

TOWARDS THE p -ADIC DERIVED HECKE ALGEBRA FOR WEIGHT ONE FORMS

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ABSTRACT. This note outlines an approach to defining p -adic Shimura classes and p -adic derived Hecke operators on the completed cohomology of modular curves, with more details to follow in a forthcoming paper by the author. After reviewing the modulo- p constructions of Harris and Venkatesh, we formulate a conjecture relating the action of p -adic derived Hecke operators on cusp forms of weight 1 and level $\Gamma_1(N)$ to the p -adic logarithm of the Stark unit for the corresponding adjoint Deligne–Serre representation. This new p -adic conjecture can be viewed as complementary to the Harris–Venkatesh conjecture.

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1. INTRODUCTION

Let $f = \sum_n a_n q^n$ be a newform of weight 1 and level $\Gamma_1(N)$, and denote its field of coefficients by $\mathbb{Q}(f) \subset \mathbb{C}$ with ring of integers $\mathbb{Z}[f]$. There is a finite Galois extension $E/\mathbb{Q}$ and an associated 3-dimensional complex representation $\text{Ad}(\rho_f)$ of $\text{Gal}(E/\mathbb{Q})$ obtained by taking the adjoint action of the Deligne–Serre representation ρ_f of f on trace-free 2×2 complex matrices. The trace-free adjoint representation has an associated dual space of units $\mathcal{U}(\text{Ad}(\rho_f)) := \text{Hom}_{\text{Gal}(E/\mathbb{Q})}(\text{Ad}(\rho_f), \mathcal{O}_E^\times \otimes \mathbb{Z}[f])$ with a $(\mathbb{Z}/p\mathbb{Z})^\times$ -regulator map

$$\text{Reg}_{(\mathbb{Z}/p\mathbb{Z})^\times} : \mathcal{U}(\text{Ad}(\rho_f)) \longrightarrow (\mathbb{Z}/p\mathbb{Z})^\times \otimes \mathbb{Z} \left[f, \frac{1}{6N} \right]$$

for each prime $p \nmid 6N$. Let Cusp denote the cuspidal divisor on the modular curve $X_1(N)$. Using constructions (especially the Shimura covering/subgroup/class) with origins in the works of Shimura [Shi63, Shi67], Mazur [Maz77], and Merel [Mer96]

among others, the study of derived Hecke operators on the coherent cohomology of modular curves

$$T_{(\mathbb{Z}/p\mathbb{Z})^\times, N} : H^0(X_1(N), \omega(-\text{Cusp})) \otimes (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow H^1(X_1(N), \omega(-\text{Cusp})) \otimes (\mathbb{Z}/p\mathbb{Z})^\times$$

was initiated by Harris and Venkatesh [HV19], who posed a conjecture relating the action of $T_{(\mathbb{Z}/p\mathbb{Z})^\times, N}$ to the action of a group of units. For the newform $f^* = \sum_n \overline{a_n} q^n$ obtained from f by complex conjugation of Fourier coefficients, the Serre duality pairing $\langle f^*, T_{(\mathbb{Z}/p\mathbb{Z})^\times, N}(f) \rangle_{\text{SD}}$ (i.e. “pseudo-eigenvalue” of $T_{(\mathbb{Z}/p\mathbb{Z})^\times, N}$) was packaged as the Harris–Venkatesh “norm” $\|f\|_{(\mathbb{Z}/p\mathbb{Z})^\times}^2 \in (\mathbb{Z}/p\mathbb{Z})^\times \otimes \mathbb{Z}[f]$ in [Zha23b], which drew an analogy with the Petersson norm $\|f\|_{\mathbb{R}}^2 = \int_{X_0(N)} |f|^2 \frac{dx dy}{y} \in \mathbb{R}$ and the conjectures of Stark [Sta71, Sta75, Sta76, Sta80].

CONJECTURE 1 (The Harris–Venkatesh conjecture). Let f be a newform of weight 1 and level $\Gamma_1(N)$. There exists a $u \in \mathcal{U}(\text{Ad}(\rho_f)) \otimes \mathbb{Q}$ and a prime p_0 such that for all primes $p \geq p_0$,

$$\|f\|_{(\mathbb{Z}/p\mathbb{Z})^\times}^2 = \text{Reg}_{(\mathbb{Z}/p\mathbb{Z})^\times}(u)$$

or equivalently,

$$\langle f^*, T_{(\mathbb{Z}/p\mathbb{Z})^\times, N}(f) \rangle_{\text{SD}} = \text{Reg}_{(\mathbb{Z}/p\mathbb{Z})^\times}(u).$$

The Harris–Venkatesh conjecture has been proved for imaginary dihedral forms in [DHRV22, Lec23, Zha23a], and for certain real dihedral forms in [DHRV22]. Numerical evidence for some exotic f has been given in [Mar21]. The Harris–Venkatesh conjecture has been refined in [Zha23b] to be compatible with the Stark conjecture (i.e. u can be taken to be the Stark unit u_{Stark}) and in [Zha23c] with explicit local data (i.e. the minimal p_0).

This note outlines key constructions and a conjecture from upcoming work of the author on p -adic derived Hecke operators. Section 2 briefly reviews the definition of the Stark unit group and the regulator maps defined on it. Section 3 reviews the modulo- p derived Hecke operator and the Harris–Venkatesh conjecture. Section 4 gives the construction of an element $\hat{f}^*$ of the completed cohomology group

$$\hat{H}^0(X_0(p^\infty), \omega(-\text{Cusp})) = \varinjlim_n H^0(X_0(p^n), \omega(-\text{Cusp})),$$

which gives rise to an element $f \cdot \hat{f}^* \in \hat{H}^0(X_0(p^\infty), \Omega)$ that will be paired with the p -adic Shimura class and serve as an analogue of the weight-2 cusp form $\text{Tr}_{\Gamma_0(p) \cap \Gamma_1(N)}^{\Gamma_0(p)}(f(z)\hat{f}^*(pz))$ in [DHRV22, Section 1.2]. Section 5 gives the construction of p -adic Shimura classes

$$\mathfrak{S}_{\mathbb{Z}_p^\times} \in \text{Hom}_{\mathbb{Z}_p^\times}(\hat{H}^0(X_0(p^\infty), \Omega), \mathbb{Z}_p^\times)$$

$$\mathfrak{S}_{\mathbb{Z}_p} \in \text{Hom}_{\mathbb{Z}_p}(\hat{H}^0(X_0(p^\infty), \Omega), \mathbb{Z}_p)$$

via projective limits of flat cohomology classes obtained from modular curve coverings

$$\pi_n : X_1(p^n) \longrightarrow X_0(p^n).$$

Since $\mathfrak{S}_{\mathbb{Z}_p^\times} = \mathfrak{S}_{(\mathbb{Z}/p\mathbb{Z})^\times} \times \mathfrak{S}_{\mathbb{Z}_p}$, we choose to focus on the $\mathbb{Z}_p$ -component that is complementary to the Harris–Venkatesh setting. Serre duality is applied to $f \cdot \widehat{f}^*$ and the Shimura class $\mathfrak{S}_{\mathbb{Z}_p}$ to define the $\mathbb{Z}_p$ derived Hecke operator

$$T_{\mathbb{Z}_p, N} : H^0(X_1(N), \omega(-\text{Cusp}))^{\text{ord}} \longrightarrow H^1(X_1(N), \omega(-\text{Cusp}))^{\text{ord}},$$

assuming ordinarity when $p = 2$ or 3 , i.e. $\text{ord}_v(\mathfrak{a}_p) = 0$ for a good prime v over p ; this gives a p -adic “norm” $\|f\|_{\mathbb{Z}_p}^2 \in \mathbb{Z}_p \otimes \mathcal{O}_v$ where $\mathcal{O}_v$ is the completion of $\mathbb{Z}[f]$ at a finite prime v over p . Inspired by the Harris–Venkatesh conjecture, we propose the following p -adic conjecture in terms of the p -adic regulator $\text{Reg}_{\mathbb{Z}_p}$, which is the p -adic logarithm of a unit after evaluation at a distinguished element $w_v \in \text{Ad}(\rho_f)$ (cf. the relationship between $\text{Reg}_{\mathbb{R}}$ and the classical logarithm in [Tat84, Zha23b]).

CONJECTURE 2 (p -adic conjecture). Let f be a newform of weight 1 and level $\Gamma_1(N)$. There exists an element $u \in \mathcal{U}(\text{Ad}(\rho_f)) \otimes \mathbb{Q}$ and a prime p_0 such that for all primes $p \geq p_0$,

$$\|f\|_{\mathbb{Z}_p}^2 = \text{Reg}_{\mathbb{Z}_p}(u),$$

or equivalently,

$$\langle f^*, T_{\mathbb{Z}_p, N}(f) \rangle = \text{Reg}_{\mathbb{Z}_p}(u).$$

In the sense that the original conjecture of Harris–Venkatesh [HV19] is over $(\mathbb{Z}/p\mathbb{Z})^\times$ and that there are isomorphisms $\mathbb{Z}_p^\times \cong \mu_{p-1} \times (1+p\mathbb{Z}_p) \cong (\mathbb{Z}/p\mathbb{Z})^\times \times \mathbb{Z}_p$, Conjecture 2 is a complement to the Harris–Venkatesh conjecture.

Remark 3. There is a unique element $u_f \in \mathcal{U}(\text{Ad}(\rho_f)) \otimes \mathbb{Q}$ associated to f by [Zha23b, Conjecture 1]. This u_f is a multiple of the Stark unit u_{Stark} from the Stark conjecture for $\text{Ad}(\rho_f)$ by a Rankin–Selberg constant $c_{f, \text{RS}}$ (cf. [Zha23b, Theorem 3], [Zha23c]). It would be nice to specify how the u in the p -adic conjecture 2 is related to u_f , as was done for the mod- p setting in [Zha23b]. The author would like to speculate that perhaps in this way, one could draw a precise local–global picture for Stark units via the derived Hecke algebra. Furthermore, the author hopes to relate the new p -adic conjecture to special values of p -adic L-functions; this would be an analogue to the Gross–Stark conjecture with units instead of p -units, perhaps answering the speculations of Rivero [Riv23, Sections 1.3 and 6].

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2. THE STARK UNIT GROUP

The “Stark unit group” defined in this section follows the treatment and notation of [Zha23b], which itself is adapted from [Tat84, HV19, DHRV22]. The elements of this group appear in the Stark conjecture, Conjecture 1, and Conjecture 2.

For a modular form f of weight 1 and level $\Gamma_1(N)$, there is a finite Galois extension $E/\mathbb{Q}$ and an associated Artin representation $\rho_f : \text{Gal}(E/\mathbb{Q}) \rightarrow \text{GL}(V)$ realized on a free module V of rank 2 over $\mathbb{Z}[f]$ by Deligne–Serre [DS74]. From ρ_f , one obtains a 3-dimensional complex representation $\text{Ad}(\rho_f)$ from the action of $\text{Gal}(E/\mathbb{Q})$ by conjugation through ρ_f on trace-free 2×2 complex matrices. For $\sigma \in \text{Gal}(E/\mathbb{Q})$, let $x_\sigma := 2\rho_f(\sigma) - \text{Tr}(\rho_f(\sigma)) \cdot \text{Id}_{2 \times 2}$ denote the trace-zero 2×2 matrix obtained from $\rho_f(\sigma)$.

The (trace-free) adjoint representation $\text{Ad}(\rho_f)$ has an integral model given by $M := \mathbb{Z}[f] \cdot \{x_\sigma \mid \sigma \in \text{Gal}(E/\mathbb{Q})\}$. There is a (dual) space of units

$$\mathcal{U}(\text{Ad}(\rho_f)) := \text{Hom}_{\text{Gal}(E/\mathbb{Q})}(M, \mathcal{O}_E^\times \otimes \mathbb{Z}[f]),$$

which is called the “Stark unit group” in [HV19] because the Stark conjectures [Sta71, Sta75, Sta76, Sta80] predict the existence of a Stark unit $u_{\text{Stark}} \in \mathcal{U}(\text{Ad}(\rho_f)) \otimes \mathbb{Q}$ that is unique up to multiplication by roots of unity and whose Stark regulator $\text{Reg}_{\mathbb{R}}(u_{\text{Stark}})$ gives the leading term of the Artin L-function $L(\text{Ad}(\rho_f), s)$ at $s = 0$. As detailed in [HV19, Lemma 2.1] and [Hor23, Corollary 2.6], the space $\mathcal{U}(\text{Ad}(\rho_f))$ is a rank-1 $\mathbb{Z}[f, \frac{1}{6N}]$ -module that does not depend on the choice of E .

For each place w of E , there is

- an embedding $\iota_w : E \hookrightarrow E_w$ where E_w is $\mathbb{R}$, $\mathbb{C}$, or a p -adic field;
- a Frobenius element Frob_w given by the p -power map or complex conjugation; and
- a distinguished element $x_{\text{Frob}_w} \in M$ that is invariant under the action of the decomposition group $\langle \text{Frob}_w \rangle \subset \text{Gal}(E/\mathbb{Q})$.

The Stark regulator map $\text{Reg}_{\mathbb{R}}$ can be defined by

- (1) picking an archimedean place w of E ,
- (2) taking the composition of the evaluation-at- x_{Frob_w} map, ι_w , and the logarithm map:

$$\text{Reg}_{\mathbb{R}} : \mathcal{U}(\text{Ad}(\rho_f)) \xrightarrow{\text{evaluation at } x_{\text{Frob}_w}} (\mathcal{O}_E^\times)^{\text{Frob}_w} \otimes \mathbb{Z}[f] \xrightarrow{\log \circ \iota_w} \mathbb{R} \otimes \mathbb{Z}[f].$$

For each prime p , there is a $(\mathbb{Z}/p\mathbb{Z})^\times$ regulator map (called “reduction of a Stark unit” in [HV19, DHRV22]) defined by

- (1) picking a non-archimedean place w of E above p ,
- (2) taking the composition of the evaluation-at- x_{Frob_w} map, ι_w , and reduction modulo the ideal of w :

$$\text{Reg}_{(\mathbb{Z}/p\mathbb{Z})^\times} : \mathcal{U}(\text{Ad}(\rho_f)) \xrightarrow{\text{evaluation at } x_{\text{Frob}_w}} (\mathcal{O}_E^\times)^{\text{Frob}_w} \otimes \mathbb{Z}[f] \xrightarrow{\iota_w} \mathcal{O}_{E_w}^\times \otimes \mathbb{Z}[f] \xrightarrow{\text{reduction mod } w} (\mathbb{Z}/p\mathbb{Z})^\times \otimes \mathbb{Z}[f].$$

Harris and Venkatesh [HV19, Section 2.8] observed that $\text{Reg}_{(\mathbb{Z}/p\mathbb{Z})^\times}$ is independent of the choice of w over p : the vector $x_{\text{Frob}_w} \in M$ is invariant under the decomposition group $\langle \text{Frob}_w \rangle$ and the maps used in defining $\text{Reg}_{(\mathbb{Z}/p\mathbb{Z})^\times}$ are $\text{Gal}(E/\mathbb{Q})$ -equivariant, so changing w corresponds to a Galois conjugation which leaves the regulator map invariant.

By omitting the “reduction modulo w ” step in the construction of $\text{Reg}_{\mathbb{Z}/p\mathbb{Z}^\times}$, we can define two p -adic regulators (with an additional application of the p -adic logarithm function): $\text{Reg}_{\mathbb{Z}_p}$ and $\text{Reg}_{\mathbb{Z}_p^\times}$.

3. THE MODULO- p SHIMURA CLASS AND DERIVED HECKE OPERATOR

This section is a condensed overview of the remaining ingredients (Shimura class, norm, and derived Hecke operator) necessary to state Conjecture 1 following the presentation in [Zha23a, Zha23b].

3.1. The modulo- p Shimura class. Let p be a prime not dividing $6N$. The Shimura covering is a finite étale Galois covering of modular curves (more precisely, the maximal étale intermediate extension) with deck group $(\mathbb{Z}/p\mathbb{Z})^\times$:

$$\begin{array}{c} X_1(p) \\ \downarrow (\mathbb{Z}/p\mathbb{Z})^\times \\ X_0(p) \end{array}$$

which induces an element $\mathfrak{S}_{(\mathbb{Z}/p\mathbb{Z})^\times, \text{ét}} \in H_{\text{ét}}^1(X_0(p), (\mathbb{Z}/p\mathbb{Z})^\times)$. After taking

- (a) the base change $X_0(p)_{\mathbb{Z}/(p-1)\mathbb{Z}} := X_0(p) \otimes \text{Spec}(\mathbb{Z}/(p-1)\mathbb{Z})$,
- (b) the pushforward of the étale sheaf $\mathbb{Z}/(p-1)\mathbb{Z} \rightarrow \mathcal{O}_{X_0(p)_{\mathbb{Z}/(p-1)\mathbb{Z}}}$,
- (c) the comparison of Zariski and étale cohomology for quasi-coherent sheaves, and
- (d) the pairing given by Serre duality,

$$\begin{array}{c} (H^1(X_0(p)_{\mathbb{Z}/(p-1)\mathbb{Z}}, \mathcal{O}) \otimes (\mathbb{Z}/p\mathbb{Z})^\times) \otimes (H^0(X_0(p)_{\mathbb{Z}/(p-1)\mathbb{Z}}, \Omega^1) \otimes (\mathbb{Z}/p\mathbb{Z})^\times) \\ \downarrow \langle \cdot, \cdot \rangle_{\text{SD}} \\ (\mathbb{Z}/p\mathbb{Z})^\times, \end{array}$$

we have a composition of the respective maps:

$$\begin{array}{ccc} \mathfrak{S}_{(\mathbb{Z}/p\mathbb{Z})^\times, \text{ét}} & \in & H_{\text{ét}}^1(X_0(p), (\mathbb{Z}/p\mathbb{Z})^\times) \\ \downarrow & & \downarrow (a) \\ & & H_{\text{ét}}^1(X_0(p), \mathbb{Z}/(p-1)\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/p\mathbb{Z})^\times \\ & & \downarrow (b) \\ & & H_{\text{ét}}^1(X_0(p)_{\mathbb{Z}/(p-1)\mathbb{Z}}, \mathbb{G}_a) \otimes (\mathbb{Z}/p\mathbb{Z})^\times \\ & & \downarrow (c) \\ \mathfrak{S}_{(\mathbb{Z}/p\mathbb{Z})^\times} & \in & H_{\text{Zar}}^1(X_0(p)_{\mathbb{Z}/(p-1)\mathbb{Z}}, \mathcal{O}) \otimes (\mathbb{Z}/p\mathbb{Z})^\times \\ \downarrow & & \downarrow (d) \\ \langle \cdot, \mathfrak{S}_{(\mathbb{Z}/p\mathbb{Z})^\times} \rangle_{\text{SD}} & \in & \text{Hom}(H^0(X_0(p), \Omega^1), (\mathbb{Z}/p\mathbb{Z})^\times). \end{array}$$

Hence, the image of $\mathfrak{S}_{(\mathbb{Z}/p\mathbb{Z})^\times, \text{ét}}$ under the first four maps furnishes an element $\mathfrak{S}_{(\mathbb{Z}/p\mathbb{Z})^\times} \in H^1(X_0(p)_{\mathbb{Z}/(p-1)\mathbb{Z}}, \mathcal{O}) \otimes (\mathbb{Z}/p\mathbb{Z})^\times$. Since there is an isomorphism $S_2(\Gamma_0(p)) \cong H^0(X_0(p), \Omega^1)$, (see [Shi94, Corollary 2.17]), we can view $\mathfrak{S}_{(\mathbb{Z}/p\mathbb{Z})^\times}$ as acting on weight-2 cusp forms via $\langle \cdot, \mathfrak{S}_{(\mathbb{Z}/p\mathbb{Z})^\times} \rangle_{\text{SD}} \in \text{Hom}(S_2(\Gamma_0(p)), (\mathbb{Z}/p\mathbb{Z})^\times)$.

3.2. The modulo- p norm. For a cusp form $f = \sum_n a_n q^n$ of weight 1 and level $\Gamma_1(N)$ with coefficients in $\mathbb{Q}(f) \subset \mathbb{C}$, applying complex conjugation to the Fourier coefficients gives the dual cusp form $f^* = \sum_n \overline{a_n} q^n$ of weight 1 and level $\Gamma_1(N)$. Define $\Gamma_{0,1}(p, N) := \Gamma_0(p) \cap \Gamma_1(N)$ and denote its modular curve by $X_{0,1}(p, N) = X_{\Gamma_{0,1}(p, N)}$.

For a prime p not dividing $6N$, the product $f(z)f^*(pz)$ is a cusp form of weight 2 and level $\Gamma_0(p) \cap \Gamma_1(N)$ whose trace $\text{Tr}_{\Gamma_{0,1}(p, N)}^{\Gamma_0(p)}(f(z)f^*(pz))$ is a cusp form of weight 2, level $\Gamma_0(p)$, trivial nebentypus, and with coefficients in $\mathbb{Z}[f, \frac{1}{6N}]$. [Zha23b] defines a Harris–Venkatesh “norm”

$$\|f\|_{(\mathbb{Z}/p\mathbb{Z})^\times}^2 := \mathfrak{S}_p\left(\text{Tr}_p^N(f(z)f^*(pz))\right) \in (\mathbb{Z}/p\mathbb{Z})^\times \otimes \mathbb{Z}\left[f, \frac{1}{6N}\right].$$

This leads to the norm formulation of Conjecture 1.

3.3. The modulo- p derived Hecke operator. Harris and Venkatesh [HV19] define a derived Hecke operator $T_{(\mathbb{Z}/p\mathbb{Z})^\times, N}$ on the space of cusp forms of weight 1 and level N coprime to p by adding $\cup \mathfrak{S}_p$ to the definition of the usual Hecke operator via the pullback and pushforward of two projection maps $\pi, \pi' : X_{\Gamma_1(N) \cap \Gamma_0(p)} \longrightarrow X_{\Gamma_1(N)}$:

$$\begin{array}{ccc} H^0(X_1(N)_{\mathbb{Z}/(p-1)\mathbb{Z}}, \omega(-\text{Cusp})) & \xrightarrow{\quad T_{(\mathbb{Z}/p\mathbb{Z})^\times, N} \quad} & H^1(X_1(N)_{\mathbb{Z}/(p-1)\mathbb{Z}}, \omega(-\text{Cusp})) \\ \otimes & & \otimes \\ (\mathbb{Z}/p\mathbb{Z})^\times & & (\mathbb{Z}/p\mathbb{Z})^\times \\ \downarrow \pi^* & & \uparrow \pi'_* \\ H^0(X_{0,1}(p, N)_{\mathbb{Z}/(p-1)\mathbb{Z}}, \omega(-\text{Cusp})) & \xrightarrow{\quad \cup \mathfrak{S}_{(\mathbb{Z}/p\mathbb{Z})^\times} \quad} & H^1(X_{0,1}(p, N)_{\mathbb{Z}/(p-1)\mathbb{Z}}, \omega(-\text{Cusp})) \\ \otimes & & \otimes \\ (\mathbb{Z}/p\mathbb{Z})^\times & & (\mathbb{Z}/p\mathbb{Z})^\times \end{array}$$

The Harris–Venkatesh norm concretely encapsulates the action of the derived Hecke operator:

$$\|f\|_{(\mathbb{Z}/p\mathbb{Z})^\times}^2 = \langle f^*, T_{(\mathbb{Z}/p\mathbb{Z})^\times, N}(f) \rangle_{\text{SD}}.$$

This leads to the derived Hecke operator formulation of Conjecture 1.

4. THE p -ADIC SYSTEM OF WEIGHT ONE FORMS

In the modulo- p setting, we applied the Shimura class $\mathfrak{S}_{(\mathbb{Z}/p\mathbb{Z})^\times}$ to the weight-2 cusp form $f(z)f^*(pz)$. In this section, we construct the p -adic analogue by looking at spaces generated by $f^*(p^n z)$. Note that this is the only section of this note in which p is allowed to be 2 or 3.

4.1. Non-vanishing and uniqueness. Let $f = \sum_n a_n q^n$ be a newform for $\Gamma_1(N)$ of weight 1 with central character ω and dual form $f^* = \sum_n \overline{a_n} q^n$. Let p be a prime that does not divide N . For the field $\mathbb{Q}(f) = \mathbb{Q}(\chi_{\rho_f})$ of coefficients of f with ring of integers $\mathbb{Z}[f]$, let v be a place of $\mathbb{Q}(f)$ over p and let $\mathcal{O}_v$ be the completion of $\mathbb{Z}[f]$ at v . Consider the intersection $\Gamma_{0,1}(p^n, N) := \Gamma_0(p^n) \cap \Gamma_1(N)$ and its modular curve $X_{0,1}(p^n, N)$.

For each integer $n \geq 0$, define the modular form $f_n^*(z) := f^*(p^n z) = f^*(z)|_1 \begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix}$. The subspace of cusp forms of weight 1 and level p^n generated by f^* over $\mathcal{O}_v$ is given by

$$V_{f^*,n} := \sum_{i=0}^n \mathcal{O}_v f_i^* \subset H^0(X_{0,1}(p^n, N)_{\mathcal{O}_v}, \omega(-\text{Cusp})),$$

where Cusp is the cuspidal divisor. The spaces $\{V_{f^*,n}\}_{n \geq 0}$ form a projective system under trace maps; take the projective limit,

$$\widehat{V}_{f^*} := \varprojlim_n V_{f^*,n}.$$

The polynomial $X^2 - a_p X + \omega(p)p$ (which should not be mistaken for the Hecke polynomial for weight 1) has a unique root α in $\mathcal{O}_v^\times$ by Hensel's lemma; it appears in calculations of the trace map between levels of our projective system.

THEOREM 4. *The space $\widehat{V}_{f^*}$ is non-zero only if v is ordinary. In this case, it is one-dimensional and generated by an element $\widehat{f^*} \in \widehat{V}_{f^*}$ whose image in $V_{f^*,n}$ is given by*

$$\widehat{f^*}_n := \alpha^{1-n} f_n^* - \alpha^{-n} \omega(p) f_{n-1}^*.$$

Remark 5. In the proof of Theorem 4, we will only use the fact that f^* has eigenvalue a_p under T_p . Thus, Theorem 4 actually holds for any old form φ generated by f^* such that the level of φ is coprime to p . Also, the condition on ordinariness (i.e. $\text{ord}_v(a_p) = 0$) is only relevant for $p = 2$ or 3 since a newform of weight 1 is ordinary at the other primes.

The key to proving Theorem 4 is the calculation of the trace map from level $n+1$ to level n .

LEMMA 6. *Consider the trace map*

$$\text{Tr}_{n+1} : V_{f^*,n+1} \longrightarrow V_{f^*,n}, \quad n \geq 1.$$

(a) *Then for all $i \leq n$,*

$$\text{Tr}_{n+1}(f_i^*) = p f_i^*;$$

(b) *and*

$$\text{Tr}_{n+1}(f_{n+1}^*) = a_p f_n^* - \omega(p) f_{n-1}^*.$$

Proof. (a): This is due to the fact that $[\Gamma_{0,1}(p^n, N) : \Gamma_{0,1}(p^{n+1}, N)] = p$.

(b): Since $\Gamma_{0,1}(p^{n+1}, N) \backslash \Gamma_{0,1}(p^n, N)$ is represented by elements $\gamma_i := \begin{pmatrix} p^{n+1} & 0 \\ p^n i N & 1 \end{pmatrix}$ for $0 \leq i \leq p-1$,

$$\text{Tr}_{n+1}(f_{n+1}^*) := \sum_{\gamma \in \Gamma_{0,1}(p^{n+1}, N) \backslash \Gamma_{0,1}(p^n, N)} f_{n+1}^*|_1 \gamma = \sum_i f^*|_1 \left(\begin{pmatrix} p^{n+1} & 0 \\ 1 & 1 \end{pmatrix} \gamma_i \right).$$

Notice that

$$\begin{pmatrix} p^{n+1} & \\ & 1 \end{pmatrix} \gamma_i = \begin{pmatrix} p^{n+1} & \\ p^{niN} & 1 \end{pmatrix} = \begin{pmatrix} p & \\ iN & 1 \end{pmatrix} \begin{pmatrix} p^n & \\ & 1 \end{pmatrix},$$

and the Hecke operator T_p is presented by the left $\Gamma_1(N)$ -cosets represented by $\begin{pmatrix} p & \\ iN & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & \\ & p \end{pmatrix}$. Then

$$\begin{aligned} \text{Tr}_{n+1}(f_{n+1}^*) &= \sum_i f^*|_1 \left(\begin{pmatrix} p^{n+1} & \\ & 1 \end{pmatrix} \gamma_i \right) \\ &= \sum_i f^*|_1 \left(\begin{pmatrix} p & \\ iN & 1 \end{pmatrix} \begin{pmatrix} p^n & \\ & 1 \end{pmatrix} \right) \\ &= \left(T_p f^* - f^*|_1 \begin{pmatrix} 1 & \\ & p \end{pmatrix} \right) \Big|_1 \begin{pmatrix} p^n & \\ & 1 \end{pmatrix} \\ &= T_p f^*|_1 \begin{pmatrix} p^n & \\ & 1 \end{pmatrix} - f^*|_1 \begin{pmatrix} p^n & \\ & p \end{pmatrix} \\ &= a_p f_n^* - \omega(p) f_{n-1}^*. \end{aligned}$$

□

Proof of Theorem 4. For any $m \geq n$, let $\text{Tr}_{m,n} : V_{f^*,m} \longrightarrow V_{f^*,n}$ denote the $(m-n)$ -fold composition of trace maps

$$\text{Tr}_{m,n} : V_{f^*,m} \xrightarrow{\text{Tr}_m} V_{f^*,m-1} \xrightarrow{\text{Tr}_{m-1}} \dots \xrightarrow{\text{Tr}_{n+1}} V_{f^*,n}.$$

Let $\widehat{V}_{f^*,n}$ denote the intersection of the images $\text{Im}(\text{Tr}_{m,n})$ over all $m \geq n$. Then the projective limit $\widehat{V}_{f^*} \neq 0$ if and only if $\widehat{V}_{f^*,n} \neq 0$ for some n .

Let π be a uniformizer of $\mathcal{O}_v$. Consider the case that $\text{ord}_v(a_p) > 1$. Then by Lemma 6, $\text{Im}(\text{Tr}_{n+2,n}) \subset \pi V_{f^*,n}$ for all $n \geq 1$. This shows that $\text{Im}(\text{Tr}_{n+2k,n}) \subset \pi^k V_{f^*,n}$, so $\widehat{V}_{f^*,n} := \bigcap_{m \geq n} \text{Im}(\text{Tr}_{m,n}) = 0$ and $\widehat{V}_{f^*} = 0$ too.

Now consider the case $\text{ord}_v(a_p) = 0$. Take the basis for $V_{f^*,n+1}$ given by,

$$\begin{aligned} (\widehat{f^*})_0 &:= f_0^*, \\ (\widehat{f^*})_i &:= \alpha^{1-i} f_i^* - \alpha^{-i} \omega(p) f_{i-1}^* \quad \text{for } 0 < i \leq n+1. \end{aligned}$$

By Lemma 6, $\text{Tr}_{n+1}(f_i^*) = p f_i^*$ for $i < n+1$ and $\text{Tr}_{n+1}(f_{n+1}^*) = a_p f_n^* - \omega(p) f_{n-1}^*$, so

$$\begin{aligned} \text{Tr}_{n+1}((\widehat{f^*})_{n+1}) &= \alpha^{1-n-1} \text{Tr}_{n+1}(f_{n+1}^*) - \alpha^{-n-1} \omega(p) \text{Tr}_{n+1}(f_n^*) \\ &= \alpha^{-n} (a_p f_n^* - \omega(p) f_{n-1}^*) - \alpha^{-n-1} \omega(p) p f_n^* \\ &= \alpha^{-n-1} (\alpha a_p - \omega(p)p) f_n^* - \alpha^{-n} \omega(p) f_{n-1}^* \\ &= \alpha^{-n-1} \alpha^2 f_n^* - \alpha^{-n} \omega(p) f_{n-1}^* \\ &= (\widehat{f^*})_n. \end{aligned}$$

Then the sequence $\{(\widehat{f^*})_n\}_{n \geq 0}$ defines an element $\widehat{f^*} \in \widehat{V}_{f^*}$ and each $\widehat{V}_{f^*,n}$ is equal to $\mathcal{O}_v \cdot (\widehat{f^*})_n$. $\square$

4.2. An extended remark on ordinary primes. The ordinariness of $\mathfrak{a}_p$ in Theorem 4 can be characterized as follows. Recall that f corresponds to a two-dimensional complex representation ρ_f by Deligne–Serre with $\mathfrak{a}_p = \text{Tr}(\rho_f(\text{Frob}_p))$. In fact, the p -th Fourier coefficient $\mathfrak{a}_p$ of f is the sum of two roots of unity whose order divides $\#\text{Im}(\rho_f)$:

$$\mathfrak{a}_p = \zeta_1 + \zeta_2 = \zeta_1 \cdot (1 - \xi),$$

where $\xi = -\frac{\zeta_2}{\zeta_1}$. It is known that nonzero $\mathfrak{a}_p$ is not invertible for some p -adic norm on $\mathbb{Q}(f)$ if and only if the order of ξ is a power of p . This shows that p is ordinary if $\mathfrak{a}_p \neq 0$ and $p \nmid 2\#\text{Im}(\rho_f)$.

When ρ_f has dihedral image, more can be said about when $\mathfrak{a}_p$ is non-zero based on the splitting of p in K . In the dihedral case, $\rho_f \cong \text{Ind}_{G_K}^{G_{\mathbb{Q}}}(\chi)$ for a finite character χ of $\text{Gal}(K^{\text{ab}}/K)$ and a quadratic number field $K/\mathbb{Q}$. Here, $L(f, s) = L(\chi, s)$ and

$$f = \sum_{\mathfrak{a} \in S_\chi} \chi(\mathfrak{a}) q^{N(\mathfrak{a})},$$

where $\mathfrak{a}$ runs through the set S_χ of ideals of $\mathcal{O}_K$ coprime to the conductor of χ . Therefore,

$$\mathfrak{a}_n = \sum_{\mathfrak{a}: N(\mathfrak{a})=n} \chi(\mathfrak{a}),$$

so $\mathfrak{a}_p = 0$ when p is inert in K and $\mathfrak{a}_p = \chi(\mathfrak{p}) + \chi(\bar{\mathfrak{p}})$ when $p\mathcal{O}_K = \mathfrak{p} + \bar{\mathfrak{p}}$.

5. THE p -ADIC SHIMURA CLASS AND DERIVED HECKE OPERATOR

In this section, we introduce the p -adic Shimura classes which play a central role in studying the action of p -adic Hecke operators on modular forms of weight 1. We then define the p -adic norm and p -adic derived Hecke operators given a p -adic Shimura class. This completes the formulation of Conjecture 2. $\mathbb{Z}_p$ is used throughout this section for notational convenience, but one may also take $\mathbb{Z}_p \otimes \mathbb{Z}[f]$ (analogously to $(\mathbb{Z}/p\mathbb{Z})^\times \otimes \mathbb{Z}[f]$).

5.1. The $\mathbb{Z}_p^\times$ Shimura class. Let p be a prime. Consider the tower of coverings of modular curves for $n \in \mathbb{N}$,

$$\pi_n : X_1(p^n) \longrightarrow X_0(p^n).$$

These covers are not étale when $n \geq 2$ even after inverting $6N$, but are finite flat (cf. [Maz77, Proposition 11.6] for the $n = 1$ case) with structure group denoted G_n and define flat cohomology classes,

$$\mathfrak{S}_{(\mathbb{Z}/p^n\mathbb{Z})^\times, \text{fl}} \in H_{\text{fl}}^1(X_0(p^n), G_n).$$

We sketch a 1-cocycle construction of the Shimura class $\mathfrak{S}_{(\mathbb{Z}/p^n\mathbb{Z})^\times, \text{fl}}$ if G_n behaves like $(\mathbb{Z}/p^n\mathbb{Z})^\times$ or some quotient $(\mathbb{Z}/p^n\mathbb{Z})_a^\times$ thereof. In the étale case, an unramified cover of connected $X = X_1(p)$ over $Y = X_0(p)$ with Galois group $(\mathbb{Z}/p\mathbb{Z})^\times$ gives a morphism from the pointed fundamental group $\pi_0(Y, y_0)$ to $(\mathbb{Z}/p\mathbb{Z})^\times$, which

gives an element of $H_{\text{ét}}^1(Y, (\mathbb{Z}/p\mathbb{Z})^\times) \cong \text{Hom}(\pi_1^{\text{ét}}(Y, y_0), (\mathbb{Z}/p\mathbb{Z})^\times)$ since the Galois group is abelian. Now for $X = X_1(p^n)$ over $Y = X_0(p^n)$ with structure group $(\mathbb{Z}/p^n\mathbb{Z})^\times$ and $n \geq 2$, the cover is not étale but is finite flat. The fiber product $X \times_Y X$ is a union of subvarieties isomorphic to X with a group action by some quotient of the original group $(\mathbb{Z}/p^n\mathbb{Z})^\times$. Fix one copy of X in $X \times_Y X$; the other copies of X come from actions of the quotient group, and $X \times_Y X$ is a group scheme over X . This gives a cochain from $X \times_Y X$ to the quotient group and an element of $H_{\text{fl}}^1(X_0(p^n), ((\mathbb{Z}/p^n\mathbb{Z})^\times))$. The compatibility of $\{\pi_n\}_{n \geq 0}$ gives an element $\mathfrak{S}_{\mathbb{Z}_p^\times, \text{fl}}$ in the (partially) complete cohomology

$$\hat{H}_{\text{fl}}^1(X_0(p^\infty), \mathbb{Z}_p^\times) := \varinjlim_n \varprojlim_m H_{\text{fl}}^1(X_0(p^n), ((\mathbb{Z}/p^m\mathbb{Z})^\times)_a).$$

One can then view the Shimura class as an element of Zariski cohomology, similar to what was done for the $n = 1$ étale case of Harris–Venkatesh [HV19, Section 3.1].

5.2. The $\mathbb{Z}_p$ Shimura class. We can obtain a $\mathbb{Z}_p$ Shimura class from a $\mathbb{Z}_p^\times$ Shimura class. Consider the natural decomposition

$$\Delta \times \mathbb{Z}_p \xrightarrow{\sim} \mathbb{Z}_p^\times,$$

where Δ is the torsion subgroup of $\mathbb{Z}_p^\times$. In particular, Δ is isomorphic to $\mu_{p-1} \cong (\mathbb{Z}/p\mathbb{Z})^\times$ when p is odd, and to μ_2 when $p = 2$. In the other component, the map $\mathbb{Z}_p \rightarrow \mathbb{Z}_p^\times$ is given by $x \mapsto (1+p)^x$ when p is odd and $x \mapsto 5^x$ when $p = 2$. Then we have a product

$$\mathfrak{S}_{\mathbb{Z}_p^\times, \text{fl}} = \mathfrak{S}_{\Delta, \text{fl}} \times \mathfrak{S}_{\mathbb{Z}_p, \text{fl}}.$$

The element $\mathfrak{S}_{\Delta, \text{fl}} = \mathfrak{S}_{\Delta, \text{ét}} \in H_{\text{ét}}^1(X_0(p), (\mathbb{Z}/p\mathbb{Z})^\times)$ gives the mod- p Shimura class previously studied by [Maz77, Mer96, HV19, Mar21, DHRV22, Hor23, Lec23, Zha23a, Zha23b, Zha23c] among others. Here, we focus on the $\mathbb{Z}_p$ Shimura class $\mathfrak{S}_{\mathbb{Z}_p, \text{fl}}$ lying in the cohomology group

$$\hat{H}_{\text{fl}}^1(X_0(p^\infty), \mathbb{Z}_p) := \varinjlim_n \varprojlim_m H_{\text{fl}}^1(X_0(p^n), \mathbb{Z}/p^m\mathbb{Z}).$$

This group has a natural map to the completed cohomology of coherent sheaves:

$$\varinjlim_n \varprojlim_m H^1(X_0(p^n)_{\mathbb{Z}/p^m\mathbb{Z}}, \mathcal{O}).$$

Using Serre duality,

$$H^1(X_0(p^n)_{\mathbb{Z}/p^m\mathbb{Z}}, \mathcal{O}) \cong \text{Hom}(H^0(X_0(p^n), \Omega), \mathbb{Z}/p^m\mathbb{Z}).$$

Thus for $\hat{H}^0(X_0(p^\infty), \Omega) := \varinjlim_n H^0(X_0(p^n), \Omega)$, the Shimura class $\mathfrak{S}_{\mathbb{Z}_p}$ can also be viewed as an element in its dual space

$$\text{Hom}_{\mathbb{Z}_p}(\hat{H}^0(X_0(p^\infty), \Omega), \mathbb{Z}_p),$$

where trace maps define the projective system.

One can perform all of these Shimura class constructions with additional level structure. It will be useful in the p -adic consideration of the Harris–Venkatesh

conjecture in the following section to use these constructions for $\Gamma_{0,1}(p^n, N) := \Gamma_0(p^n) \cap \Gamma_1(N)$ and its modular curve $X_{0,1}(p^n, N)$ instead of $\Gamma_0(p^n)$ and $X_0(p^n)$.

5.3. The p -adic norm. We apply Theorem 4 to the dual form $f^* = \sum_n \bar{a}_n q^n$ of f . Assuming that $\text{ord}_v(\bar{a}_p) = 0$, we have the element $\hat{f}^* \in \hat{H}^0(X_0(p^\infty), \omega(-\text{Cusp}))$ and in particular

$$f \cdot \hat{f}^* \in \hat{H}^0(X_0(p^\infty), \Omega).$$

Recall that $\mathcal{O}_v$ is the completion of $\mathbb{Z}[f]$ at v ; given a $\mathbb{Z}_p$ -Shimura class $\mathfrak{S}_{\mathbb{Z}_p, \mathfrak{f}} \in \text{Hom}_{\mathbb{Z}_p}(\hat{H}^0(X_0(p^\infty), \Omega), \mathbb{Z}_p)$, we can define a p -adic period of f ,

$$\|f\|_{\mathbb{Z}_p}^2 := \mathfrak{S}_{\mathbb{Z}_p}(f \cdot \hat{f}^*) \in \mathcal{O}_v.$$

Concretely, what we have done in Section 5 is the construction of the class $\mathfrak{S}_{\mathbb{Z}/p^n\mathbb{Z}, \mathfrak{f}} \in H^1(X_{0,1}(p^{n+1}, N), \mathbb{Z}/p^n\mathbb{Z})$ using the étale cover $X_1(p^n) \rightarrow X_0(p^n)$, and then the demonstration that this class induces an element $\mathfrak{S}_{\mathbb{Z}/p^n\mathbb{Z}} \in \text{Hom}(H^0(X_{0,1}(p^{n+1}, \Omega), \mathbb{Z}/p^n\mathbb{Z})$ through which we obtain

$$\|f\|_{\mathbb{Z}_p}^2 = \varprojlim_n \mathfrak{S}_{\mathbb{Z}/p^n\mathbb{Z}} \left((\alpha^{1-n} f_n^*(z) - \alpha^{-n} \omega(p) f_{n-1}^*(z)) f(z) dz \right).$$

With these constructions in place, we are now ready to formulate Conjecture 2 in terms of the p -adic norm: we ask whether

$$\|f\|_{\mathbb{Z}_p}^2 = \text{Reg}_{\mathbb{Z}_p}(u)$$

for some element $u \in \mathcal{U}(\text{Ad}(\rho_f)) \otimes \mathbb{Q}$ and all sufficiently large primes p . In other words, the $\mathbb{Z}_p$ Shimura class should act on weight 1 forms compatibly with the $(\mathbb{Z}/p\mathbb{Z})^\times$ Shimura class in the Harris–Venkatesh conjecture, and there is a combined equality for all sufficiently large primes p :

$$\|f\|_{\mathbb{Z}_p^*}^2 = \text{Reg}_{\mathbb{Z}_p^*}(u).$$

5.4. The p -adic derived Hecke operator. Define the subspace of p -adic ordinary elements in $H^i(X_1(N), \omega(-\text{Cusp})) \otimes \mathbb{Z}_p$ to be

$$H^i(X_1(N), \omega(-\text{Cusp}))^{\text{ord}} := \bigcap_{n \geq 0} T_p^n H^i(X_1(N), \omega(-\text{Cusp})) \otimes \mathbb{Z}_p.$$

In this space, we have the action of the invertible operator T_p and $\langle p \rangle$. There is a unique invertible operator A on $H^i(X_1(N), \omega(-\text{Cusp}))^{\text{ord}}$ satisfying the equation:

$$A^2 - T_p A + \langle p \rangle p = 0.$$

Such an A can be either defined by the binomial formula,

$$A = \frac{T_p}{2} \left(1 + \sqrt{1 - 4\langle p \rangle p / T_p^2} \right) = T_p - \sum_{n \geq 1} \frac{1 \cdot 3 \cdots (2n-3)}{2^{1-n} n!} \left(\frac{\langle p \rangle p}{T_p^2} \right)^n,$$

or by the limit $A = \lim_{n \rightarrow \infty} A_n$ with A_n the invertible operator defined by

$$\begin{aligned} A_0 &= T_p, \\ A_n &= T_p - \frac{\langle p \rangle p}{A_{n-1}}, \quad n > 0. \end{aligned}$$

Serre duality gives a perfect pairing:

$$\langle \cdot, \cdot \rangle_{\text{SD}} : H^0(X_1(N), \omega(-\text{Cusp}))^{\text{ord}} \otimes H^1(X_1(N), \omega(-\text{Cusp}))^{\text{ord}} \longrightarrow \mathbb{Z}_p.$$

Using the operator A and $f_n(z) := f(p^n z)$, we have a lifting

$$\begin{aligned} H^0(X_1(N), \omega(-\text{Cusp}))^{\text{ord}} &\longrightarrow \varprojlim_n H^0(X_{0,1}(p^n, N), \omega(-\text{Cusp})) \\ f &\longmapsto \widehat{f} := \varprojlim_n (A^{1-n} f_n - A^{-n} \langle p \rangle p f_{n-1}). \end{aligned}$$

Then we have a derived Hecke operator

$$T_{\mathbb{Z}_p, N} : H^0(X_1(N), \omega(-\text{Cusp}))^{\text{ord}} \longrightarrow H^1(X_1(N), \omega(-\text{Cusp}))^{\text{ord}}$$

defined by this lifting and $\cup \mathfrak{S}_{\mathbb{Z}_p}$ such that for any $f_1, f_2 \in H^0(X_1(N), \omega(-\text{Cusp}))^{\text{ord}}$,

$$\mathfrak{S}_{\mathbb{Z}_p}(f_1 \widehat{f}_2) = \langle f_2, T_{\mathbb{Z}_p, N}(f_1) \rangle_{\text{SD}}.$$

We need only consider new forms $f_1 = f, f_2 = f^*$; in this case, the prediction that

$$\|f\|_{\mathbb{Z}_p}^2 = \text{Reg}_{\mathbb{Z}_p}(\mathfrak{u}),$$

together with the pairing identity above, yields the derived Hecke operator formulation of Conjecture 2 in terms of $\langle f^*, T_{\mathbb{Z}_p, N}(f) \rangle_{\text{SD}} = \text{Reg}(\mathfrak{u})$.

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