma_3}\overline{\sigma_2^2}\overline{\sigma_1} - 4\overline{\sigma_3^2}\overline{\sigma_2})\sigma_3 - 4\overline{\sigma_2}\overline{\sigma_1^3} + 20\overline{\sigma_3}\overline{\sigma_1^2} + 4\overline{\sigma_2^2}\overline{\sigma_1} - 28\overline{\sigma_3}\overline{\sigma_2})\sigma_2 + \\
& 2\overline{\sigma_3}\overline{\sigma_2^2}\sigma_3^2 + (4\overline{\sigma_2^2}\overline{\sigma_1^2} - 10\overline{\sigma_3}\overline{\sigma_2}\overline{\sigma_1} - 4\overline{\sigma_2^3} + 12\overline{\sigma_3^2})\sigma_3 - 2\overline{\sigma_1^3} - 10\overline{\sigma_2}\overline{\sigma_1} + 24\overline{\sigma_3})\sigma_1^2 + ((2\overline{\sigma_3^2}\overline{\sigma_1^2} - 12\overline{\sigma_3^2}\overline{\sigma_2})\sigma_3^2 + \\
& ((2\overline{\sigma_3^2}\overline{\sigma_2}\overline{\sigma_1} + 2\overline{\sigma_3^3})\sigma_3 - 8\overline{\sigma_3}\overline{\sigma_1^3} + 2\overline{\sigma_2^2}\overline{\sigma_1^2} + 6\overline{\sigma_3}\overline{\sigma_2}\overline{\sigma_1} + 22\overline{\sigma_3^2})\sigma_2^2 + (-4\overline{\sigma_3^2}\overline{\sigma_2^2}\sigma_3^2 + (24\overline{\sigma_3}\overline{\sigma_2}\overline{\sigma_1^2} + (-4\overline{\sigma_2^3} - \\
& 38\overline{\sigma_3^2})\overline{\sigma_1} - 6\overline{\sigma_3}\overline{\sigma_2^2})\sigma_3 + 2\overline{\sigma_1^4} - 22\overline{\sigma_3}\overline{\sigma_1})\sigma_2 + (-16\overline{\sigma_3}\overline{\sigma_2^2}\overline{\sigma_1} + 2\overline{\sigma_2^4} + 6\overline{\sigma_3^2}\overline{\sigma_2})\sigma_2^2 + (-2\overline{\sigma_2}\overline{\sigma_1^3} - 16\overline{\sigma_3}\overline{\sigma_1^2} + \\
& 12\overline{\sigma_2^2}\overline{\sigma_1} - 6\overline{\sigma_3}\overline{\sigma_2})\sigma_3 + 4\overline{\sigma_1^2})\sigma_1 + 4\overline{\sigma_3^3}\sigma_4^2 + (-2\overline{\sigma_3^3}\overline{\sigma_1}\sigma_3 - 2\overline{\sigma_3}\overline{\sigma_2}\overline{\sigma_1^2} - 10\overline{\sigma_3^2}\overline{\sigma_1})\sigma_2^2 + (2\overline{\sigma_3^3}\overline{\sigma_2}\sigma_3^2 + (8\overline{\sigma_3^2}\overline{\sigma_1^2} + \\
& 4\overline{\sigma_3}\overline{\sigma_2^2}\overline{\sigma_1} + 10\overline{\sigma_3^2}\overline{\sigma_2})\sigma_3 - 2\overline{\sigma_2}\overline{\sigma_1^3} + 10\overline{\sigma_3}\overline{\sigma_1^2})\sigma_2^2 + ((-20\overline{\sigma_3^2}\overline{\sigma_2}\overline{\sigma_1} - 2\overline{\sigma_3}\overline{\sigma_2^3} + 18\overline{\sigma_3^3})\sigma_3^2 + (-2\overline{\sigma_3}\overline{\sigma_1^3} + 4\overline{\sigma_2^2}\overline{\sigma_1^2} - \\
& 28\overline{\sigma_3}\overline{\sigma_2}\overline{\sigma_1} - 18\overline{\sigma_3^2})\sigma_3)\sigma_2 + 12\overline{\sigma_3^2}\overline{\sigma_2^2}\sigma_3^3 + (2\overline{\sigma_3}\overline{\sigma_2}\overline{\sigma_1^2} + (-2\overline{\sigma_2^3} + 18\overline{\sigma_3^2})\overline{\sigma_1} + 6\overline{\sigma_3}\overline{\sigma_2^2})\sigma_3^2 + (-8\overline{\sigma_2}\overline{\sigma_1^2} + \\
& 36\overline{\sigma_3}\overline{\sigma_1})\sigma_3)\sigma_2 + (-4\sigma_3\sigma_1^3 + \sigma_2^2\sigma_1^2 + 18\sigma_3\sigma_2\sigma_1 - 4\sigma_2^3 - 27\sigma_3^2)\overline{\sigma_1^2} + ((-2\sigma_2 - 12\overline{\sigma_1}\sigma_3)\sigma_1^3 + (4\overline{\sigma_1}\sigma_2^2 - 2\overline{\sigma_2}\sigma_3\sigma_2 + \\
& 12\overline{\sigma_3}\sigma_2^2 + 6\sigma_3)\sigma_1^2 + ((-2\overline{\sigma_3}\sigma_3 + 8)\sigma_2^2 + 54\overline{\sigma_1}\sigma_3\sigma_2 - 18\overline{\sigma_2}\sigma_3^2)\sigma_1 - 16\overline{\sigma_1}\sigma_3^2 + 12\overline{\sigma_2}\sigma_3\sigma_2^2 + (-18\overline{\sigma_3}\sigma_3^2 - 36\sigma_3)\sigma_2 - \\
& 54\overline{\sigma_1}\sigma_3^2)\overline{\sigma_1^2} + (\sigma_1^4 + (-6\overline{\sigma_1}\sigma_2 + (-12\overline{\sigma_1^2} - 10\overline{\sigma_2})\sigma_3)\sigma_1^3 + ((6\overline{\sigma_1^2} + 2\overline{\sigma_2})\sigma_2^2 + ((-6\overline{\sigma_2}\overline{\sigma_1} + 16\overline{\sigma_3})\sigma_3 - 4)\sigma_2 + \\
& (24\overline{\sigma_3}\overline{\sigma_1} + \overline{\sigma_2^2})\sigma_3^2 + 24\overline{\sigma_1}\sigma_3)\sigma_1^2 + (-2\overline{\sigma_3}\sigma_3^2 + (-6\overline{\sigma_3}\overline{\sigma_1}\sigma_3 + 22\overline{\sigma_1})\sigma_2^2 + (4\overline{\sigma_3}\overline{\sigma_2}\sigma_3^2 + (54\overline{\sigma_1^2} + 20\overline{\sigma_2})\sigma_3)\sigma_2 - \\
& 12\overline{\sigma_3^2}\sigma_3^3 + (-36\overline{\sigma_2}\overline{\sigma_1} - 30\overline{\sigma_3})\sigma_3^2 - 12\sigma_3)\sigma_1 + (-24\overline{\sigma_1^2} - 4\overline{\sigma_2})\sigma_3^2 + (\overline{\sigma_3^2}\sigma_3^2 + (36\overline{\sigma_2}\overline{\sigma_1} - 14\overline{\sigma_3})\sigma_3 + 4)\sigma_2^2 + \\
& ((-36\overline{\sigma_3}\overline{\sigma_1} - 12\overline{\sigma_2^2})\sigma_3^2 - 90\overline{\sigma_1}\sigma_3)\sigma_2 + 18\overline{\sigma_3}\overline{\sigma_2}\sigma_3^3 + (-27\overline{\sigma_1^2} - 18\overline{\sigma_2})\sigma_3^2)\overline{\sigma_1^2} + (2\overline{\sigma_1}\sigma_4^2 + ((-6\overline{\sigma_1^2} - 4\overline{\sigma_2})\sigma_2 + \\
& (-4\overline{\sigma_1^3} - 20\overline{\sigma_2}\overline{\sigma_1} - 2\overline{\sigma_3})\sigma_3)\sigma_1^3 + ((4\overline{\sigma_1^3} + 6\overline{\sigma_2}\overline{\sigma_1} + 4\overline{\sigma_3})\sigma_2^2 + ((-6\overline{\sigma_2}\overline{\sigma_1^2} - 4\overline{\sigma_2^2})\sigma_3 - 4\overline{\sigma_1})\sigma_2 + \\
& (12\overline{\sigma_3}\overline{\sigma_1^2} + 2\overline{\sigma_2^2}\overline{\sigma_1} + 20\overline{\sigma_3}\overline{\sigma_2})\sigma_3^2 + (30\overline{\sigma_1^2} + 16\overline{\sigma_2})\sigma_3)\sigma_1^2 + (-6\overline{\sigma_3}\overline{\sigma_1}\sigma_3^2 + ((-6\overline{\sigma_3}\overline{\sigma_1^2} + 2\overline{\sigma_3}\overline{\sigma_2})\sigma_3 + 18\overline{\sigma_1^2} + \\
& 8\overline{\sigma_2})\sigma_2^2 + ((8\overline{\sigma_3}\overline{\sigma_2}\overline{\sigma_1} - 26\overline{\sigma_3^2})\sigma_3^2 + (18\overline{\sigma_1^3} + 44\overline{\sigma_2}\overline{\sigma_1} - 20\overline{\sigma_3})\sigma_3 - 8)\sigma_2 + (-12\overline{\sigma_3}\overline{\sigma_1} - 2\overline{\sigma_3}\overline{\sigma_2^2})\sigma_3^3 + (-18\overline{\sigma_2}\overline{\sigma_1^2} - \\
& 54\overline{\sigma_3}\overline{\sigma_1} - 28\overline{\sigma_2^2})\sigma_3^2 - 42\overline{\sigma_1}\sigma_3)\sigma_1 + (2\overline{\sigma_3^2}\sigma_3 - 16\overline{\sigma_1^3} - 12\overline{\sigma_2}\overline{\sigma_1})\sigma_2^2 + (2\overline{\sigma_3^2}\overline{\sigma_1}\sigma_3^2 + (36\overline{\sigma_2}\overline{\sigma_1^2} - 26\overline{\sigma_3}\overline{\sigma_1} + 12\overline{\sigma_2^2})\sigma_3 + \\
& 12\overline{\sigma_1})\sigma_2^2 + (-2\overline{\sigma_3^2}\overline{\sigma_2}\sigma_3^2 + (-18\overline{\sigma_3}\overline{\sigma_1^2} - 24\overline{\sigma_2^2}\overline{\sigma_1} - 8\overline{\sigma_3}\overline{\sigma_2})\sigma_3^2 + (-72\overline{\sigma_1^2} - 44\overline{\sigma_2})\sigma_3)\sigma_2 + 4\overline{\sigma_3^3}\sigma_4^2 + (18\overline{\sigma_3}\overline{\sigma_2}\overline{\sigma_1} + \\
& 4\overline{\sigma_2^3} + 24\overline{\sigma_3^2})\sigma_3^3 - 6\overline{\sigma_3}\sigma_3^2 + 32\sigma_3^2)\overline{\sigma_1^2} + ((\overline{\sigma_1^2} + 2\overline{\sigma_2})\sigma_4^2 + ((-2\overline{\sigma_1^3} - 8\overline{\sigma_2}\overline{\sigma_1} - 2\overline{\sigma_3})\sigma_2 + (-10\overline{\sigma_2}\overline{\sigma_1^2} - 2\overline{\sigma_3}\overline{\sigma_1} - \\
& 8\overline{\sigma_2^2})\sigma_3 - 2\overline{\sigma_1})\sigma_1^3 + ((\overline{\sigma_1^4} + 6\overline{\sigma_2}\overline{\sigma_1^2} + 8\overline{\sigma_3}\overline{\sigma_1} + \overline{\sigma_2^2})\sigma_2^2 + ((-2\overline{\sigma_2}\overline{\sigma_1^3} + 16\overline{\sigma_3}\overline{\sigma_1^2} - 8\overline{\sigma_2^2}\overline{\sigma_1} + 24\overline{\sigma_3}\overline{\sigma_2})\sigma_3 + 4\overline{\sigma_1^2} - \\
& 4\overline{\sigma_2})\sigma_2 + (\overline{\sigma_2^2}\overline{\sigma_1^2} + 20\overline{\sigma_3}\overline{\sigma_2}\overline{\sigma_1} + 2\overline{\sigma_2^3} + \overline{\sigma_3^2})\sigma_3^2 + (12\overline{\sigma_1^3} + 36\overline{\sigma_2}\overline{\sigma_1} - 14\overline{\sigma_3})\sigma_3 + 4)\sigma_1^2 + ((-6\overline{\sigma_3}\overline{\sigma_1^2} - 2\overline{\sigma_3}\overline{\sigma_2})\sigma_3^2 + \\
& ((-2\overline{\sigma_3}\overline{\sigma_1^3} + 4\overline{\sigma_3}\overline{\sigma_2}\overline{\sigma_1} - 16\overline{\sigma_3^2})\sigma_3 + 2\overline{\sigma_1^3} + 14\overline{\sigma_2}\overline{\sigma_1} + 4\overline{\sigma_3})\sigma_2^2 + ((4\overline{\sigma_3}\overline{\sigma_2}\overline{\sigma_1^2} - 26\overline{\sigma_3^2}\overline{\sigma_1} + 2\overline{\sigma_3}\overline{\sigma_2^2})\sigma_3^2 + (28\overline{\sigma_2}\overline{\sigma_1^2} - \\
& 30\overline{\sigma_3}\overline{\sigma_1} + 2\overline{\sigma_2^2})\sigma_3 - 16\overline{\sigma_1})\sigma_2 + (-2\overline{\sigma_3}\overline{\sigma_2^2}\overline{\sigma_1} - 10\overline{\sigma_3^2}\overline{\sigma_2})\sigma_3^2 + (-24\overline{\sigma_3}\overline{\sigma_1^2} - 30\overline{\sigma_2^2}\overline{\sigma_1} - 40\overline{\sigma_3}\overline{\sigma_2})\sigma_3^2 + (-42\overline{\sigma_1^2} - \\
& 4\overline{\sigma_2})\sigma_3)\sigma_1 + \overline{\sigma_3^2}\sigma_4^2 + (4\overline{\sigma_3^2}\overline{\sigma_1}\sigma_3 - 4\overline{\sigma_1^4} - 12\overline{\sigma_2}\overline{\sigma_1^2} - 2\overline{\sigma_3}\overline{\sigma_1})\sigma_3^2 + ((\overline{\sigma_3^2}\overline{\sigma_1^2} - 4\overline{\sigma_3^2}\overline{\sigma_2})\sigma_3^2 + (12\overline{\sigma_2}\overline{\sigma_1^3} - 10\overline{\sigma_3}\overline{\sigma_1^2} + \\
& 24\overline{\sigma_2^2}\overline{\sigma_1} - 14\overline{\sigma_3}\overline{\sigma_2})\sigma_3 + 13\overline{\sigma_1^2})\sigma_2^2 + ((-2\overline{\sigma_3^2}\overline{\sigma_2}\overline{\sigma_1} + 12\overline{\sigma_3^3})\sigma_3^3 + (-12\overline{\sigma_2^2}\overline{\sigma_1^2} - 12\overline{\sigma_3}\overline{\sigma_2}\overline{\sigma_1} - 12\overline{\sigma_2^2} + 30\overline{\sigma_3^2})\sigma_3^2 + \\
& (-18\overline{\sigma_1^3} - 70\overline{\sigma_2}\overline{\sigma_1} + 12\overline{\sigma_3})\sigma_3)\sigma_2 + \overline{\sigma_1^2}\overline{\sigma_2^2}\sigma_3^4 + ((4\overline{\sigma_2^3} + 12\overline{\sigma_3^2})\overline{\sigma_1} + 22\overline{\sigma_3}\overline{\sigma_2^2})\sigma_3^3 + (18\overline{\sigma_2}\overline{\sigma_1^2} - 6\overline{\sigma_3}\overline{\sigma_1} + 13\overline{\sigma_2^2})\sigma_3^2 + \\
& 48\overline{\sigma_1}\sigma_3)\overline{\sigma_1^2} + (2\overline{\sigma_2}\overline{\sigma_1}\sigma_4^2 + ((-4\overline{\sigma_2}\overline{\sigma_1^2} - 2\overline{\sigma_3}\overline{\sigma_1} - 2\overline{\sigma_2^2})\sigma_2 + (-8\overline{\sigma_2^2}\overline{\sigma_1} - 2\overline{\sigma_3}\overline{\sigma_2})\sigma_3 - 2\overline{\sigma_1^2})\sigma_1^3 + ((2\overline{\sigma_2}\overline{\sigma_1^3} + \\
& 4\overline{\sigma_3}\overline{\sigma_1^2} + 2\overline{\sigma_2^2}\overline{\sigma_1} + 4\overline{\sigma_3}\overline{\sigma_2})\sigma_2^2 + ((-4\overline{\sigma_2^2}\overline{\sigma_1^2} + 24\overline{\sigma_3}\overline{\sigma_2}\overline{\sigma_1} - 2\overline{\sigma_2^3} + 2\overline{\sigma_3^2})\sigma_3 + 4\overline{\sigma_1^3} - 8\overline{\sigma_3})\sigma_2 + (2\overline{\sigma_2^2}\overline{\sigma_1} + \\
& 8\overline{\sigma_3}\overline{\sigma_2^2})\sigma_3^2 + (20\overline{\sigma_2}\overline{\sigma_1^2} - 16\overline{\sigma_3}\overline{\sigma_1} + 10\overline{\sigma_2^2})\sigma_3 + 4\overline{\sigma_1})\sigma_1^2 + ((-2\overline{\sigma_3}\overline{\sigma_1^3} - 4\overline{\sigma_3}\overline{\sigma_2}\overline{\sigma_1} - 2\overline{\sigma_3^2})\sigma_3^2 + ((2\overline{\sigma_3}\overline{\sigma_2}\overline{\sigma_1^2} - \\
& 16\overline{\sigma_3^2}\overline{\sigma_1} + 4\overline{\sigma_3}\overline{\sigma_2^2})\sigma_3 - 2\overline{\sigma_1^4} + 4\overline{\sigma_2}\overline{\sigma_1^2} + 12\overline{\sigma_3}\overline{\sigma_1})\sigma_2^2 + ((2\overline{\sigma_3}\overline{\sigma_2^2}\overline{\sigma_1} - 20\overline{\sigma_3^2}\overline{\sigma_2})\sigma_3^2 + (4\overline{\sigma_2}\overline{\sigma_1^3} - 10\overline{\sigma_3}\overline{\sigma_1^2} + 6\overline{\sigma_2^2}\overline{\sigma_1} - \\
& 28\overline{\sigma_3}\overline{\sigma_2})\sigma_3 - 10\overline{\sigma_1^2})\sigma_2 - 2\overline{\sigma_3}\overline{\sigma_2^2}\sigma_3^3 + (-2\overline{\sigma_2^2}\overline{\sigma_1^2} - 38\overline{\sigma_3}\overline{\sigma_2}\overline{\sigma_1} - 10\overline{\sigma_2^3} + 18\overline{\sigma_3^2})\sigma_3^2 + (-12\overline{\sigma_1^3} - 22\overline{\sigma_2}\overline{\sigma_1} + \\
& 36\overline{\sigma_3})\sigma_3)\sigma_1 + 2\overline{\sigma_3^2}\overline{\sigma_1}\sigma_2^2 + ((2\overline{\sigma_3^2}\overline{\sigma_1^2} - 2\overline{\sigma_3^2}\overline{\sigma_2})\sigma_3 - 4\overline{\sigma_2}\overline{\sigma_1^3} - 4\overline{\sigma_3}\overline{\sigma_1^2})\sigma_3^2 + ((-4\overline{\sigma_3^2}\overline{\sigma_2}\overline{\sigma_1} + 12\overline{\sigma_3^3})\sigma_3^2 + (2\overline{\sigma_3}\overline{\sigma_1^3} + \\
& 12\overline{\sigma_2^2}\overline{\sigma_1^2} - 6\overline{\sigma_3}\overline{\sigma_2}\overline{\sigma_1} + 6\overline{\sigma_3^2})\sigma_3 + 6\overline{\sigma_1^3})\sigma_2^2 + (2\overline{\sigma_3^2}\overline{\sigma_2^2}\sigma_3^3 + (-4\overline{\sigma_3}\overline{\sigma_2}\overline{\sigma_1^2} + (-12\overline{\sigma_2^3} + 6\overline{\sigma_3^2})\overline{\sigma_1} + 10\overline{\sigma_3}\overline{\sigma_2^2})\sigma_3^2 + \\
& (-28\overline{\sigma_2}\overline{\sigma_1^2} - 6\overline{\sigma_3}\overline{\sigma_1})\sigma_3)\sigma_2 + (2\overline{\sigma_3}\overline{\sigma_2^2}\overline{\sigma_1} + 4\overline{\sigma_2^4} + 18\overline{\sigma_3^2}\overline{\sigma_2})\sigma_3^3 + (12\overline{\sigma_3}\overline{\sigma_1^2} + 22\overline{\sigma_2^2}\overline{\sigma_1} - 18\overline{\sigma_3}\overline{\sigma_2})\sigma_3^2 + 24\overline{\sigma_1^2}\sigma_3)\overline{\sigma_1^2} + \\
& \overline{\sigma_2^2}\sigma_4^2 + ((-2\overline{\sigma_2^2}\overline{\sigma_1} - 2\overline{\sigma_3}\overline{\sigma_2})\sigma_2 - 2\overline{\sigma_2^2}\sigma_3 - 2\overline{\sigma_2}\overline{\sigma_1} + 4\overline{\sigma_3})\sigma_1^3 + ((\overline{\sigma_2^2}\overline{\sigma_1^2} + 4\overline{\sigma_3}\overline{\sigma_2}\overline{\sigma_1} + \overline{\sigma_3^2})\sigma_2^2 + ((-2\overline{\sigma_2^2}\overline{\sigma_1} + \\
& 8\overline{\sigma_3}\overline{\sigma_2^2})\sigma_3 + 4\overline{\sigma_2}\overline{\sigma_1^2} - 10\overline{\sigma_3}\overline{\sigma_1})\sigma_2 + \overline{\sigma_2^4}\sigma_3^2 + (8\overline{\sigma_2^2}\overline{\sigma_1} - 6\overline{\sigma_3}\overline{\sigma_2})\sigma_3 + \overline{\sigma_1^2})\sigma_1^2 + ((-2\overline{\sigma_3}\overline{\sigma_2}\overline{\sigma_1^2} - 2\overline{\sigma_3^2}\overline{\sigma_1})\sigma_2^2 + \\
& ((4\overline{\sigma_3}\overline{\sigma_2^2}\overline{\sigma_1} - 10\overline{\sigma_3^2}\overline{\sigma_2})\sigma_3 - 2\overline{\sigma_2}\overline{\sigma_1^3} + 8\overline{\sigma_3}\overline{\sigma_1^2})\sigma_2^2 + (-2\overline{\sigma_3}\overline{\sigma_2^2}\sigma_3^2 + (4\overline{\sigma_2^2}\overline{\sigma_1^2} - 26\overline{\sigma_3}\overline{\sigma_2}\overline{\sigma_1} + 18\overline{\sigma_3^2})\sigma_3 - 2\overline{\sigma_1^3})\sigma_2 + \\
& (-2\overline{\sigma_2^3}\overline{\sigma_1} - 6\overline{\sigma_3}\overline{\sigma_2^2})\sigma_3^2 + (-10\overline{\sigma_2}\overline{\sigma_1^2} + 18\overline{\sigma_3}\overline{\sigma_1})\sigma_3)\sigma_1 + \overline{\sigma_3^2}\overline{\sigma_1^2}\sigma_4^2 + ((-2\overline{\sigma_3^2}\overline{\sigma_2}\overline{\sigma_1} + 4\overline{\sigma_3^3})\sigma_3 - 2\overline{\sigma_3}\overline{\sigma_1^3})\sigma_2^2 + \\
& (\overline{\sigma_3^2}\overline{\sigma_2^2}\sigma_3^2 + (8\overline{\sigma_3}\overline{\sigma_2}\overline{\sigma_1^2} - 6\overline{\sigma_3^2}\overline{\sigma_1})\sigma_3 + \overline{\sigma_1^4})\sigma_2^2 + ((-10\overline{\sigma_3}\overline{\sigma_2^2}\overline{\sigma_1} + 18\overline{\sigma_3^2}\overline{\sigma_2})\sigma_3^2 + (-2\overline{\sigma_2}\overline{\sigma_1^3} - 6\overline{\sigma_3}\overline{\sigma_1^2})\sigma_3)\sigma_2 + \\
& 4\overline{\sigma_3}\overline{\sigma_2^2}\sigma_3^3 + (\overline{\sigma_2^2}\overline{\sigma_1^2} + 18\overline{\sigma_3}\overline{\sigma_2}\overline{\sigma_1} - 27\overline{\sigma_3^2})\sigma_3^2 + 4\overline{\sigma_1^3}\sigma_3 = 0.
\end{aligned}$$

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