

An example of unirational varieties in characteristic p .

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This note is to supplement a computation in my talk at Kyoto whose content can be found in [1].

Let X denote the Fermat variety of dimension $2r$ and of degree n , defined by the equation

$$(1) \quad \sum_{i=1}^{2r+2} x_i^n = 0$$

in the projective space $\mathbb{P}^{2r+1}$ of characteristic $p \neq 2$. When $n \not\equiv 0 \pmod{p}$, X is a non-singular irreducible variety.

An irreducible variety (defined over a field k) is called unirational if its function field is contained in a purely transcendental extension of k . For simplicity, we assume k to be algebraically closed.

Proposition. Assume that $p^\nu \equiv -1 \pmod{n}$ for some integer ν . Then the Fermat variety X is unirational.

Proof. Put $q = p^\nu$ and $q + 1 = n \cdot a$ with some integer a .

Then the map

$$(2) \quad (x_i) \longrightarrow (x_i^a)$$

defines a surjective morphism of the Fermat variety of degree $q+1$ onto that of degree n .

Hence it suffices to prove the case $n = q+1$.

By a change of coordinates, we rewrite (1) as

$$(3) \quad \sum_{i=1}^{r+1} (x_{2i-1}^{q+1} - x_{2i}^{q+1}) = 0.$$

Putting

$$(4) \quad \begin{cases} y_{2i-1} = x_{2i-1} + x_{2i} \\ y_{2i} = x_{2i-1} - x_{2i} \end{cases} \quad (1 \leq i \leq r+1),$$

we have

$$(5) \quad \sum_{i=1}^{r+1} y_{2i-1} y_{2i} (y_{2i-1}^{q-1} + y_{2i}^{q-1}) = 0.$$

Setting $y_{2r+2} = 1$, we consider $y_1, \dots, y_{2r+1}$ as the inhomogeneous coordinates on X . The function field $K = k(X)$ is given by

$$(6) \quad K = k(y_1, \dots, y_{2r+1}).$$

Put

$$(7) \quad \begin{cases} y_2 = y_1 \cdot u \\ y_{2i} = y_{2i-1} \cdot t_{2i} \cdot u \\ y_{2r+1} = u \cdot v \end{cases} \quad (2 \leq i \leq r).$$

Then we have

$$(8) \quad K = k(y_1, u, v, y_{2i-1}, t_{2i} \quad (2 \leq i \leq r))$$

with the relation

$$(9) \quad y_1^{q+1}(1+u^{q-1}) + \sum_{i=2}^r y_{2i-1}^{q+1} t_{2i} (1+u^{q-1} t_{2i}^{q-1}) + v(u^{q-1} v^{q-1} + 1) = 0,$$

i.e.

$$(10) \quad u^{q-1}(y_1^{q+1} + \sum_{i=2}^r y_{2i-1}^{q+1} t_{2i}^q + v^q) = -(y_1^{q+1} + \sum_{i=2}^r y_{2i-1}^{q+1} t_{2i} + v).$$

Let

$$(11) \quad \begin{cases} (y_{2i-1})^{1/q} = t_{2i-1} & (1 \leq i \leq r) \\ t_2 = u(t_1^{q+1} + \sum_{i=2}^r t_{2i-1}^{q+1} t_{2i} + v). \end{cases}$$

Then the field

$$(12) \quad \begin{aligned} K' &= k(t_1, u, v, t_{2i-1}, t_{2i} \quad (2 \leq i \leq r)) \\ &= k(t_1, t_2, v, t_{2i-1}, t_{2i} \quad (2 \leq i \leq r)) \end{aligned}$$

is a purely inseparable extension of K , (8).

Now the relation (10) becomes

$$(13) \quad t_2^{q-1}(t_1^{q+1} + \sum_{i=2}^r t_{2i-1}^{q+1} t_{2i} + v) = -(t_1^{q(q+1)} + \sum_{i=1}^r t_{2i-1}^{q(q+1)} t_{2i} + v).$$

This shows that v is a rational function of $t_1, \dots, t_{2r}$, and

hence

$$K' = k(t_1, t_2, \dots, t_{2r})$$

is a purely transcendental extension of k of dimension $2r$.

This proves the unirationality of X , q.e.d.

Reference

- [1] T. Shioda, An example of unirational surfaces in characteristic p , Math. Ann. (to appear).
- [2] O. Zariski, On Castelnuovo's criterion of rationality $p_{2r} = P_2 = 0$ of an algebraic surface, Illinois J. Math. 2 (1958) ^a 303-315