

G.A.G.A. affine by polynomial growth

by

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§0. Introduction

P. Deligne and G. Malsiniotis studied G.A.G.A. on a non-singular separated scheme of finite type over $\mathbb{C}$. Their method is dependent on L^2 estimates for the $\bar{\partial}$ operator by Hörmander. On the other hand, using the Čech theory and Cousin integrals, N. Sasakura studied a polynomial growth cochain complex for a p.g. coherent sheaf. We shall define a polynomial growth cochain complex for a locally free sheaf (i.e. a vector bundle) by his techniques in [2] and give G.A.G.A. similar to the one due to [1]. This is a generalization of the theorem (3.4) in [2].

The contents of the various sections are as follows. In §1 we summarise some results in [2], which we shall use in the last section. In §2 we shall define a polynomial growth cochain complex for locally free sheaf over a smooth affine variety over $\mathbb{C}$. In §3 we shall give G.A.G.A. (Th. (3.0)) for a vector bundle over a smooth affine variety.

The idea of this paper is much in debt to Professor N. Sasakura. Professor T. Kori has encouraged me in my studies. I wish to thank them.

§1. Preliminaries

Let X be a smooth affine variety over $\mathbb{C}$. Unless we say otherwise, we regard X as the analytic variety (with the underlying analytic structure). Moreover, $\mathcal{O}_X$ = the structure sheaf of the analytic variety. Let $X \subset \mathbb{C}^N$ be an embedding of X and $g = |z| + 1$ a growth function increasing at infinity, where z is the coordinate of $\mathbb{C}^N$.

Definition(1.0)

By a polynomial growth (or simply p.g.) covering of X of size $\sigma = (\sigma_1, \sigma_2)$ ($\sigma_1 \geq 1, \sigma_2 \geq 1$) with respect to g , we mean the collection as follows:

$\hat{A}_\sigma(X; g)$ (or simply, $= \hat{A}_\sigma(X)$) $= \{\tilde{U}_\sigma(Q; g) = X \cap U_\sigma(Q; g); Q \in X\}$, where $U_\sigma(Q; g)$ is the disc, whose center is Q and radius is $\frac{1}{\sigma_1} g(Q)^{-\sigma_2}$.

Definition(1.1)

By a p.g. cochain complex of $\mathcal{O}_X^k$ for $\hat{A}_\sigma(X)$, we mean the subgroup of the cochain complex $C^*(\hat{A}_\sigma(X), \mathcal{O}_X^k)$ as follows:

$C^*(\hat{A}_\sigma(X), \mathcal{O}_X^k; g)_{p.g.} = \{\varphi \in C^*(\hat{A}_\sigma(X), \mathcal{O}_X^k); \exists \alpha, \beta \geq 1, |\varphi| \leq \alpha(|z| + 1)^\beta\}$
 , where $|\varphi|$ runs through all the components of φ
 , where $\mathcal{O}_X^k$ is a direct sum of $\mathcal{O}_X$.

Theorem(1.2)

Let X be as above. Then $H^q(\varinjlim C^*(\hat{A}_\sigma(X), \mathcal{O}_X^k)_{p.g.}) = 0$ for $q \geq 1$
 $H^0(\varinjlim C^*(\hat{A}_\sigma(X), \mathcal{O}_X^k)_{p.g.}) = \Gamma(X_{\text{alg}}, \mathcal{O}_{X_{\text{alg}}}^k)$

, where $\mathcal{O}_{X_{\text{alg}}}$ is a structure sheaf as an algebraic variety.

We shall extend it to a locally free sheaf in the last section.

2. Polynomial growth cochain complexes for locally free sheaves.

(i) Let $X, \mathcal{O}_X$ and g be the same as those in §1.

Proposition(2.1)

Let $X \subset \mathbb{C}^{N'}$ be another embedding of X and $g = |z'| + 1$ a growth function, where z' is the coordinate of $\mathbb{C}^{N'}$. Then

$$\lim_{\sigma \rightarrow 0} C^*(\hat{\mathcal{H}}_{\sigma}(X), \mathcal{O}_X^k; g)_{p.g} = \lim_{\sigma \rightarrow 0} C^*(\hat{\mathcal{H}}_{\sigma}(X), \mathcal{O}_X^k; g')_{p.g}.$$

In order to prove this proposition we need the following Lemma.

Lemma(2.2)

Let $H(z)$ be a polynomial of $z = (z_1, \dots, z_N)$, then there exist

$\alpha_1 > 0, \alpha_2 > 0, \beta_1 > 0, \beta_2 > 0$ such that

$$(*) \quad \alpha_1 (|z| + 1)^{\alpha_1} \leq |H(z)| + 1 \leq \beta_1 (|z| + 1)^{\beta_1}.$$

proof of (2.2)

We may write $H(z) = \sum_{|\nu| \leq n} a_{\nu} z_1^{\nu_1} \dots z_N^{\nu_N}$. Then we have

$$(a) \quad |H(z)| \leq \sum_{|\nu| \leq n} |a_{\nu}| z_1^{\nu_1} \dots z_N^{\nu_N} \leq \max_{|\nu| \leq n} |a_{\nu}| \cdot (|z| + 1)^n.$$

We derive the inequality of the right hand side in (*) from (a).

On the other hand we divide the numerator and the denominator of the following function by $|z|^n$,

$$(b) \quad \frac{(|z| + 1)^n}{\left| \sum_{|\nu| \leq n} a_{\nu} z_1^{\nu_1} \dots z_N^{\nu_N} \right| + 1}$$

And $|z| \rightarrow \infty$, then (b) has a non-zero limit, so that there exist

$r > 0$ and $M' > 0$ such that

$$(c) \quad (|z| + 1)^n \leq M' (|H(z)| + 1) \quad \text{for } |z| > r.$$

Now we have

$$(d) \quad (|z| + 1)^n \leq (r + 1)^n \quad \text{for } |z| \leq r.$$

Then we get the following inequality by (c) and (d):

$$(e) \quad \frac{1}{M} (|z| + 1)^n \leq |H(z)| + 1, \text{ where } M = \max((r + 1)^n, M').$$

Q.E.D

proof of (2.1)

By the assumption of the embedding of X there exist polynomials $f_i(z) \in \mathbb{C}[z_1, \dots, z_N]$ ($i = 1, \dots, N$) such that

$$(a) \quad z'_i = f_i(z) \quad (i = 1, \dots, N').$$

From (a) and Lemma(2.2) we have

$$(b) \quad \alpha_1(|z| + 1)^{\alpha_2} \leq |z'| + 1 \leq \beta_1(|z| + 1)^{\beta_2} \text{ on } X.$$

We take an element $\varphi \in C^*(\hat{A}_G(X), \mathcal{O}_X^k; g)_{p.g.}$. By the definition of $C^*(\hat{A}_G(X), \mathcal{O}_X^k; g)_{p.g.}$ and (b) we get

$$(c) \quad |\varphi| = \gamma_1(|z'| + 1)^{\gamma_2}.$$

(c) shows that φ is contained in $C^*(\hat{A}_G(X), \mathcal{O}_X^k; g')_{p.g.}$.

Q.E.D.

Prop.(2.1) shows that p.g cochain complex is independent of any embedding of X .

Let E be an algebraic vector bundle over X and $X = \bigcup_{i \in I} X_{f_i}$ ($= \bigcup_{i \in I} X_i$ simply) an affine open covering, where E is trivial over each X_i and f_i ($i \in I$) is a polynomial. Taking a subset J of I , X_J denotes $X_{f_1} \cap \dots \cap X_{f_s} = X_{f_1, \dots, f_s}$. There is a wellknown diagram:

$$\begin{array}{ccc} \mathbb{C}^N - \{f_1, \dots, f_s = 0\} & \longrightarrow & S = \{1 - f_1(z)w_1 = 0 \dots 1 - f_s(z)w_s = 0\} \subset \mathbb{C}^N \times \mathbb{C}^s \\ (z) \longmapsto & & (z) \times \left(\frac{1}{f_1(z)}, \dots, \frac{1}{f_s(z)} \right) \end{array}$$

$$(*) \quad \bigcup_{X_J} \longrightarrow \tilde{X}_J = S \cap \pi^{-1}(X_J)$$

, where $\pi; \mathbb{C}^N \times \mathbb{C}^s \longrightarrow \mathbb{C}^N$ is a projection and (w) is the coordinate of $\mathbb{C}^s$. Since X_J and $\tilde{X}_J$ are isomorphic each other in (*),

$g_J = 1 + |z| + \frac{1}{|f_1(z)|} + \dots + \frac{1}{|f_s(z)|}$ is a growth function on X_J . E is trivial over X_J and we can regard E as an analytic vector bundle, so

$$(**) \quad E|_{X_J} \cong \mathcal{O}_X^k|_{X_J}.$$

From (**) we can define the p.g. cochain complex $C^*(\hat{\mathcal{A}}_\sigma(X_J; g_J), E)_{p.g.}$. This p.g. cochain complex is independent of the way of taking any isomorphism in (**) and uniquely determined by Prop.(2.1).

Proposition(2.3)

Let K be a subset of J , then $X_J \subset X_K$ and there is a restriction map:

$$\text{res}_{KJ}; C^*(\hat{\mathcal{A}}_\sigma(X_J; g_J), E)_{p.g.} \longrightarrow C^*(\hat{\mathcal{A}}_\sigma(X_K; g_K), E)_{p.g.}$$

proof

Because

$$\begin{cases} g_K(z) = 1 + |z| + \frac{1}{|f_1(z)|} + \dots + \frac{1}{|f_s(z)|} \\ g_J(z) = 1 + |z| + \frac{1}{|f_1(z)|} + \dots + \frac{1}{|f_s(z)|} + \frac{1}{|f_t(z)|}, \end{cases}$$

we have $g_K(Q) \leq g_J(Q)$ for $\forall Q \in X_J$. This means that

(a) $U_\sigma(Q; g_J) \supset U_\sigma(Q; g_K)$, where $U_\sigma(Q; g_J)$ is the disc, whose center is Q and radius is $\frac{1}{g_J} g_J^{-\sigma_2}$. We have the following refining map by (a):

$$(b) \quad r; \hat{\mathcal{A}}_\sigma(X_K; g_K) \xrightarrow{\quad \quad \quad} \hat{\mathcal{A}}_\sigma(X_J; g_J)$$

$\downarrow \quad \quad \quad \downarrow$

$$\tilde{U}_\sigma(Q; g_K) = U_\sigma(Q; g_K) \cap X_K \mapsto \tilde{U}_\sigma(Q; g_J) = U_\sigma(Q; g_J) \cap X_J.$$

(b) induces the following restriction map:

$$\text{res}_{KJ}; C^*(\hat{\mathcal{A}}_\sigma(X_J; g_J), E)_{p.g.} \longrightarrow C^*(\hat{\mathcal{A}}_\sigma(X_K; g_K), E)_{p.g.}$$

Q.E.D.

(ii) Let X, E and g_J be as above. We make a double complex in the following manner:

$$K_{\sigma, p.g}^{pq} \subseteq \bigoplus_{|J|=q+1}^{\text{alt.}} C^p(\hat{\mathcal{A}}_{\sigma}(X_J; g_J), E)_{p.g.}$$

Differentials δ_1 and δ_2 can be defined as follows:

$$\delta_1: K_{\sigma, p.g}^{pq} \longrightarrow K_{\sigma, p.g}^{p+1q}$$

$\downarrow$

$\varphi \longmapsto \delta_1 \varphi$

$$\text{where } (\delta_1 \varphi)_{j_0 \dots j_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \text{res}(\varphi_{j_0 \dots \hat{j}_k \dots j_{p+1}}) \text{ and } \bigoplus^{\text{alt.}}$$

denotes an alternating sum.

$$\delta_2: K_{\sigma, p.g}^{pq} \longrightarrow K_{\sigma, p.g}^{pq+1}$$

$\downarrow$

$\varphi \longmapsto \delta_2 \varphi$

$$\text{, where } (\delta_2 \varphi)_{i_0 \dots i_{q+1}} = \sum_{k=0}^{q+1} (-1)^k \text{res}(i_0 \dots i_{q+1})(i_0 \dots \hat{i}_k \dots i_{q+1})(\varphi_{i_0 \dots \hat{i}_k \dots i_{q+1}})$$

and $\text{res}(i_0 \dots i_{q+1})(i_0 \dots \hat{i}_k \dots i_{q+1})$ is the restriction map in

Prop.(2.3). On the other hand we have the following refining map:

$$(a) \quad r_i: \hat{\mathcal{A}}_{\sigma}(X_i; g_i) \longrightarrow \hat{\mathcal{A}}_{\sigma}(X; g)$$

$$X_i \cap U_{\sigma}(Q; g_i) \hookrightarrow X \cap U_{\sigma}(Q; g),$$

where $g_i = 1 + |z| + \frac{1}{|f_i(z)|}$ and $g = 1 + |z|$. From (a) we have the following diagram:

$$C^p(\hat{\mathcal{A}}_{\sigma}(X; g), E) \xrightarrow{r = \bigoplus r_i} \bigoplus_i C^p(\hat{\mathcal{A}}_{\sigma}(X_i; g_i), E)$$

$\downarrow \delta_2$

$$\bigoplus_{i_0 i_1}^{\text{alt.}} C^p(\hat{\mathcal{A}}_{\sigma}(X_{i_0 i_1}; g_{i_0 i_1}), E)$$

, where $C^p(\hat{\mathcal{A}}_{\sigma}(X; g), E)$, $C^p(\hat{\mathcal{A}}_{\sigma}(X_i; g_i), E)$ and $C^p(\hat{\mathcal{A}}_{\sigma}(X_{i_0 i_1}; g_{i_0 i_1}), E)$ are the usual cochain complexes and

$$g_{i_0 i_1} = 1 + |z| + \frac{1}{|f_{i_0}(z)|} + \frac{1}{|f_{i_1}(z)|}$$

Proposition(2.3)

We take the inductive limit of the above diagram:

$$\begin{array}{ccc} \varinjlim_{\sigma} C^p(\hat{A}_{\sigma}(X), E) & \xrightarrow{r = \oplus r_i} & \bigoplus_i \varinjlim_{\sigma} C^p(\hat{A}_{\sigma}(X_i), E) \\ & & \downarrow \delta_2 \\ & & \varinjlim_{i_0 i_1}^{alt.} C^p(\hat{A}_{\sigma}(X_{i_0 i_1}), E) \end{array}$$

Then $\text{Ker } \delta_2 = \text{Im } r$.

proof

$\text{Ker } \delta_2 \supset \text{Im } r$ is trivial.

First we show that $\text{Ker } \delta_2 \subset \text{Im } r$ in case $p=0$. We take an element

$\varphi = (\varphi_i) \in \bigoplus_i C^0(\hat{A}_{\sigma}(X_i), E)$. If we can take $\sigma' \geq \sigma$ ($\sigma'_1 \geq \sigma_1, \sigma'_2 \geq \sigma_2$)

which is independent of $Q \in X$ such that $\bigwedge_{Q \in X}$ there exists $i = i_Q \in I$ for $\forall Q \in X$ and

(a) $\tilde{U}_{\sigma'}(Q; g) \subset \tilde{U}_{\sigma}(Q; g_i)$, we may take $\Psi = \{\Psi_Q\}_Q \in C^0(\hat{A}_{\sigma}(X), E)$ such

that $\Psi_Q = \varphi_{i_Q} | \tilde{U}_{\sigma'}(Q; g)$. On the other hand for another correspondence

$j = j_Q$ as above we can define Ψ'_Q as $\varphi_{j_Q} | \tilde{U}_{\sigma'}(Q; g_j)$, then there exists

$i \in I$ such that $\varphi_{i_Q} = \varphi_{j_Q}$ on $\tilde{U}_{\sigma}(X_{ij}; g_{ij})$ which is a cocycle condition (i.e. $\delta_2 \varphi = 0$). This means that $\Psi'_Q = \Psi_Q$. Therefore the

$\{\Psi_Q\}_Q \in C^0(\hat{A}_{\sigma}(X), E)$ is independent of the above correspondence,

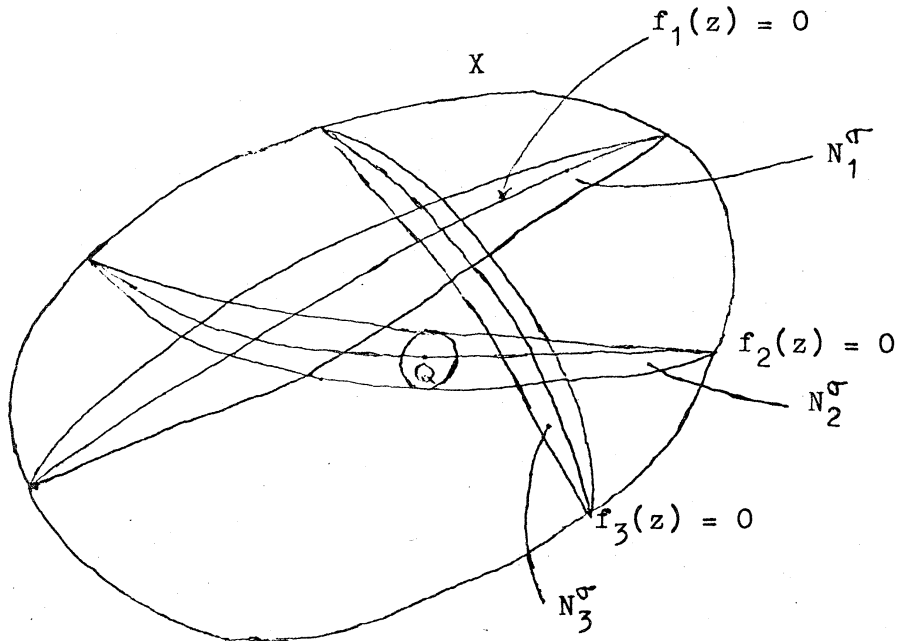
$X \ni Q \longmapsto i = i_Q \in I$. Then we shall define the following manner:

$$X \ni Q \longrightarrow i_Q \in I,$$

where this i_Q is i such that $g_i(Q) = \min_{j \in I} \{g_j(Q)\}$. Since I is finite,

we can choose it. The existence of $\sigma' \geq \sigma$ which is independent of

$Q \in X$ is derived from the following figure:



, where $N_1^\sigma = \bigcup_{Q \in \{f_i=0\}} U_0(Q;g)$ and $g = 1 + |z|$. When we choose sufficiently large σ , $X = \bigcup_{i \in I} X_i = \bigcup_{i \in I} N_i^\sigma$. We can prove $\text{Ker } \delta_2 \subset \text{Im } r$ for the p -th cochain complex $C^p(\hat{\mathcal{A}}_\sigma(X), E)$ in the same manner.

Q.E.D.

Now we can define a $p.g.$ cochain complex $C^*(X, E)_{p.g}$ for X .

Definition(2.4)

$C^p(X, E)_{p.g} = \varprojlim_{\sigma} (\text{Im } r)_{p.g}^{p, \sigma}$,
 where $(\text{Im } r)_{p.g}^{p, \sigma} \subset \bigoplus_i C^p(\hat{\mathcal{A}}_\sigma(X_i; g_i), E)_{p.g}$. Since
 $(\text{Im } r)^{p, \sigma} \subset \bigoplus_i C^p(\hat{\mathcal{A}}_\sigma(X_i; g_i), E)$ and $C^p(\hat{\mathcal{A}}_\sigma(X_i; g_i), E)_{p.g}$ has defined,
 we can attach $(\text{Im } r)^{p, \sigma}$ to $\bigwedge^a_{p.g}$ condition.

Remark(2.5)

As r is injective, $\text{Im } r$ is uniquely defined. (i.e. $C^p(X, E)_{p.g}$ is independent of an affine open covering $X = \bigcup_{i \in I} X_i$.)

Proposition(2.6)

$C^*(X, E)_{p.g}$ is a complex in the usual manner.

This is trivial by the definition.

§3. The comparison and vanishing theorem

Let X and E be as above and $C^*(X, E)_{p.g}$ the one in the def.(2.4).

Theorem(3.0)

$$H^q(C^*(X, E)_{p.g}) = 0 \quad \text{for } q \geq 1,$$

$$H^0(C^*(X, E)_{p.g}) \cong [(X_{\text{alg.}}, E_{\text{alg.}})].$$

When we regard E as an algebraic vector bundle, we write $E_{\text{alg.}}$.

In order to prove the Th.(3.0) we think the following diagram;

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & \downarrow & & \downarrow \\
 0 & \longrightarrow & C^0(X, E)_{p.g} & \xrightarrow{i_2} & K_{p.g}^{00} & \longrightarrow & K_{p.g}^{01} \longrightarrow \\
 & & \downarrow & & \downarrow \delta_1 & & \downarrow \delta_2 \\
 0 & \longrightarrow & C^1(X, E)_{p.g} & \xrightarrow{i_2} & K_{p.g}^{10} & \longrightarrow & K_{p.g}^{11} \longrightarrow \\
 & & \vdots & & \vdots & & \vdots \\
 0 & \longrightarrow & C^p(X, E)_{p.g} & \xrightarrow{i_2} & K_{p.g}^{p0} & \longrightarrow & K_{p.g}^{p1} \longrightarrow \\
 & & \downarrow & & \downarrow \delta_1 & & \downarrow \delta_2
 \end{array}$$

(*)

, where $\mathcal{U} = \{X_i\}_{i \in I}$ is an affine open covering and

$$K_{p.g}^{pq} = \varinjlim_{\sigma} K_{\sigma, p.g}^{pq}.$$

By the simple observation we have the following:

$$(a) K_{p.g}^{0q} = \varinjlim_{\sigma} \bigoplus_{i_0 \dots i_q}^{\text{alt.}} C^0(\hat{\mathcal{A}}_{\sigma}(X_{i_0 \dots i_q}), E)_{p.g} \longleftrightarrow \bigoplus_{i_0 \dots i_q}^{\text{alt.}} \Gamma(X_{i_0 \dots i_q}, E)_{p.g}$$

$$(b) \bigoplus_{i_0 \dots i_q}^{\text{alt.}} \Gamma(X_{i_0 \dots i_q}, E)_{p.g} = \bigoplus_{i_0 \dots i_q}^{\text{alt.}} \Gamma(X_{i_0 \dots i_q}, E_{\text{alg.}})$$

$$= C^q(\mathcal{U}, E_{\text{alg.}}).$$

We derive the first equality in (b) from the Sasakura's th.(1.2) and the second equality in (b) from the Serre's vanishing theorem of affine varieties. Therefore i_1 is injective from (a) and (b). Also i_2 is injective by Prop.(2.3). On the other hand

$$H^p(K_{p.g}^{*q}) = \bigoplus_{i_0 \dots i_q}^{\text{alt.}} \varinjlim_{\sigma} H^p(C^*(\hat{\mathcal{A}}_{\sigma}(X_{i_0 \dots i_q}), E)_{p.g}) = 0 \quad \text{for } p \geq 1$$

by the Sasakura's vanishing theorem(1.2). This means that (1) in the following proposition is true.

Proposition(3.1)

The complexes,

$$(1) K_{p.g}^{0q} \longrightarrow K_{p.g}^{1q} \longrightarrow \dots \longrightarrow K_{p.g}^{pq} \longrightarrow \dots \quad \text{for } q \geq 0$$

$$(2) K_{p.g}^{p0} \longrightarrow K_{p.g}^{p1} \longrightarrow \dots \longrightarrow K_{p.g}^{pq} \longrightarrow \dots \quad \text{for } p \geq 0,$$

are exact.

We must prove (2).

Lemma(3.2)

Let $\mathcal{U} = \{X_i\}_{i \in I}$ be as above. Then $H^1(K_{p.g}^{p*}) = 0$ for $p \geq 0$.

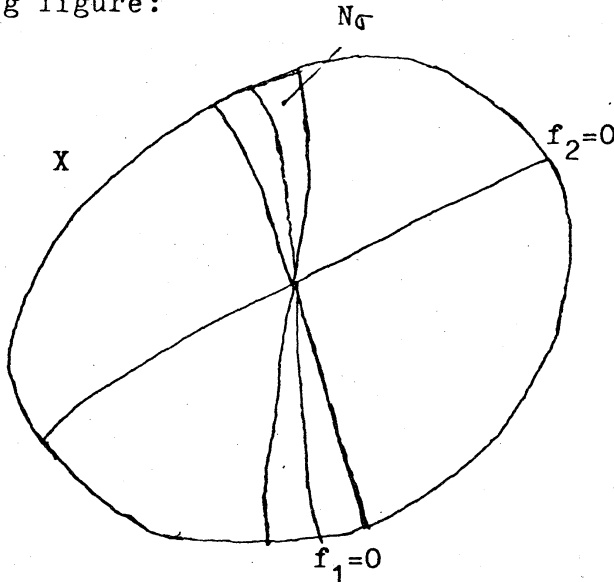
proof of Lemma(3.2)

We shall prove it by the induction on the number of sheets of the covering $\mathcal{U}$.

(i) When $\#I = 2$, $\mathcal{U} = \{X_1, X_2\}$.

We define a neighborhood N_σ of $\{f_1=0\}$ in the following manner:

$N_\sigma = \bigcup_{Q \in V_1 - V_{12}} U_\sigma(Q; d(Q, V_{12}))$, where $V_1 = \{f_1=0\}$ and $V_{12} = \{f_1=0\} \cap \{f_2=0\}$ and $d(Q, V_{12})$ is a distance from Q to V_{12} and $U_\sigma(Q; d(Q, V_{12}))$ is the disc whose center is Q and radius $\frac{1}{\sigma_1} d(Q, V_{12})^{+\sigma_2}$. Then we have the following figure:



We take an element $\varphi_{12} \in C^p(\hat{\mathcal{R}}_\sigma(X_{12}), E)_{p.g}$, then we give

φ_1 and φ_2 in the following:

$$\begin{aligned} \varphi_1 & \begin{cases} = -\varphi_{12} & \text{on } N_\sigma \\ = 0 & \text{on } X - N_\sigma \end{cases} \\ \varphi_2 & \begin{cases} = 0 & \text{on } N_\sigma \\ = \varphi_{12} & \text{on } X - N_\sigma. \end{cases} \end{aligned}$$

Then we have $\varphi_{12} = \varphi_2 - \varphi_1$. When we take sufficiently large $\sigma' (\geq \sigma)$,

$\varphi_i \in C^p(\hat{\mathcal{R}}_{\sigma'}(X_i), E)_{p.g}$ ($i=1,2$) by the Lojasiewicz inequality and the triangle inequality. Therefore $H^1(K_{p.g}^{p*}) = 0$ for $\forall p \geq 0$.

(ii) When $\#I=3$, $\mathcal{U} = \{X_1, X_2, X_3\}$ and we take

$$\varphi = (\varphi_{12}, \varphi_{13}, \varphi_{23}) \in \bigoplus_{1,j}^{\text{alt.}} C^P(\hat{A}_\tau(X_{1j}), E)_{p.g.}$$

Applying the assumption of the induction to $\{X_1, X_2\}$, there exist

$\varphi_1 \in C^P(\hat{A}_\sigma(X_1), E)_{p.g.}$ and $\varphi_2 \in C^P(\hat{A}_\sigma(X_2), E)_{p.g.}$ such that

$\varphi_{12} = \varphi_2 - \varphi_1$. By the cocycle condition, $\varphi_{23} - \varphi_{13} + \varphi_{12} = 0$, we have $(\varphi_{23} + \varphi_2) - (\varphi_{13} + \varphi_1) = 0$. We define newly as follows:

$$\tilde{\varphi}_{23} = \varphi_{23} - \varphi_2, \quad \tilde{\varphi}_{13} = \varphi_{13} - \varphi_1, \quad \tilde{\varphi}_{12} = 0.$$

Since $(\tilde{\varphi}_{13}, \tilde{\varphi}_{23})$ is a cocycle for (X_{13}, X_{23}) , we can define $\tilde{\varphi}_3$ as follows:

$$\tilde{\varphi}_3 \begin{cases} = \varphi_{13} & \text{on } X_{13} \\ = \varphi_{23} & \text{on } X_{23} \\ = 0 & \text{on } X_3 - (X_{13} \cup X_{23}). \end{cases}$$

This cochain $\tilde{\varphi}_3$ is not contained in $C^P(\hat{A}_\sigma(X_3), E)_{p.g.}$. Now we define

$\varphi_3 \in C^P(\hat{A}_\sigma(X_3), E)_{p.g.}$ in the following manner:

$$\begin{aligned} \varphi_3 & \begin{cases} = \tilde{\varphi}_3 & \text{on } X_3 - N \\ = 0 & \text{on } N \end{cases} \\ \varphi_1 & \begin{cases} = \tilde{\varphi}_3 & \text{on } N - X_1^c \\ = 0 & \text{on otherwise} \end{cases} \\ \varphi_2 & \begin{cases} = \tilde{\varphi}_3 & \text{on } N - X_3^c \\ = 0 & \text{otherwise} \end{cases} \end{aligned}$$

, where X_1^c is a complement of X_1 . Then $\varphi_3 \in C^P(\hat{A}_\sigma(X_3), E)_{p.g.}$ and $H^1(K_{p.g.}^{P*}) = 0$ for $p \geq 0$.

(iii) When $\#I=n$, we can show that $H^1(K_{p.g.}^{P*}) = 0$ for $p \geq 0$ in the same manner as (ii).

Q.E.D.

Let $\mathcal{U}^n = \{X_1, \dots, X_n\}$ be a covering of X and $K_{p.g.}^{P*}(\mathcal{U}^n)$ as $K_{p.g.}^{P*}$ for $\mathcal{U}^n$.

Lemma(3.3)

We assume that $H^1(K_{p.g}^{p*}(\mathcal{N}^n)) = 0$ for $i = q-1$, q ($q-1 \geq 1$).

Then $H^q(K_{p.g}^{p*}(\mathcal{N}^{n+1})) = 0$.

proof

We take an element $\varphi_{n+1} \in Z^q(K_{p.g}^{p*}(\mathcal{N}^{n+1}))$. We restrict φ_{n+1} to $\mathcal{N}^n$ and $\varphi_n \in Z^q(K_{p.g}^{p*}(\mathcal{N}^n))$ denotes it. By the assumption of the induction there is $\Psi_n \in K_{p.g}^{pq-1}(\mathcal{N}^n)$ such that $\delta_2 \Psi_n = \varphi_n$. Now we define $\Psi_{n+1} \in K_{p.g}^{pq-1}(\mathcal{N}^{n+1})$ as follows:

$$\begin{cases} \Psi_{n+1}(X_{i_1} \wedge \dots \wedge X_{i_q}) = \Psi_n(X_{i_1} \wedge \dots \wedge X_{i_q}) & (1 \leq i_1 < \dots < i_q = n) \\ \Psi_{n+1}(X_{i_1} \wedge \dots \wedge X_{i_q}) = 0 & (i_q = n+1). \end{cases}$$

We define an element $\varphi'_{n+1} \in Z^q(K_{p.g}^{p*}(\mathcal{N}^{n+1}))$ by

$$(*) \quad \varphi'_{n+1} = \varphi_{n+1} - \delta_2 \Psi_{n+1} \begin{cases} = 0 & \text{on } X_{i_1} \wedge \dots \wedge X_{i_{q+1}} \quad (i_{q+1} \leq n) \\ = \varphi_{n+1}(X_{i_1} \wedge \dots \wedge X_{i_q} \wedge X_{n+1}) & (i_q \leq n). \end{cases}$$

The number of the sheets of the covering $\mathcal{B}^n = \{X_i \wedge X_{n+1}; i=1 \dots n\}$

is n . From (*) we can regard φ'_{n+1} as an element φ'_n of $Z^{q-1}(K_{p.g}^{p*}(\mathcal{B}^n))$ by using the covering $\mathcal{B}^n$. By the assumption of the induction there exists $\tilde{\varphi}_n \in K_{p.g}^{pq-2}(\mathcal{B}^n)$ such that $\delta_2 \tilde{\varphi}_n = \varphi'_n$. We pull back $\tilde{\varphi}_n$ and we have $\tilde{\varphi}_{n+1} \in K_{p.g}^{pq-1}(\mathcal{N}^{n+1})$.

Q.E.D.

proof of (2) in the Proposition(3.1)

By $H^2(K_{p.g}^{p*}(\mathcal{N}^2)) = 0$, Lemma(3.2) and Lemma(3.3) we have $H^2(K_{p.g}^{p*}(\mathcal{N}^3)) = 0$. By $H^2(K_{p.g}^{p*}(\mathcal{N}^3)) = 0$, Lemma(3.2) and Lemma(3.3) we have $H^2(K_{p.g}^{p*}(\mathcal{N}^4)) = 0$. By using Lemma(3.2), (3.3) repeatedly we have (2).

Q.E.D.

proof of Th.(3.0)

By using Prop.(3.1) and the commutative diagram (*) we have

$$(a) \quad H^p(C^*(X, E)_{p.g.}) \cong H^p(C^*(\mathcal{V}, E_{alg.})) \quad \text{for } p \geq 0.$$

By the Serre's vanishing theorem we get

$$(b) \quad H^p(C^*(\mathcal{V}, E_{alg.})) = 0 \quad \text{for } p \geq 1.$$

Therefore this completes the proof by (a) and (b).

Q.E.D.

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