

# Eigenvalue Asymptotics for the Schrödinger Operator with Asymptotically Flat Magnetic Fields and Decreasing Electric Potential

SHIRAI, S. and IWATSUKA, A.

Faculty of Sciences, Osaka University  
Faculty of Sciences, Kyoto University

## 1 Introduction

We investigate the asymptotic distribution of eigenvalues of the two dimensional Schrödinger operator with an electromagnetic potential. We consider the operator in  $L^2(\mathbf{R}^2)$  of the form :

$$H_V = -\frac{\partial^2}{\partial x_1^2} + \left( \frac{1}{i} \frac{\partial}{\partial x_2} - b(x_1) \right)^2 + V(x_1, x_2),$$

where  $(0, b(x_1))$  is the (magnetic) vector potential which gives a perturbed constant magnetic field and  $V(x_1, x_2)$  is the (electric) scalar potential decaying at infinity.

First, we shall consider the magnetic field  $B(x_1)$  obeying :

(B.1)  $B(x_1) \in C^2(\mathbf{R}; \mathbf{R})$ , real-valued  $C^2$ -functions on  $\mathbf{R}$ . Moreover,  $B(x_1)$  is a monotone increasing in  $x_1$  and there exist positive numbers  $B_{\pm} > 0$  such that

$$B_- < B_+,$$
$$\lim_{x_1 \rightarrow \pm\infty} B(x_1) = B_{\pm}.$$

Under the assumption (B.1), we define the vector potential  $b(x_1)$  as follows.

$$b(x_1) = \int_0^{x_1} B(t) dt.$$

In the case where  $V(x_1, x_2) \equiv 0$ , the spectrum of  $H_0$  has a band structure if (B.1) holds (See, [Iwa]):

$$\sigma(H_0) = \sigma_{ac}(H_0) = \bigcup_{n=1}^{\infty} [\Lambda_n^-, \Lambda_n^+],$$

$$\Lambda_n^{\pm} = (2n - 1)B_{\pm}.$$

(B.2) $_{\pm}$  In addition to (B.1),  $B(x_1) \in \mathcal{B}^{\infty}(\mathbf{R})$ , moreover, there exists  $M > 0$  such that, for each  $\alpha \in N \cup \{0\}$ ,

$$|\partial_1^{\alpha}(B_{\pm} - B(x_1))| \leq C_{M\alpha} \langle x_1 \rangle^{-M} \quad \text{as } x_1 \rightarrow \pm\infty$$

holds for some constant  $C_{M\alpha}$ , where

$$\mathcal{B}^{\infty}(\mathbf{R}) = \{f \in C^{\infty}(\mathbf{R}) \mid \text{for each } \alpha, \|\partial^{\alpha} f\|_{\infty} < \infty\},$$

and  $\|\cdot\|_{\infty}$  denotes the usual  $L^{\infty}$ -norm.

(B.3) In addition to (B.1), assume that  $B(x_1)$  fulfills the following conditions:

$$\begin{aligned} B_+ &< 3B_- , \\ \|\partial_1 B\|_{\infty} &\leq B_+ - B_- , \\ (B_+ - B_-)(1 + \frac{1}{\sqrt{3B_- - B_+}}) &< \frac{B_+ + B_-}{6} . \end{aligned}$$

(V.1)  $V(x_1, x_2) \in C^{\infty}(\mathbf{R}^2; \mathbf{R})$ , real-valued  $C^{\infty}$ -functions on  $\mathbf{R}^2$ , and there exists  $m > 0$  such that

$$|\partial_1^{\alpha} \partial_2^{\beta} V(x_1, x_2)| \leq C_{\alpha\beta} \langle x_1, x_2 \rangle^{-m-\alpha-\beta}$$

holds for some positive constant  $C_{\alpha\beta}$  independent of  $(x_1, x_2)$  in  $\mathbf{R}^2$ .

Here  $\partial_1, \partial_2$  denotes  $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$  respectively and  $\langle x_1, x_2 \rangle = (1 + x_1^2 + x_2^2)^{\frac{1}{2}}$ .

It is well-known that the operator  $H_V$  defined on  $C_0^{\infty}(\mathbf{R}^2)$  is essentially self-adjoint and  $V$  is a relatively compact perturbation with respect to  $H_0$  ([L-S]). Thus one expects that  $H_V$  have discrete spectra in the spectral gaps of  $H_0$  and they accumulate at most to the tips of the gap. In the case where  $b(x_1)$  is the vector potential which gives a constant magnetic field,

the eigenvalue asymptotics around the essential spectrum tips is investigated ([Rai1], [Rai2]).

For  $\mu > 0, a_0 \in \mathbf{R}$ , define

$$\nu_{+}^{\pm}(\mu; a_0) = \frac{1}{2\pi} \text{vol}\{(x_1, x_2) \in \mathbf{R}^2 | x_1 > a_0, \pm V(x_1, x_2) > \mu\}.$$

and

$$\nu_{-}^{\pm}(\mu; a_0) = \frac{1}{2\pi} \text{vol}\{(x_1, x_2) \in \mathbf{R}^2 | -x_1 > a_0, \pm V(x_1, x_2) > \mu\}.$$

For simplicity, we denote  $\nu_{+}^{+}(\mu; a_0), \nu_{-}^{-}(\mu; a_0)$  by  $\nu_{+}(\mu; a_0), \nu_{-}(\mu; a_0)$  respectively.

For a positive, decreasing function  $f$ , we say that  $f$  satisfies (T) if

(T) there exist positive numbers  $\gamma, \gamma', \mu_0$  such that

$$\frac{f(\mu_1)}{f(\mu_2)} \leq \left( \frac{\mu_2}{\mu_1} \right)^{\gamma} \quad (1.1)$$

holds for  $\mu_1, \mu_2 \in (0, \mu_0)$  with  $\mu_1 < \mu_2$ . Moreover

$$f(\mu) \geq \gamma' \mu^{-\frac{2}{m}}$$

holds for  $\mu \in (0, \mu_0)$ .

Let  $S$  be a self-adjoint operator in a Hilbert space, and suppose  $S$  has purely discrete spectra in an open interval  $(a, b) \subset \mathbf{R}$ . Then  $N((a, b)|S)$  denotes the total multiplicity of the eigenvalues of  $S$  lying on  $(a, b)$ , i.e.,

$$N((a, b)|S) = \dim(\text{Ran } E_S(a, b))$$

where  $E_S(a, b)$  denotes the spectral projection of  $S$  on  $(a, b)$ .

We devote ourself to get the asymptotics at some specific gap such that

$$\Lambda_n^{+} < \Lambda_{n+1}^{-}$$

holds where  $\Lambda_0^{+} = 0$ . Thus we shall consider such a gap.

One of the main theorems is:

**Theorem 1.1** Suppose that (V.1), (B.2)<sub>+</sub> (resp. (B.2)<sub>-</sub>) and (B.3) with  $m < M$ . Moreover suppose that  $\nu_{+}(\mu; a_0)$  (resp.  $\nu_{-}(\mu; a_0)$ ) satisfies (T) and  $\nu_{+}^{-}(\mu; a_0)$  (resp.  $\nu_{-}^{+}(\mu; a_0)$ ) satisfies (1.1). Then we have

$$\begin{aligned} N((\Lambda_n^{+} + \mu, M_n)|H_V) &= B_{+}\nu_{+}(\mu; a_0)(1 + o(1)) \quad \text{as } \mu \downarrow 0, \\ (\text{resp. } N((M_n, \Lambda_{n+1}^{-} - \mu)|H_V) &= B_{-}\nu_{-}(\mu; a_0)(1 + o(1)) \quad \text{as } \mu \downarrow 0, ) \end{aligned}$$

where we put  $M_n = \frac{\Lambda_n^{+} + \Lambda_{n+1}^{-}}{2}$

**Remark 1.1** (i) As  $\mu \downarrow 0$ , the asymptotic behavior of  $\nu_{\pm}^{\pm}(\mu; a_0)$  does not depend on a choice of  $a_0 > 0$ . (A similar assertion holds for  $\nu_{\pm}^{\pm}(\mu; a_0)$ .)

(ii) In case  $V(x_1, x_2)$  is non-negative (resp. non-positive), it follows from the proof that the assumption on  $\nu_{-}(\mu; a_0)$  (resp.  $\nu_{+}(\mu; a_0)$ ) is not needed.

In the case where the scalar potential  $V(x_1, x_2)$  decays slowly, i.e., of order  $m$  with  $0 < m < 1$ , satisfying the assumption (V.2) with the constant  $m$ , some of assumptions on  $V(x_1, x_2)$  and  $B(x_1)$  can be weakened:

(V.2)  $V(x_1, x_2) \in C^2(\mathbf{R}^2; \mathbf{R})$  and there exist  $m, m', C > 0$  such that

$$\begin{aligned} 0 < m < 1 \quad , \quad 2m < m' \quad , \\ |V(x_1, x_2)| &\leq C \langle x_1; x_2 \rangle^{-m} \quad , \\ |\partial_1 V(x_1, x_2)| + |\partial_2 V(x_1, x_2)| &\leq C \langle x_1; x_2 \rangle^{-m'} \quad . \end{aligned}$$

(B.4) $_{\pm}$  In addition to (B.1), there exist constants  $M, M', C$  such that

$$\begin{aligned} M' &> 3M \\ |B(x_1) - B_{\pm}| &\leq C \langle x_1 \rangle^{-M} \quad \text{as } x_1 \rightarrow \pm\infty, \\ |\partial_1 B(x_1)| &\leq C \langle x_1 \rangle^{-M'} \quad \text{as } x_1 \rightarrow \pm\infty. \end{aligned}$$

The other of the main theorems is:

**Theorem 1.2** Suppose that (V.2) and (B.4) $_{+}$  (resp. (B.4) $_{-}$ ) hold with  $M > m$ . And suppose that  $\nu_{+}(\mu; a_0)$  (resp.  $\nu_{-}(\mu; a_0)$ ) satisfies (T). Then we have the same eigenvalue asymptotics as in Theorem 1.1.

We shall give only a proof of Theorem 1.2 in the following sections. Theorem 1.2 can be prove using the min-max principle and estimates of the number of eigenvalues of self-adjoint operators associated with suitable quadratic forms derived from the results in [Col].

## 2 Direct integral decomposition

[Iwa] proved that  $H_0$  is unitarily equivalent to the self-adjoint operator  $L$  acting in  $L^2(\mathbf{R}_{x_1} \times \mathbf{R}_{\xi})$  that has the (constant fiber) direct integral decomposition (see, e.g., [R-S4]) :

$$L = \int_{\mathbf{R}_{\xi}}^{\oplus} L(\xi) d\xi, \quad (2.1)$$

using the partial Fourier transformation

$$(\mathcal{F}u)(x_1, \xi) = (2\pi)^{-\frac{1}{2}} \int e^{-ix_2 \xi} u(x_1, x_2) dx_2 \quad (2.2)$$

which is a unitary operator from  $L(\mathbf{R}_{x_1} \times \mathbf{R}_{x_2})$  to  $L(\mathbf{R}_{x_1} \times \mathbf{R}_\xi)$ . Here for each  $\xi$  in  $\mathbf{R}$ ,  $L(\xi)$  is a second-order ordinary differential operator in  $L^2(\mathbf{R}_{x_1})$  of the form :

$$L(\xi) = -\frac{d^2}{dx_1^2} + (\xi - b(x_1))^2. \quad (2.3)$$

**Lemma 2.1** ([Iwa]) *Assume that (B.1) holds. Let  $\xi$  be a real number. Then there exists a complete orthonormal system  $\{\varphi_n(x_1, \xi)\}_{n=1}^\infty$  in  $L^2(\mathbf{R}_{x_1})$  of eigenfunctions for  $L(\xi)$  :*

$$L(\xi)\varphi_n(x_1, \xi) = \lambda_n(\xi)\varphi_n(x_1, \xi), \quad (2.4)$$

$$0 < \lambda_1(\xi) < \lambda_2(\xi) < \lambda_3(\xi) < \dots \rightarrow \infty, \quad (2.5)$$

so that for  $n \in \mathbf{N}$

- (i) each  $\lambda_n(\xi)$  is non-degenerate, and depends analytically on  $\xi$ ,
- (ii)  $\lambda_n(\xi)$  is monotone increasing in  $\xi$ , and  $\lim_{\xi \rightarrow \pm\infty} \lambda_n(\xi) = \Lambda_n^\pm$ ,
- (iii)  $\varphi_n(\cdot, \xi) \in D(L(0))$  and depends analytically on  $\xi$  with respect to the graph norm  $\|u\|_{1,0} \equiv (\|u\|^2 + \|L(0)u\|^2)^{\frac{1}{2}}$ , where  $\|\cdot\|$  stands for the  $L^2$ -norm.
- (iv)  $\varphi_n(x_1, \xi)$  is a real-valued continuous function of  $x_1$  and  $\xi$ , and, moreover  $\varphi_n(x_1, \xi)$  is infinitely differentiable in  $x_1$  for each  $\xi$  and is analytic in  $\xi$  for each  $x_1$ .

*Proof.* See lemma 2.3 and a remark at the end of [Iwa].  $\square$

Now we consider the following assumption on the eigenvalues  $\{\lambda_n(\xi)\}$ :

- (A.1) There exists a constant  $C > 0$  such that for  $j, k \in \mathbf{N}$ ,  $j \neq k$ ,
- $$|\lambda_j(\xi) - \lambda_k(\xi)| \geq C \text{ holds for all } \xi \in \mathbf{R}.$$

Although it is not trivial whether the (non-constant) magnetic fields satisfying this condition (A.1) in addition to (B.2) exist, but the following lemma gives an answer. We shall give the proof in Sect.12.

**Lemma 2.2** (B.3) *implies (A.1).*

Hence we get Theorem 1.1 if only we prove the following theorem:

**Theorem 2.3** *Under the same assumptions as in Theorem 1.1, except that (B. 3) is replaced by (A.1), we have the same eigenvalue asymptotics.*

### 3 Proof of Theorem 2.3

In the proof of Theorem 2.3, we denote the variables  $(x_1, x_2)$  by  $(x, y)$  for notational convenience. Corresponding to this,  $\partial_1, \partial_2$  shall be replaced by  $\partial_x, \partial_y$  etc. And we shall often denote by  $C$  various (positive) constants appeared in estimates. In the case where we want to specify the dependence of some constants, we shall denote them by  $C_\varepsilon, C(\eta)$  or  $C_{\alpha, \beta}$  etc.

Using the partial Fourier transformation  $\mathcal{F}$  defined by (2.2), we consider the operator  $L_V$  as follows.

$$L_V = L + \mathcal{F}V\mathcal{F}^{-1} \quad \text{in } L^2(\mathbf{R}_x \times \mathbf{R}_\xi), \quad (3.1)$$

where  $V$  stands for the multiplication operator by  $V(x, y)$  in  $L^2(\mathbf{R}_x \times \mathbf{R}_y)$  (Generally we shall use the notation  $f$  to the multiplication operator  $f(x)$  acting in a function space throughout this paper).

In the sequel we denote  $\mathcal{F}V\mathcal{F}^{-1}$  by  $\tilde{V}$ .

**Lemma 3.1** ([Iwa]) *Assume (B.1) holds. For each  $n \in \mathbf{N}$ , let  $\mathcal{H}_n$  be the closed subspace of  $L^2(\mathbf{R}_x \times \mathbf{R}_\xi)$  defined by*

$$\mathcal{H}_n = \{\varphi_n(x, \xi)f(\xi) | f(\xi) \in L^2(\mathbf{R}_\xi)\} \quad (3.2)$$

where  $\varphi_n(x, \xi)$  is as in Lemma 2.1. Then we have

- (i)  $L^2(\mathbf{R}_x \times \mathbf{R}_\xi) = \sum_n \oplus \mathcal{H}_n$  (the orthogonal sum of Hilbert spaces).
- (ii)  $L$  is reduced by  $\mathcal{H}_n$ .
- (iii)  $L|_{\mathcal{H}_n}$  (restriction of  $L$  to  $\mathcal{H}_n$ ) is unitarily equivalent to the operator of multiplication by  $\lambda_n(\xi)$  on  $L^2(\mathbf{R}_\xi)$ .

where  $\varphi_n(x, \xi)$  and  $\lambda_n(\xi)$  is as in Lemma 2.1.

*Proof.* See [Iwa], Lemma 2.5.  $\square$

We define the operator

$$T_n : L^2(\mathbf{R}_\xi) \longrightarrow \mathcal{H}_n \left( \hookrightarrow L^2(\mathbf{R}_x \times \mathbf{R}_\xi) \right)$$

by

$$(T_n f)(x, \xi) = \varphi_n(x, \xi)f(\xi) \quad (3.3)$$

for  $f(\xi) \in L^2(\mathbf{R}_\xi)$  (then we can find

$$T_n^* : \mathcal{H}_n \longrightarrow L^2(\mathbf{R}_\xi)$$

by

$$(T_n^* F)(\xi) = \int_{\mathbf{R}_x} \varphi_n(x, \xi) F(x, \xi) dx \quad (3.4)$$

for  $F(x, \xi) \in \mathcal{H}_n$ , and define  $P_n : L^2(\mathbf{R}_x \times \mathbf{R}_\xi) \longrightarrow L^2(\mathbf{R}_x \times \mathbf{R}_\xi)$  by

$$(P_n u)(x, \xi) = \varphi_n(x, \xi) \int_{\mathbf{R}_x} \varphi_n(x, \xi) u(x, \xi) dx \quad (3.5)$$

for  $u(x, \xi) \in L^2(\mathbf{R}_x \times \mathbf{R}_\xi)$ .

Note that  $P_n$  is the orthogonal projection with the range  $\mathcal{H}_n$  and  $T_n$  is a unitary operator from  $L^2(\mathbf{R}_\xi)$  to  $\mathcal{H}_n$  which gives the equivalence stated in Lemma 2.1(iii). Furthermore  $T_n T_n^* P_n = P_n$  on  $L^2(\mathbf{R}_x \times \mathbf{R}_\xi)$  holds.

**Lemma 3.2**  $\tilde{V} P_n$  is a compact operator on  $L^2(\mathbf{R}_x \times \mathbf{R}_\xi)$ .

*Proof.* We can find that

$$\tilde{V} P_n = \mathcal{F} V \mathcal{F}^{-1} P_n \quad (3.6)$$

$$= \mathcal{F} V (H_0 - i)^{-1} \mathcal{F}^{-1} \mathcal{F} (H_0 - i) \mathcal{F}^{-1} P_n \quad (3.7)$$

$$= \mathcal{F} V (H_0 - i)^{-1} \mathcal{F}^{-1} (L - i) P_n. \quad (3.8)$$

Then  $\tilde{V} P_n$  is compact, since  $V(H_0 - i)^{-1}$  is compact (see [A-H-S], Theorem 2.6) and  $(L - i)P_n$  is bounded by Lemma 3.1(ii) and the closed graph theorem.

□

Set

$$K = -i(P_n \tilde{V} - \tilde{V} P_n)$$

and denote by  $K_+, K_-$  the positive and negative part of  $K$  respectively so that  $K = K_+ - K_-$ ,  $|K| = K_+ + K_-$ . Further, for  $\varepsilon > 0$  denote by  $Y_n^\pm(\varepsilon)$  the operator associated with the quadratic form

$$(Y_n^\pm(\varepsilon)u, u) = \|i\varepsilon K_+^{\frac{1}{2}} Q_n u \pm \varepsilon K_+^{\frac{1}{2}} P_n u\|^2 + \|i\varepsilon K_-^{\frac{1}{2}} Q_n u \mp \varepsilon K_-^{\frac{1}{2}} P_n u\|^2 \quad (3.9)$$

for  $u \in L^2(\mathbf{R}^2)$ , where  $Q_n = I - P_n$ . Throughout this paper,  $(\cdot, \cdot)$  stands for the standard inner product of  $L^2$ . It is easy to see, by the definition, that  $Y_n^\pm(\varepsilon)$  are compact and nonnegative self-adjoint operators. And direct computations lead us to

$$L_V = P_n L_V P_n + Q_n (L_V \pm \varepsilon^{-2} |K|) Q_n \pm \varepsilon^2 P_n |K| P_n \mp Y_n^\pm(\varepsilon). \quad (3.10)$$

Let us prepare a useful inequality called the Weyl-Ky Fan inequality. We shall frequently make use of it to estimate the upper bound of the number of eigenvalues:

**Lemma 3.3** ([Rai1]) *Let  $A_0, A_1$  be bounded self-adjoint operators acting on a Hilbert space. Assume  $A_1$  is compact and set  $A = A_0 + A_1$ . Then the estimates*

$$\begin{aligned} N((\mu_1, \mu_2) | A_0) &\leq N((\mu_1 - \tau_1, \mu_2 + \tau_2) | A) \\ &\quad + N((\tau_1, \infty) | -A_1) + N((\tau_2, \infty) | A_1) \end{aligned}$$

hold for each interval  $(\mu_1, \mu_2) \subset \mathbf{R}$  and every  $\tau_1 > 0, \tau_2 > 0$ .

*Proof.* See [Rai1], Lemma 5.4.  $\square$

Applying Lemma 3.3 to the former of (3.10) twice (first, with  $A_1 = Y_n^+(\varepsilon)$ ,  $A_0 = L_V$ ,  $\mu_1 = \Lambda_n^+ + \mu$ ,  $\mu_2 = M_n$ ,  $\tau_1 = \frac{\varepsilon}{2}\mu$ ,  $\tau_2 = \frac{\varepsilon}{2}$ , and, second, with  $A_1 = P_n|K|P_n$ ,  $A = P_n L_V P_n + Q_n(L_V + \varepsilon^{-2}|K|)Q_n$ ,  $\nu_1 = \Lambda_n^+ + (1 - \frac{\varepsilon}{2})\mu$ ,  $\mu_2 = M_n + \frac{\varepsilon}{2}$ ,  $\tau_1 = \frac{\varepsilon}{2}\mu$ ,  $\tau_2 = \frac{\varepsilon}{2}$ , and using the non-negativity of  $Y_n^+(\varepsilon)$  and  $\varepsilon^2 P_n|K|P_n$ , we find

$$\begin{aligned}
& N((\Lambda_n^+ + \mu, M_n)|L_V) \\
\leq & N((\Lambda_n^+ + (1 - \varepsilon)\mu, M_n + \varepsilon)|P_n L_V P_n) \\
& + N((\Lambda_n^+ + (1 - \varepsilon)\mu, M_n + \varepsilon)|Q_n(L_V + \varepsilon^{-2}|K|)Q_n) \\
& + N((\frac{\mu}{2\varepsilon}, \infty)|P_n|K|P_n) \\
& + N((\frac{\varepsilon}{2}, \infty)|Y_n^+(\varepsilon))
\end{aligned} \tag{3.11}$$

where we also used, at the second inequality, the fact that  $P_n L_V P_n + Q_n(L_V + \varepsilon^{-2}|K|)Q_n$  is a direct sum of two operators, for  $P_n$  and  $Q_n = I - P_n$  are orthogonal projections.

To obtain the converse inequality, apply Lemma 3.3 again to the latter half of (3.10) twice (first with  $A_0 = -\varepsilon^2 P_n L_V P_n + Q_n(L_V + |K|)Q_n$ ,  $A_1 = Y_n^-(\varepsilon)$ ,  $\tau_1 = \frac{\varepsilon}{2}\mu$ ,  $\tau_2 = \frac{\varepsilon}{2}$ , and second, with  $A_1 = -\varepsilon^2 P_n|K|P_n$ ,  $A = L_V$ ,  $\tau_1 = \frac{\varepsilon}{2}\mu$ ,  $\tau_2 = \frac{\varepsilon}{2}$ ). Then we get the following estimate as before:

$$\begin{aligned}
& N((\Lambda_n^+ + \mu, M_n)|L_V) \\
\geq & N((\Lambda_n^+ + (1 + \varepsilon)\mu, M_n - \varepsilon)|P_n L_V P_n) \\
& + N((\Lambda_n^+ + (1 + \varepsilon)\mu, M_n - \varepsilon)|Q_n(L_V - \varepsilon^{-2}|K|)Q_n) \\
& - N((\frac{\mu}{2\varepsilon}, \infty)|P_n|K|P_n) \\
& - N((\frac{\varepsilon}{2}, \infty)|Y_n^-(\varepsilon)) .
\end{aligned} \tag{3.12}$$

In what follows we treat only the asymptotics of  $N((\Lambda_n^+ + \mu, M_n)|L_V)$ , since we can prove the case of minus sign of Theorem 2.3 (, as we shall meet later, also in the case of Theorem 1.2) in the same way with obvious modifications.

**Lemma 3.4** For  $\mu > 0$ ,

$$N((\mu, \infty)|P_n|K|P_n) \leq 2N((\frac{\mu^2}{4}, \infty)|T_n^* \tilde{V}^2 T_n)$$

holds.

*Proof.* Set

$$E = P_n \tilde{V}, \quad K' = P_n \tilde{V} + \tilde{V} P_n,$$



then we have

$$K^2 \leq K^2 + (K')^2 = 2(E^*E + EE^*). \quad (3.13)$$

By the variational principle, it is easily seen that  $\mu_k(P_n|K|P_n) \leq \mu_k(|K|)$  where  $\mu_k(\cdot)$  stands for the  $k$ -th eigenvalue, of decreasing order, counting multiplicity and  $\langle \psi_1, \dots, \psi_l \rangle^\perp$  is shorthand for  $\{\varphi | (\varphi, \psi_k) = 0, k = 1, \dots, l\}$ . Then it follows that

$$\begin{aligned} N((\mu, \infty)|P_n|K|P_n) &\leq N((\mu, \infty)||K|) \\ &\leq N((\mu^2, \infty)|K^2) \\ &= N((\frac{\mu^2}{2}, \infty)|E^*E + EE^*) \end{aligned} \quad (3.14)$$

where we used (3.13) at the third inequality.

We choose  $\varphi_1, \dots, \varphi_N$  (resp.  $\varphi_{N+1}, \dots, \varphi_{N+M}$ ) to be an orthonormal basis of  $\text{Ran}E^{(1)}(\frac{\mu^2}{4}, \infty)$  (resp.  $\text{Ran}E^{(2)}(\frac{\mu^2}{4}, \infty)$ ), where we denote the spectral projection of the self-adjoint operator  $E^*E$  (resp.  $EE^*$ ) by  $E^{(1)}(\cdot)$  (resp.  $E^{(2)}(\cdot)$ ). Now let  $\varphi$  be arbitrary element such that  $\varphi \in \langle \varphi_1, \dots, \varphi_N \rangle^\perp$  and  $\|\varphi\| = 1$ . Then

$$(\varphi, (E^*E + EE^*)\varphi) \leq \frac{\mu^2}{4} + \frac{\mu^2}{4} = \frac{\mu^2}{2}$$

holds. From this inequality and the variational principle, we get

$$\mu_{N+M+1}(E^*E + EE^*) \leq \frac{\mu^2}{2}.$$

Henceforth, it follows that

$$\begin{aligned} &N((\frac{\mu^2}{2}, \infty)|E^*E + EE^*) \\ &\leq N + M \\ &= N((\frac{\mu^2}{4}, \infty)|E^*E) + N((\frac{\mu^2}{4}, \infty)|EE^*). \end{aligned}$$

By considering the canonical form of compact operators  $E^*E$  and  $EE^*$ , we conclude that two terms in the R.H.S. of the above inequality are equal. Finally the statement of the lemma follows from the fact that  $P_n \tilde{V}^2 P_n|_{\mathcal{H}_n}$  is unitarily equivalent to  $T_n^* \tilde{V}^2 T_n$ .  $\square$

**Lemma 3.5** *For  $\varepsilon > 0$  small enough, there exists  $C_1(\varepsilon) > 0$  independent of  $\mu$  such that*

$$N((\Lambda_n^+ + (1 \pm \varepsilon)\mu, M_n \pm \varepsilon)|Q_n(L_V - \varepsilon^{-2}|K|)Q_n) \leq C_1(\varepsilon)$$

*holds.*

*Proof.* Using the fact that  $L$  is reduced by  $\text{Ran} Q_n$ ,

$$\begin{aligned} & Q_n(L_V - \varepsilon^{-2}|K|)Q_n(Q_n L Q_n - i)^{-1} \\ = & Q_n(L_V - \varepsilon^{-2}|K|)(L - i)^{-1}(L - i)(Q_n L Q_n - i)^{-1}Q_n \\ & \pm \varepsilon^{-2}Q_n|K|(Q_n L Q_n - i)^{-1}Q_n . \end{aligned}$$

We observe that  $\tilde{V}(L - i)^{-1}$  is compact, as commented in the proof of Lemma 3.2, and  $(L - i)(Q_n L Q_n - i)^{-1}Q_n$  is bounded by the closed graph theorem, and that the last term of the R.H.S. is compact owing to  $|K|$ . Therefore  $Q_n(L_V - \varepsilon^{-2}|K|)Q_n$  is relatively compact with respect to  $Q_n L Q_n$ . Finally,

$$\begin{aligned} & \sigma_{ess}(Q_n L Q_n) \cap (\Lambda_n^+ + (1 \pm \varepsilon)\mu, M_n \pm \varepsilon) \\ = & \bigcup_{j \neq n} [\Lambda_j^-, \Lambda_j^+] \cap (\Lambda_n^+ + (1 \pm \varepsilon)\mu, M_n \pm \varepsilon) \\ = & \emptyset , \end{aligned}$$

holds for  $\varepsilon > 0$  small enough. Putting together these facts, we come to the conclusion.  $\square$

We state a key proposition without proof. This can be proved using the asymptotic estimate of the number of eigenvalues of pseudodifferential operators of negative order ([D-R]) :

**Proposition 3.6** (i) Assume (V.1), (B.2) and (A.1) hold. Moreover assume that  $\nu_{\pm}^{\pm}(\mu)$  satisfy the condition (T). Then we have

$$N((\Lambda_n^+ + \mu, M_n)|T_n^* L_V T_n) = B_+ \nu_+(\mu)(1 + o(1)) \quad \text{as } \mu \downarrow 0 .$$

(ii) Under the same assumptions as (i), we have

$$\lim_{\varepsilon \downarrow 0} \limsup_{\mu \downarrow 0} N((\frac{\mu^2}{\varepsilon}, \infty)|T_n^* \tilde{V}^2 T_n) / \nu_+(\mu) = 0 .$$

Now let us set about a proof of one of main theorems.

*Proof of Theorem 2.3.* Since  $Y_n^{\pm}(\varepsilon)$  is compact, for each  $\varepsilon > 0$ , there exists a constant  $C_2(\varepsilon) > 0$  independent of  $\mu$ , it is derived that

$$N((\frac{\varepsilon}{2}, \infty)|Y_n^{\pm}(\varepsilon)) \leq C_2(\varepsilon) . \quad (3.15)$$

Putting together (3.11), (3.12), (3.15), lemma 3.4, and lemma 3.5,

$$\begin{aligned} \pm N((\Lambda_n^+ + \mu, M_n)|L_V) & \leq \pm N((\Lambda_n^+ + (1 - \varepsilon)\mu, M_n + \varepsilon)|T_n^* L_V T_n) \\ & + N((\frac{\mu^2}{8\varepsilon^2}, \infty)|T_n^* \tilde{V}^2 T_n) \\ & \pm C_1(\varepsilon) + C_2(\varepsilon) . \end{aligned}$$

Furthermore, by Proposition 3.6,

$$\pm \lim_{\varepsilon \downarrow 0} \limsup_{\mu \downarrow 0} N((\Lambda_n^+ + \mu, M_n)|L_V) / B_+ \nu_+(\mu) \leq \pm 1$$

holds where we also used (T). This proves Theorem 2.3.  $\square$

## References

- [A-H-S] Avron, J., Herbst, I. and Simon, B., Schrödinger operators with magnetic fields I. General interactions Duke Math. J., **45**(4) 847-883 (1978).
- [Col] Colin de Verdiere, Yves., L'asymptotique de Weyl pour les bouteilles magnétiques, Commun. Math. Phys., **105** 327-335 (1986).
- [D-R] Dauge, M. and Robert, D., Weyl's formula for a class of pseudodifferential operators with negative order on  $L^2(\mathbf{R}^n)$ , Lecture notes in Math. **1256** 91-122 (1987).
- [Iwa] Iwatsuka, A., Examples of absolutely continuous Schrödinger operators in magnetic fields, Publ. RIMS, Kyoto Univ., **21** 385-401 (1985).
- [L-S] Leinfelder, H. and Simader, C. G., Schrödinger operators with singular magnetic potentials, Math. Z., **176** 1-19 (1981).
- [Rai1] Raikov, G. D., Eigenvalue asymptotics for the Schrödinger operator with homogeneous magnetic potential and decreasing electric potential I. Behaviour near the essential spectrum tips, Comm. in P.D.E., **15**(3) 407-434 (1990).
- [Rai2] ———, Border-line eigenvalue asymptotics for the Schrödinger operator with electromagnetic potential, Integral Eq. and Operator Theory, **14** 875-888 (1991).
- [R-S4] Reed, M and Simon, B., *Methods of Modern Mathematical Physics*, Vol. IV, Academic Press, New York, 1978.