

NON-DEGENERATE BILINEAR FORMS AND FIBER FUNCTORS

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ABSTRACT. We shall review a categorical approach to the basic quantum group $SL_q(2, \mathbb{C})$ or $SU_q(2)$ based on the preprint S. Yamagami, Fiber functors on Temperley-Lieb categories, arXiv:math.QA/0405517.

1. CLASSICAL TANNAKA DUALITY

The notion of tensor category is traced back to the celebrated work of T. Tannaka on a duality theory of compact groups.

Given a compact group G , let

$\text{Rep}(G)$ = the category of finite-dimensional unitary representations of G ,

which is referred to as the **Tannaka dual** of G and provides a typical example of tensor categories:

- Given two objects (representations) V, W , another object $V \otimes W$ is associated functorially so that $(U \otimes V) \otimes W = U \otimes (V \otimes W)$.
- There is a special object I , the trivial representation of G , which satisfies $I \otimes V = V \otimes I$ for any V .
- We have the operation of taking contragredient representation $V \rightarrow V^*$, which is categorically characterized by the existence of morphisms $\epsilon : V \otimes V^* \rightarrow I$ and $\delta : I \rightarrow V^* \otimes V$ such that the compositions

$$\begin{aligned} V &\xrightarrow{1_V \otimes \delta} V \otimes V^* \otimes V \xrightarrow{\epsilon \otimes 1_V} V, \\ V^* &\xrightarrow{\delta \otimes 1_{V^*}} V^* \otimes V \otimes V^* \xrightarrow{1_{V^*} \otimes \epsilon} V^* \end{aligned}$$

are identities.

The Tannaka dual has the special feature that it is realized as a subcategory of Vec , the category of finite-dimensional vector spaces, or Hilb , the category of finite-dimensional Hilbert spaces.

The celebrated **Tannaka duality** states that the group G itself is recovered by looking at the categorical information on representations. Here are some important generalizations to quantum groups.

Unitary version (Woronowicz, 1988):

$$\begin{array}{c} \text{Compact quantum groups} \\ \Longleftrightarrow \text{rigid } C^*\text{-tensor categories } \subset \mathcal{Hilb}. \end{array}$$

Algebraic version (Ulbrich, 1990):

$$\begin{array}{c} \text{Algebraic quantum groups} \\ \Longleftrightarrow \text{rigid tensor categories } \subset \mathcal{Vec}. \end{array}$$

Typical examples are $SU_q(2)$ ($q \in \mathbb{R}^\times$) and $SL_q(2, \mathbb{C})$ ($q \in \mathbb{C}^\times$).

2. FIBER FUNCTORS

We shall here split the relevant information in Tannaka duality into two parts. Given an abstract tensor category $\mathcal{T}$, a **fiber functor** on $\mathcal{T}$ is, by definition, a faithful tensor functor $F : \mathcal{T} \rightarrow \mathcal{Vec}$. Each object X , which is considered to an abstract label, produces a finite-dimensional vector space $F(X)$ in such a way that $F(X \otimes Y) = F(X) \otimes F(Y)$. In other words, a fiber functor is a kind of representation of an abstract tensor category $\mathcal{T}$ in terms of the concrete tensor category $\mathcal{Vec}$ or $\mathcal{Hilb}$.

Viewing quantum groups this way, we are naturally lead to the problem of their representation theoretical classifications: Again we have two stages.

- Classify abstract tensor categories.
- Classify fiber functors on a tensor category up to equivalences.

Here ‘equivalences’ are with respect to a natural equivalence, say $\{\varphi_X\}$, preserving tensor products.

$$\begin{array}{ccc} F(X) & \xrightarrow{\varphi_X} & F'(X) \\ F(f) \downarrow & & \downarrow F'(f), \\ F(Y) & \xrightarrow{\varphi_Y} & F'(Y) \end{array} \quad \varphi_{X \otimes Y} = \varphi_X \otimes \varphi_Y.$$

3. TEMPERLEY-LIEB CATEGORIES

Generally classification is a difficult problem for tensor categories because it consists of determining moduli for non-linear equations. We shall here restrict ourselves to the simple but fundamental case of Temperley-Lieb category $\mathcal{TL}_d$ ($d \in \mathbb{C}^\times$ being a complex parameter), which is the linearization of the monoidal category of Kauffman’s

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monoids and turns out to be the representation category of the quantum group $SL_q(2, \mathbb{C})$.

$$\mathcal{TL}_d \equiv \mathcal{Rep}(SL_q(2, \mathbb{C})), \quad \text{with } d = -q - q^{-1}.$$

By definition, **Kauffman's monoids** are isotopy classes of planar strings stretched out between upper and lower boundaries of a strip. Let $K_{m,n}$ be the set of Kauffman's monoids with m and n vertices placed on upper and lower boundaries respectively. Here is a figure for the case $K_{2,4}$.



Objects in $\mathcal{TL}_d$ are labeled by the natural numbers $\{0, 1, 2, \dots\}$ with hom-sets given by $\mathcal{TL}_d(n, m) = \mathbb{C}[K_{m,n}]$, the free vector space generated by the set $K_{m,n}$. We regard each n as representing n -th tensor product of the object V labeled by 1:

$$n \iff \overbrace{V \otimes \dots \otimes V}^{n\text{-times}}.$$

The structure of tensor category is then defined as follows:

- The operation of composition is given by the concatenation of monoids with each closed circle replaced by the complex number d .

- Tensor product is defined by the horizontal juxtaposition of base monoids:

Note here that

$$\mathcal{TL}_d \not\cong \mathcal{TL}_{d'} \quad \text{if } d \neq d'.$$

With this geometrical presentation of Temperley-Lieb categories, we have

Theorem

$$\bigsqcup_{d \in \mathbb{C}^\times} (\{\text{fiber functors on } \mathcal{TL}_d\} / \sim) \cong \{\text{non-degenerate bilinear forms}\} / \sim$$

$$B \sim B' \iff B' = {}^t T B T \quad \text{for some invertible linear map } T.$$

Basic arcs in Kauffman's monoids are repaced with non-degenerate bilinear forms by fiber functors.

With a choice of linear basis, non-degenerate bilinear forms are represented by invertible matrices and the above classification problem can be dealt with by the following result:

Williamson-Wall Theorem: Let

$$\Theta : GL(n, \mathbb{C}) \ni B \mapsto {}^t B^{-1} B \in GL(n, \mathbb{C}).$$

Then

$$GL(n, \mathbb{C}) / \sim \cong \Theta(GL(n, \mathbb{C})) / \text{similarity}.$$

An invertible matrix M belongs to $\Theta(GL(n, \mathbb{C}))$ if and only if

- (1) $\mu_M(z) = \mu_M(z^{-1})$ for $z \in \mathbb{C}^\times$,
- (2) $\mu_M^{(k)}(1)$ is even for even k ,
- (3) $\mu_M^{(k)}(-1)$ is even for odd k .

Here

$$\mu_M(z) = (\mu_M^{(1)}(z), \mu_M^{(2)}(z), \dots),$$

denotes the multiplicity function of the matrix M : $\mu_M^{(k)}(z)$ is the multiplicity of z -Jordan block of size k in M . Note here that the parameter is related to $\Theta(B)$ by the formula

$$d = \text{trace}(\Theta(B)) = \text{trace}({}^t B^{-1} B).$$

Example: We have the following identifications.

$$GL(2, \mathbb{C}) / \sim = \left(\begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix} \right) / (q \leftrightarrow q^{-1}) \sqcup \left(\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \right)$$

with

$$\begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix} \longleftrightarrow \text{SL}_{q^{-1}}(2, \mathbb{C}),$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \longleftrightarrow \text{Woronowicz' Hopf algebra}.$$

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More generally, algebraic quantum groups of Dubois-Violette and Launer (1990), together with their representation categories, are classified in this way.

4. UNITARY FIBER FUNCTORS

A C^* -tensor category is, by definition, a tensor category with a compatible C^* -structure. It is well-known that the Temperley-Lieb category $\mathcal{TL}_d$ is a C^* -tensor category if and only if $d \in \mathbb{R}$ and $|d| \geq 2$. Moreover the C^* -structure is unique up to natural equivalences.

We can work out a similar characterization of unitary fiber functors on the Temperley-Lieb C^* -category $\mathcal{TL}_d$:

$$\{\text{unitary fiber functors on } \mathcal{TL}_d\} / \sim \cong \left\{ \Phi : V \rightarrow \bar{V}; \Phi^{-1} = \frac{d}{|d|} \bar{\Phi} \right\} / \sim,$$

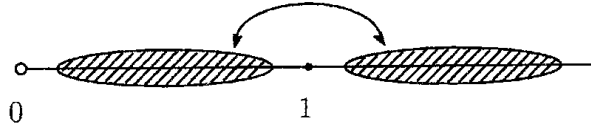
where the equivalence relation $\sim$ is defined by $\Phi \sim {}^t U \Phi U$ with U a unitary operator on V .

In terms of an eigenvalue-list of the positive part $|\Phi|$ of Φ , we have the following description:

Theorem:

An equivalence class of unitary fiber functors $\Leftrightarrow$ an unordered sequence $\{h_j > 0\}$ such that

$$\begin{aligned} \{h_j^{-1}\} &= \{h_j\}, \quad \text{tr}(|\Phi|^2) = |d|, \\ (d/|d|)^m &= 1, \quad m = \dim \ker(|\Phi| - 1). \end{aligned}$$



- (i) $n = 2$: For each $|d| \geq 2$, $\exists$ a unique

$$\{h, h^{-1}\}, h \geq 1, h^2 + h^{-2} = |d|.$$

- (ii) $n = 3$: For $d \geq 3$, a new choice

$$\{h, h^{-1}, 1\}, h \geq 1, h^2 + h^{-2} + 1 = d.$$

- (iii) $n = 2k$: For $|d| \geq n$,

$$\{h_j, h_j^{-1}\}, h_j \geq 1, \sum_j (h_j^2 + h_j^{-2}) = |d|.$$

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(iv) $n = 2k + 1$: For $d \geq n$,

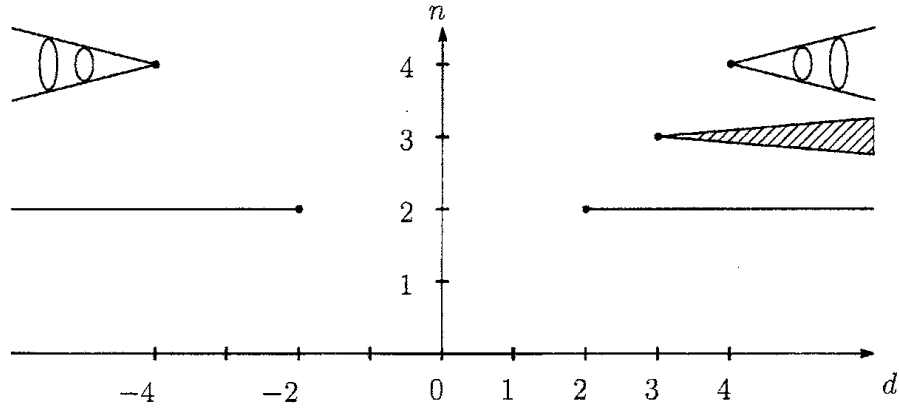
$$\{h_j, h_j^{-1}, 1\}, h_j \geq 1, \sum_j (h_j^2 + h_j^{-2}) = d - 1.$$

Set $t_j = h_j^2 + h_j^{-2} - 2$,

$$\sum_j t_j = |d| - 2k \text{ or } \sum_j t_j = d - 2k - 1.$$

Then the parameter space (moduli) of unitary fiber functors is the $k - 1$ -dimensional simplex

$$\{(t_1, \dots, t_k); t_j \geq 0, \sum_j t_j = r\} / S_k.$$



In this way, we have multiparameter families of compact quantum groups, which turns out to be Wang-Banica's universal quantum group of orthogonal type.