

Cuntz の Canonical Endomorphism の エントロピーについて

大阪教育大学 長田 まりゑ (Marie CHODA)

Cuntz 環 $\mathcal{O}_n$ の生成元を $\{S_i\}_{i=1}^n$ とした時、所謂 Cuntz の canonical endomorphism Φ は、

$$\Phi(x) = \sum_{i=1}^n S_i x S_i^*, \quad (x \in \mathcal{O}_n)$$

によって、定義される。

Voiculescu の topological entropy は、nuclear C^* -環の automorphism に対して、定義された概念であるが、その定義と、ここで用いる性質は、彼の結果をそのまま、適用する事により、*-endomorphism に対しても、有効である。

ここでは、Cuntz の canonical endomorphism Φ の $\mathcal{O}_n$ の unique $\log n$ -KMS state ϕ に対する Connes-Narnhofer-Thirring の entropy $h_\phi(\Phi)$ と、Voiculescu の topological entropy $ht(\Phi)$ との間の、次の関係を報告する：

$$h_\phi(\Phi) = ht(\Phi) = \log n.$$

この証明と基本的に同じやり方で、 $n \times n$ 行列環の無限テンソル積 A の shift automorphism α_n に対して、 $A \rtimes_{\alpha_n} \mathbb{Z}$ の implementing unitary を u とした時、

$$h_\tau(Ad_u) = ht(Ad_u) = \log n$$

も導き出す事ができる。但し、 τ は $A \rtimes_{\alpha_n} \mathbb{Z}$ の tracial state である。

2. Preliminaries

2.1. Let H_0 be a Hilbert space of dimension $n < \infty$. Put $H_i = H_0$, $i \in \mathbb{Z}$. For two integers i and j with $i < j$, we put

$$H_{[i,j]} = H_i \otimes H_{i+1} \otimes \cdots \otimes H_j.$$

Let $\{\delta(i) : i = 1, \dots, n\}$ be an orthonormal basis of H_0 . The embedding $H_{[i,j]} \hookrightarrow H_{[i-1,j+1]}$ is given by $\xi \in H_{[i,j]} \rightarrow \delta(1) \otimes \xi \otimes \delta(1) \in H_{[i-1,j+1]}$. We denote by $\mathcal{H}_i$ the inductive limit of $\{H_{[i,i+j]} : j = 0, 1, \dots\}$ and by $\mathcal{H}$ the inductive limit of the increasing sequence $\{\mathcal{H}_i : i = 0, -1, \dots\}$.

Given $k, l \in \mathbb{Z}$ $k < l$, let

$$W_{[k,l]}^n = \{\mu = (\mu_k, \dots, \mu_l) : \mu_i \in \{1, \dots, n\}, (k \leq i \leq l)\}.$$

Let $\mu \in W_{[k,l]}^n$ and $\nu \in W_{[l+1,m]}^n$. We put

$$\mu \cdot \nu = (\mu_k, \dots, \mu_l, \nu_{l+1}, \dots, \nu_m).$$

Further, let

$$W_0^n = \{0\}, \quad W_{[0,\infty]}^n = \bigcup_{k=0}^{\infty} W_{[0,k]}^n \quad \text{and} \quad W_{\infty}^n = \bigcup_{k=0}^{\infty} W_{[-k,k]}^n.$$

The shift $\alpha : i \in \mathbb{Z} \rightarrow i + 1$ induces the mapping on W_{∞}^n , which we denote by the same notation α .

For $\mu \in W_{[k,l]}^n$, we put

$$\delta(\mu) = \delta(\mu_k) \otimes \dots \otimes \delta(\mu_l) \in H_{[k,l]}.$$

Then $\{\delta(\mu) : \mu \in W_{[k,l]}^n\}$ is an orthonormal basis in $H_{[k,l]}$.

Let $A_0 = B(H_0)$ and $\{e(i, j) : i, j = 1, \dots, n\}$ be the matrix unit of A_0 with respect to the orthonormal basis $\{\delta(i) : i = 1, \dots, n\}$. We denote the trace $(1/n)\text{Tr}$ of A_0 by τ_0 . Put $A_i = A_0$, ($i \in \mathbb{Z}$) and $\tau_i = \tau_0$. For two integers $i < j$, let

$$A_{[i,j]} = A_i \otimes A_{i+1} \otimes \dots \otimes A_j.$$

For $\mu, \nu \in W_{[k,l]}^n$, we put

$$e(\mu, \nu) = e(\mu_k, \nu_k) \otimes \dots \otimes e(\mu_l, \nu_l) \in A_{[k,l]}.$$

Then $\{e(\mu, \nu) : \mu, \nu \in W_{[k,l]}^n\}$ is a matrix units of $A_{[k,l]}$.

2.2. We apply the entropy of Connes-Narnhofer-Thirring and Voiculescu's topological entropy to both of automorphisms and unital $*$ -endomorphisms on C^* -algebras. To fix notations, we recall the definition of the topological entropy. Let B be a nuclear C^* -algebra with unity. Let $CAP(B)$ be triples (ρ, η, C) , where C is a finite dimensional C^* -algebra, and $\rho : B \rightarrow C$ and $\eta : C \rightarrow B$ are unital completely positive maps. Let Ω be the set of finite subsets of B . For an $\omega \in \Omega$, put

$$rcp(\omega; \delta) = \inf\{\text{rank } C : (\rho, \eta, C) \in CAP(B), \|\eta \cdot \rho(a) - a\| < \delta, a \in \omega\},$$

where $\text{rank } C$ means the dimension of a maximal abelian self-adjoint subalgebra of C . For a unital $*$ -endomorphism β of B , put

$$ht(\beta, \omega; \delta) = \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log rcp(\omega \cup \beta(\omega) \cup \dots \cup \beta^{N-1}(\omega); \delta)$$

and

$$ht(\beta, \omega) = \sup_{\delta > 0} ht(\beta, \omega; \delta).$$

Then the topological entropy $ht(\beta)$ of β is defined by

$$ht(\beta) = \sup_{\omega \in \Omega} ht(\beta, \omega).$$

Assume that there exists an increasing sequence $(\omega_j)_{j \in \mathbb{N}}$ of finite subsets of B such that the linear span of $\cup_{j \in \mathbb{N}} \omega_j$ is dense in B . Even in the case of $*$ -endomorphisms which are not automorphisms, by the obvious analogues of [V : 4.3 Proposition], $ht(\cdot)$ is obtained as the following form which we use later:

$$ht(\beta) = \sup_{j \in \mathbb{N}} ht(\beta, \omega_j).$$

Let ϕ be a state of B with $\phi \cdot \beta = \phi$. The essential relation between $ht(\beta)$ and Connes-Narnhofer-Thirring entropy $h_\phi(\beta)$ is by [V: 4.6 Proposition]

$$h_\phi(\beta) \leq ht(\beta).$$

3. Entropy of Cuntz's canonical endomorphism

3.1. To compute the entropy of Φ , we recall some of the representation for the Cuntz algebra $\mathcal{O}_n$ as a crossed product in [Cu1], (cf., [Ch2, I2, P, R]). We use the same notations as in §2.1.

For a $j \in \mathbb{Z}$, $\mathcal{A}_j$ is given as the infinite tensor product:

$$\mathcal{A}_j = \bigotimes_{i=j}^{\infty} A_i.$$

Define embeddings

$$\mathcal{A}_j \hookrightarrow \mathcal{A}_{j-1} \hookrightarrow \mathcal{A}_{j-2} \hookrightarrow \dots$$

by $x \in \mathcal{A}_j \rightarrow e(1, 1) \otimes x \in \mathcal{A}_{j-1}$. The inductive limit of this sequence is denoted by $\mathcal{A}$. Since two embeddings $\mathcal{A}_j \hookrightarrow \mathcal{A}_{j-1}$ and $\mathcal{H}_i \hookrightarrow \mathcal{H}_{i-1}$ are compatible, we can consider $\mathcal{A}$ acting faithfully on $\mathcal{H}$.

The automorphism σ of $\mathcal{A}$ is induced by the shift $\alpha : i \in \mathbb{Z} \rightarrow i + 1$. Then the crossed product $\mathcal{A} \rtimes_{\sigma} \mathbb{Z}$ acts faithfully on the Hilbert space

$$K = \sum_{i \in \mathbb{Z}} \bigoplus u^i \mathcal{H},$$

where u is the implementing unitary in $\mathcal{A} \rtimes_{\sigma} \mathbb{Z}$ for the automorphism σ of $\mathcal{A}$. Let p be the unit of $\mathcal{A}_0 \subset \mathcal{A} \rtimes_{\sigma} \mathbb{Z}$ and put

$$w = up.$$

We remark $u^j p = w^j$.

Then Cuntz algebra $\mathcal{O}_n$ is reresented as $p(\mathcal{A} \rtimes_{\sigma} \mathbb{Z})p$, which is the C^* subalgebra $C^*(\mathcal{A}_0, w)$ of $(\mathcal{A} \rtimes_{\sigma} \mathbb{Z})$ generated by $\{\mathcal{A}_0, w\}$. There exists a conditional expectation E of $C^*(\mathcal{A}_0, w)$ onto $\mathcal{A}_0$ with $E(w^j) = 0$ for all $j = 1, 2, \dots$. The unique tracial state τ of $\mathcal{A}_0$ is extended to the state ϕ of $C^*(\mathcal{A}_0, w)$ by $\phi = \tau \cdot E$. Then ϕ is the unique log n -KMS state of $C^*(\mathcal{A}_0, w)$ ([DP]).

3.2. Since

$$\sigma^j(p)(\mathcal{H}) = \mathcal{H}_j, \quad j \in \mathbb{Z},$$

the algebra $p(\mathcal{A} \rtimes_{\sigma} \mathbb{Z})p$ is acting faithfully on

$$pK = \sum_{i \in \mathbb{Z}} \bigoplus u^i \mathcal{H}_{-i}.$$

The restriction $\sigma|_{\mathcal{A}_0}$ of σ to $\mathcal{A}_0$ is the one sided non commutative Bernoulli shift. Cuntz's canonical inner endomorphism Φ of $\mathcal{O}_n$ is nothing but the extension of $\sigma|_{\mathcal{A}_0}$ to the Cuntz algebra $C^*(\mathcal{A}_0, w)$ which maps

$$a \rightarrow \sigma(a), \quad (a \in \mathcal{A}_0), \quad \text{and} \quad w \rightarrow vw,$$

where

$$v = \sum_{j=1}^n e((j, 1), (1, j)) \in A_{[0,1]},$$

([Cu2], cf. [Ch2]).

3.3. Let $k, m \in \mathbb{N}$. We define

$$K(k, m) = \sum_{l=-k}^k \bigoplus u^l H_{[-l, -l+m]}$$

and we denote the orthogonal projection of K onto $K(k, m)$ by $Q(k, m)$. The set $\{u^j \delta(\mu) : -k \leq j \leq k, \mu \in W_{[-j, -j+m]}^n\}$ is an orthonormal basis of $K(k, m)$. We denote by $E((j, \mu), (l, \nu))$ the partial isometry in $B(K(k, m))$ such that

$$E((j, \mu), (l, \nu)) : u^l \delta(\nu) \rightarrow u^j \delta(\mu), \quad (\mu \in W_{[-j, -j+m]}^n, \nu \in W_{[-l, -l+m]}^n).$$

Then the set

$$\mathcal{E}(k, m) = \{E((j, \mu), (l, \nu)) : -k \leq j, l \leq k, \mu \in W_{[-j, -j+m]}^n, \nu \in W_{[-l, -l+m]}^n\}$$

is a matrix units of $B(K(k, m))$.

3.4. Let $k, m \in \mathbb{N}$. We define the completely positive unital linear map

$$\varphi_{k,m} : p(\mathcal{A} \rtimes_{\sigma} \mathbb{Z})p \rightarrow B(K(k, m))$$

by

$$\varphi_{k,m}(x) = Q(k, m)xQ(k, m)|_{K(k, m)}.$$

For two integers a and b with $a < b$, we let

$$\omega_{a,b} = \{e(\mu, \nu)w^j : 0 \leq j \leq a \text{ and } \mu, \nu \in W_{[0,b]}^n\}.$$

3.5. We define the linear map

$$\psi_{k,m} : B(K(k, m)) \rightarrow p(\mathcal{A} \rtimes_{\sigma} \mathbb{Z})p$$

by

$$\psi_{k,m}(E((j, \mu), (l, \nu))) = \frac{1}{2k+1} pu^j e(\mu, \nu) u^{*l} p,$$

for $E((j, \mu), (l, \nu)) \in \mathcal{E}(k, m)$. Then we have that $\psi_{k,m}$ is a unital completely positive map. Since $u^j p = w^j$, we have

$$\psi_{k,m} \cdot \varphi_{k,m}(e(\mu, \nu)w^j) = \frac{2k-j+1}{2k+1} e(\mu, \nu)w^j,$$

for all $e(\mu, \nu)w^j \in \omega_{a,b}$, $a \leq k$ and $b \leq m$.

Applying Voiculescu's definition of topological entropy to these completely positive maps and the above increasing sequence of finite sets $(\omega_{a,b})_{a,b}$, we have the following Theorem.

3.6. Theorem. *Let Φ be Cuntz's canonical inner endomorphism of $\mathcal{O}_n$. Then*

$$ht(\Phi) = \log n.$$

3.7. Corollary. *Let ϕ be the unique $\log n$ -KMS state of $\mathcal{O}_n$. Then*

$$h_{\phi}(\Phi) = \log n.$$

Proof. Let τ be the unique tracial state of $\mathcal{A}_0$ and E be the conditional expectation of $p(\mathcal{A} \rtimes_{\sigma} \mathbb{Z})p$ onto $\mathcal{A}_0$, then $\phi = \tau \cdot E$. Hence $\phi \cdot \Phi = \phi$. This relation implies, by the endomorphism version of [V: 4.6 Proposition],

$$\log n = h_{\tau}(\sigma|_{\mathcal{A}_0}) \leq h_{\phi}(\Phi) \leq ht(\Phi) = \log n.$$

Therefore $h_\phi(\Phi) = \log n$. $\square$

REFERENCES

- [Ch1] M. Choda : Entropy for $*$ -endomorphisms and relative entropy for subalgebras, *J. Operator Theory*, **25** (1991), 125-140.
- [Ch2] M. Choda : Canonical $*$ -endomorphisms and simple C^* -algebras, *J. Operator Theory*, **33** (1995), 235-248.
- [Ch3] M. Choda : Extension algebras of II_1 factors via $*$ -endomorphisms, *Int. J. Math.*, **5** (1994), 635-655.
- [CNT] A. Connes, H. Narnhofer and W. Thirring : Dynamical entropy of C^* algebras and von Neumann algebras, *Commun. Math. Phys.*, **112** (1987), 691-719.
- [CS] A. Connes and E. Størmer : Entropy of II_1 von Neumann algebras, *Acta Math.*, **134** (1975), 289-306.
- [Cu1] J. Cuntz : Simple C^* -algebras generated by isometries, *Commun. Math. Phys.*, **57** (1977), 173-185.
- [Cu2] J. Cuntz : Regular actions of Hopf algebras on the C^* -algebra generated by a Hilbert space, in "Operator algebras, mathematical physics, and low dimensional topology (Istanbul, 1991)" 87-100, *Res. Notes Math.* **5**, Wellesley, MA, (1993).
- [HS] U. Haagerup and E. Størmer : Equivalence of normal states on von Neumann algebras and flow of weights, *Adv. Math.*, **83** (1990), 180-262.
- [Hu] T. Hudetz : Topological entropy for appropriately approximated C^* -algebras, *J. Math. Phys.*, **35** (1994), 4303-4333.
- [I1] M. Izumi : Canonical extension on factors, in *Subfactors*, H. Araki et al. (eds.), World Scientific, Singapore, (1994), 129-138.
- [I2] M. Izumi : Subalgebras of infinite C^* -algebras with finite indices. II. Cuntz-Krieger algebras, Preprint.
- [L1] R. Longo : Index of subfactors and statistics of quantum fields. I, *Commun. Math. Phys.*, **126** (1989), 145-155.
- [L2] R. Longo : Duality for Hopf algebras and for subfactors, *Commun. Math. Phys.*, **159** (1994), 133-150.
- [LR] R. Longo and J.E. Roberts: A theory of dimension, Preprint.
- [OP] D. Olesen and G. K. Pedersen : Some C^* -dynamical systems with a single KMS state, *Math. Scand.*, **42** (1978), 111-118.
- [P] W. Paschke : The crossed product of a C^* -algebra by an endomorphism, *Proc. Amer. Math. Soc.*, **80** (1980), 113-118.
- [R] M. Rørdam : Classification of certain infinite simple C^* -algebras, *J. Funct. Anal.*, **131** (1995), 415-458.

- [S] E. Størmer : Entropy in operator algebras, Preprint in University of Oslo (1994).
- [T] K. Thomsen : Topological entropy for endomorphisms of local C^* -algebras, *Comm. Math. Phys.*, **164** (1994), 181-193.
- [V] D. Voiculescu : Dynamical approximation entropies and topological entropy in operator algebras, *Comm. Math. Phys.*, **170** (1995), 249-281.