

Stability of direct images of cotangent bundles by Frobenius morphisms

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Introduction

$k = \bar{k}$: a field, $\text{char}(k) = p > 0$
 X : nonsing. proj. var./ k , $\dim X = n$
 $F = F_X$: the absolute Frobenius morphism of X

Theorem (H. Lange-C. Pauly (2003)).

X : nonsing. proj. curve/ k , $g(X) \geq 2$
 $\mathcal{E}$: line bundle on X
 $\implies F_*\mathcal{E}$: stable

a natural generalization

Problem.



X : nonsing. proj. var. of general type/ k
 H : numerically positive divisor on X
 $\mathcal{E}$: semistable vector bundle on X w.r.t. H
 $\implies F_*\mathcal{E}$ is semistable w.r.t. H ?

Results

Theorem (K-Sumihiro (2006)).

X : nonsing. proj. surf./ k
 H : num. pos. div./ X s.t.
 $|mH|$: base pt free
and contains nonsing. member ($m \gg 0$)
 $K_X H > 0$, Assume Ω_X^1 : semistable w.r.t. H
 $\implies F_*\mathcal{L}$ is semistable w.r.t. H
for any line bundle $\mathcal{L}$ on X .

Higher dimensional case (surface)

Theorem (K-Sumihiro (2006), Mehta-Paully, Sun).

X : nonsing. proj. curve/ k , $g(X) \geq 2$
 $\mathcal{E}$: semistable (resp. stable) vector bundle on X
 $\implies F_*\mathcal{E}$ is semistable (resp. stable).

Higher rank case

Main Theorem

(K-Sumihiro (2007))

X : nonsing. proj. surf./ k
 H : num. pos. div./ X s.t.
 $|mH|$: base pt free
and contains nonsing. member ($m \gg 0$)
 $K_X H > 0$, Assume Ω_X^1 : semistable w.r.t. H
 $\implies F_*(\Omega_X^1 \otimes \mathcal{L})$ is semistable w.r.t. H
for any line bundle $\mathcal{L}$ on X .

Higher dimensional and higher rank case
(surface, cotangent bundle)

Main tools

Canonical filtrations



A useful filtration of $F^*F_*\mathcal{O}_X$

$\varphi: F^*F_*\mathcal{O}_X \rightarrow \mathcal{O}_X$: natural surjection

$F^*F_*\mathcal{O}_X$: $\mathcal{O}_X$ -algebra

$I := \ker \varphi$

So we get

$F^*F_*\mathcal{O}_X \supset I \supset I^2 \supset I^3 \supset \dots$

$I^\bullet$: canonical filtration of $F^*F_*\mathcal{O}_X$

$\mathcal{E}$: a vector bundle on X

$W^i := F^*F_*\mathcal{E} \cdot I^i$

$W^\bullet$: canonical filtration of $F^*F_*\mathcal{E}$

Canonical connections



$\nabla: F^*F_*\mathcal{O}_X \rightarrow F^*F_*\mathcal{O}_X \otimes \Omega_X^1$

An application

An application for the geography of nonsing. proj. minimal surf. of gen. type

X : nonsing. proj. minimal surf. of gen. type/ k
Assume Ω_X^1 : semistable w.r.t. K_X

(1) (Bogomolov's inequality)
 Ω_X^1 : strongly semistable
($\iff (F^k)^*\Omega_X^1$: semistable $\forall k$) w.r.t. K_X ,
 $\implies c_1^2(X) \leq 4c_2(X)$.

(2) $(F^{k-1})^*\Omega_X^1$: semistable w.r.t. K_X
and $(F^k)^*\Omega_X^1$: not semistable w.r.t. $K_X \exists k$,
 $\implies c_1^2(X) \leq \frac{4p^{2k}}{p^{2k} - (p-1)^2} c_2(X)$.

In particular, $c_2(X) > 0$.