

Holomorphic and Singular Solutions of Non Linear Singular Partial Differential Equations

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In this note, I will report some results on holomorphic and singular solutions of singular partial differential equations of the following three cases:

1. linear case;
2. non linear first order case;
3. non linear higher order case.

1 Linear case

First of all, let us survey my result in the case of linear Fuchsian case.

Let $(t, x) = (t, x_1, \dots, x_n) \in C_t \times C_x^n$ and let us consider

$$(E_1) \quad \left(t \frac{\partial}{\partial t}\right)^m u = \sum_{\substack{j+|\alpha| \leq m \\ j < m}} a_{j,\alpha}(t, x) \left(t \frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u + f(t, x),$$

where $m \in N^*(= \{1, 2, \dots\})$, $\alpha = (\alpha_1, \dots, \alpha_n) \in N^n (= \{0, 1, 2, \dots\}^n)$, $|\alpha| = |\alpha_1| + \dots + |\alpha_n|$ and

$$\left(\frac{\partial}{\partial x}\right)^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

Assume the following conditions:

A₁) $a_{j,\alpha}(t, x)$ and $f(t, x)$ are holomorphic near the origin;

A₂) $a_{j,\alpha}(0, x) \equiv 0$, if $|\alpha| > 0$.

Then, (E₁) is called a Fuchsian type equation with respect to t . The indicial polynomial $C(\rho, x)$ is defined by

$$C(\rho, x) = \rho^m - \sum_{j < m} a_{j,0}(0, x) \rho^j$$

and the characteristic exponents $\rho_1(x), \dots, \rho_m(x)$ are defined by the roots of $C(\rho, x) = 0$.

Definition of $\tilde{\mathcal{O}}$. $\tilde{\mathcal{O}}$ is the set of all functions $u(t, x)$ satisfying the following : there are $\varepsilon > 0$ and $r > 0$ such that $u(t, x)$ is holomorphic in $\{(t, x) \in \mathcal{R}(C \setminus \{0\}) \times C^n ; 0 < |t| < \varepsilon \text{ and } |x| \leq r\}$, where $\mathcal{R}(C \setminus \{0\})$ is the universal covering space of $C \setminus \{0\}$.

THEOREM 1 (Tahara [1]). Denote by $\mathcal{S}$ the set of all $\tilde{\mathcal{O}}$ -solutions of (E₁). Then, if $\rho_i(0) \notin \mathbf{N}$ ($1 \leq i \leq m$) and $\rho_i(0) - \rho_j(0) \notin \mathbf{Z}$ ($1 \leq i \neq j \leq m$) hold, we have

$$\mathcal{S} = \{U(\varphi_1, \dots, \varphi_m) ; (\varphi_1, \dots, \varphi_m) \in (C\{x\})^m\},$$

where $U(\varphi_1, \dots, \varphi_m)$ is an $\tilde{\mathcal{O}}$ -solution of (E₁) depending on $(\varphi_1, \dots, \varphi_m) \in (C\{x\})^m$ which can be taken arbitrarily and having an expansion of the following form:

$$\begin{aligned} U(\varphi_1, \dots, \varphi_m) = & \sum_{i=0}^{\infty} u_i(x) t^i \\ & + \sum_{i=1}^m \sum_{j=0}^{\infty} \sum_{k=0}^{mj} \phi_{i,j,k}(x) t^{\rho_i(x)+j} (\log t)^k \end{aligned}$$

with $\phi_{i,0,0}(x) = \varphi_i(x)$ ($i = 1, \dots, m$).

2 Non linear first order case

Next, I will report a result for non linear first order equation of the following form:

$$(E_2) \quad t \frac{\partial u}{\partial t} = F(t, x, u, \frac{\partial u}{\partial x}),$$

where $(t, x) \in C_t \times C_x^n$ and $\frac{\partial u}{\partial x} = (\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n})$.

Put $v = (v_1, \dots, v_n)$ and assume the following:

- B₁) $F(t, x, u, v)$ is holomorphic near the origin ;
- B₂) $F(0, x, 0, 0) \equiv 0$ near $x = 0$;
- B₃) $\frac{\partial F}{\partial v_i}(0, x, 0, 0) \equiv 0$ for $i = 1, \dots, n$.

Then, (E₂) is called an equation of Briot-Bouquet type with respect to t (in [3]). Put

$$\rho(x) = \frac{\partial F}{\partial u}(0, x, 0, 0).$$

Definition of $\widetilde{\mathcal{O}}_+$. We denote by $\widetilde{\mathcal{O}}_+$ the set of all $u(t, x)$ satisfying the following i) and ii):

- i) There are $r > 0$ and a positive-valued continuous function $\varepsilon(s)$ on $\mathbf{R}_s$ such that $u(t, x)$ is a holomorphic function on

$$\{(t, x) \in \mathcal{R}(C \setminus \{0\}) \times C^n ; 0 < |t| < \varepsilon(\arg t), |x| \leq r\};$$

- ii) There is an $a > 0$ such that for any $\theta > 0$ we have

$$\max_{|x| \leq r} |u(t, x)| = O(|t|^a)$$

as $t \longrightarrow 0$ under the condition $|\arg t| < \theta$.

THEOREM 2 (Gérard-Tahara [4]). Denote by S_+ the set of all $\tilde{O}_+$ -solutions of (E_2) . Then, if $\rho(0) \notin N^*$ holds, we have:

$$S_+ = \begin{cases} \{u_0\}, & \text{when } \operatorname{Re} \rho(0) \leq 0, \\ \{u_0\} \cup \{U(\varphi); 0 \neq \varphi(x) \in C\{x\}\}, & \text{when } \operatorname{Re} \rho(0) > 0, \end{cases}$$

where u_0 is the unique holomorphic solution of (E_2) and $U(\varphi)$ is an $\tilde{O}_+$ -solution of (E_2) having an expansion of the following form:

$$U(\varphi) = \sum_{i \geq 1} u_i(x) t^i + \sum_{\substack{i+2j \geq k+2 \\ j \geq 1}} \varphi_{i,j,k}(x) t^{i+j\rho(x)} (\log t)$$

with $\varphi_{0,1,0}(x) = \varphi(x)$ which can be taken arbitrarily.

3 Non linear higher order case

Lastly, I will report a generalization of the result in section 2 to higher order case.

Let us consider

$$(E_3) \quad \left(t \frac{\partial}{\partial t}\right)^m u = F(t, x, \left\{ \left(t \frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u \right\}_{\substack{j+|\alpha| \leq m \\ j < m}}),$$

where $(t, x) \in C_t \times C_x^n$ and $m \in N^*$. Put

$$z = \{z_{j,\alpha}\}_{\substack{j+|\alpha| \leq m \\ j < m}}$$

and assume the following conditions:

- C₁) $F(t, x, z)$ is holomorphic near the origin ;
- C₂) $F(0, x, 0) \equiv 0$ near $x = 0$;
- C₃) $\frac{\partial F}{\partial z_{j,\alpha}}(0, x, 0) \equiv 0$ near $x = 0$, if $|\alpha| > 0$.

Note the following: 1) if $m = 1$, (E_3) is nothing but (E_2) ; 2) if (E_3) is linear, (E_3) is nothing but (E_1) . Thus, (E_3) includes both cases (E_1) and (E_2) .

Put

$$C(\rho, x) = \rho^m - \sum_{j < m} \frac{\partial F}{\partial z_{j,0}}(0, x, 0) \rho^j$$

and denote by $\rho_1(x), \dots, \rho_m(x)$ the roots of $C(\rho, x) = 0$ in ρ . Set

$$\mu = \text{the cardinal of } \{i; \operatorname{Re} \rho_i(0) > 0\}.$$

If $\mu = 0$, this implies that $\operatorname{Re} \rho_i(0) \leq 0$ for all $i = 1, \dots, m$. When $\mu \geq 1$, by a renumeration we may assume

$$\begin{cases} \operatorname{Re} \rho_i(0) > 0, & \text{for } 1 \leq i \leq \mu, \\ \operatorname{Re} \rho_i(0) \leq 0, & \text{for } \mu + 1 \leq i \leq m. \end{cases}$$

Then we have:

THEOREM 3 (Gérard-Tahara [5]). *Denote by $\mathcal{S}_+$ the set of all $\widetilde{\mathcal{O}}_+$ -solutions of (E_3) . Then we have:*

(I) *If $\mu = 0$, we have*

$$\mathcal{S}_+ = \{u_0\},$$

where u_0 is the unique holomorphic solution of (E_3) .

(II) *If $\mu \geq 1$, under the additional conditions:*

- 1) $\rho_i(0) \neq \rho_j(0)$ for $1 \leq i \neq j \leq \mu$,
- 2) $C(1, 0) \neq 0$,
- 3) $C(i + j_1 \rho_1(0) + \dots + j_\mu \rho_\mu(0), 0) \neq 0$ for any $(i, j) \in \mathbb{N} \times \mathbb{N}^\mu$ satisfying $i + |j| \geq 2$,

we have

$$\mathcal{S}_+ = \{U(\varphi_1, \dots, \varphi_\mu); (\varphi_1, \dots, \varphi_\mu) \in (C\{x\})^\mu\},$$

where $U(\varphi_1, \dots, \varphi_\mu)$ is an $\widetilde{\mathcal{O}}_+$ -solution of (E_3) depending on $(\varphi_1, \dots, \varphi_\mu) \in (C\{x\})^\mu$ which can be taken arbitrarily and having an expansion of the following form:

$$U(\varphi_1, \dots, \varphi_\mu) = \sum_{i \geq 1} u_i(x) t^i$$

$$+ \sum_{\substack{i+2m|j|\geq k+2m \\ |j|\geq 1}} \phi_{i,j,k}(x) t^{i+j_1\rho_1(x)+\dots+j_\mu\rho_\mu(x)} (\log t)^k$$

with $\phi_{0,e_p,0}(x) = \varphi_p(x)$ ($p = 1, \dots, \mu$) where $e_1 = (1, 0, \dots, 0), \dots, e_\mu = (0, \dots, 0, 1) \in N^\mu$.

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