

博 士 論 文

Deformation and degeneration of p -divisible groups
and their applications

(p 可除群の変形と退化およびその応用)

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Deformation and degeneration of p -divisible groups and their applications

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To my family.

Abstract

The theory of p -divisible groups is studied in various contexts of arithmetic geometry and applied to the study of p -adic cohomological properties of algebraic varieties, including abelian varieties, $K3$ surfaces, and Shimura varieties.

Nygaard-Ogus introduced the notion of a class of characteristic 0 liftings, called quasi-canonical liftings, and proved that the Tate conjecture holds for $K3$ surfaces over a finite field admitting a quasi-canonical lifting. Furthermore, they showed that, when the characteristic is at least 5, a $K3$ surface of finite height over a finite field admits a quasi-canonical lifting, using the deformation theory of p -divisible groups. In the first part of this thesis, we give another construction of quasi-canonical liftings of such $K3$ surfaces, which is valid even when the characteristic is 3.

Faltings-Chai constructed a toroidal compactification of the moduli stack of principally polarized abelian varieties, called Siegel modular stack, by collecting semi-abelian degenerations of abelian schemes. Kato introduced the notion of log p -divisible groups, which captures degeneration of p -divisible groups, and it is important to understand log p -divisible groups to study p -adic properties of toroidal compactifications of Siegel modular stacks (or more generally Shimura varieties). In the second part of this thesis, we study some extension properties around log p -divisible groups over log regular bases. Moreover, we prove a log version of the results of Oort-Zink on slope filtrations of p -divisible groups and give a construction of log p -divisible groups associated with semi-abelian degenerations of abelian schemes.

Philosophically, the Shimura variety associated with a Shimura datum (G, X) is regarded as the moduli space of “motives with G -structure of type X ”. Imai-Kato-Youcis constructed realizations of “the universal motives with G -structure of type X ” in the framework of prismatic cohomology theory, called prismatic realizations. In the third part of this thesis, we construct the extension of prismatic realizations over toroidal compactifications of Shimura varieties by establishing log prismatic Dieudonné theory, which is a log version of prismatic Dieudonné theory of Anschütz-Le Bras. Moreover, we apply our construction to p -adic comparison isomorphisms for p -adic étale cohomology of automorphic p -adic local systems on Shimura varieties. This result generalizes Lovering’s result proved in the case where Shimura varieties are proper.

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CHAPTER 1

Introduction

1.1. Outline of this thesis

The theory of p -divisible groups is deeply related to various objects in arithmetic geometry, including abelian varieties, $K3$ surfaces, and Shimura varieties. In this thesis, we focus on deformation and degeneration of p -divisible groups. More specifically, we obtain the following results:

- (1) We prove that, for a $K3$ surface of finite height over a finite field of odd characteristic, there exists a quasi-canonical lifting of it. See Chapter 2 for detail. This result is based on the published paper [Ino24].
- (2) We prove that, for a log p -divisible group over a log regular base, an isogeny from its generic fiber to a completely slope divisible log p -divisible group uniquely extends to an isogeny over the whole base. See Chapter 3 for detail. This result is based on the preprint [Ino23].
- (3) We canonically associate a log p -divisible group with a semi-abelian scheme over a regular base whose restriction to the complement of a normal crossings divisor is an abelian scheme. See Chapter 4 for detail. This result is based on the preprint [Ino25c].
- (4) We construct functors associating with log prismatic F -crystals coefficient objects in other p -adic cohomology theories and study their properties. See Chapter 5 for detail. This result is based on the preprint [Ino25b].
- (5) We establish an equivalence between the category of log p -divisible groups and the category of log prismatic Dieudonné crystals over a class of p -adic log formal schemes that are important for applications. Furthermore, we apply it to the construction of log prismatic realization on toroidal compactifications of canonical integral models of Shimura varieties of Hodge type. See Chapter 6 for detail. This result is based on the preprint [Ino25a].

In the rest of this chapter, we shall give precise statements of our results. At the beginning of each chapter, a more detained introduction will be provided.

1.2. Notation and conventions

- (1) Unless otherwise specified, p is a prime number.
- (2) All rings are associative, unitary, and commutative.
- (3) All monoids are commutative. For a monoid M , let $\overline{M} := M/M^\times$, and let M^{gp} be an abelian group associated to M .
- (4) For a monoid P and an integer $n \geq 1$, let $P^{1/n}$ denote the monoid P with $P \rightarrow P^{1/n}$ mapping p to p^n . The colimit of $P^{1/n}$ with respect to $n \geq 1$ is denoted by $P_{\mathbb{Q}_{\geq 0}}$.
- (5) A log (formal) scheme is a triple $(X, \mathcal{M}_X, \alpha)$ consisting of a (formal) scheme X and a sheaf of monoids $\mathcal{M}_X$ on the étale site $X_{\text{ét}}$ with a monoid map $\alpha: \mathcal{M}_X \rightarrow \mathcal{O}_X$ inducing $\alpha^{-1}(\mathcal{O}_X^\times) \xrightarrow{\sim} \mathcal{O}_X^\times$. When there is no possibility of confusion, we simply

- write $(X, \mathcal{M}_X)$ for a log (formal) scheme $(X, \mathcal{M}_X, \alpha)$. A morphism $(X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y)$ is called *strict* if the natural map $f^*\mathcal{M}_Y \rightarrow \mathcal{M}_X$ is an isomorphism.
- (6) A *prelog ring* is a triple (A, M, α) consisting of a ring A , a monoid M , and a monoid map $\alpha: M \rightarrow A$, where A is regarded as a monoid with respect to multiplication. We simply write (A, M) for (A, M, α) when there is no possibility of confusion. For a prelog ring (A, M) , the associated log scheme is denoted by $(\mathrm{Spec}(A), M)^a$.
 - (7) Let $(X, \mathcal{M}_X)$ be a log (formal) scheme and $f: Y \rightarrow X$ be a morphism of (formal) schemes. Unless otherwise specified, we write $\mathcal{M}_Y$ for the pullback log structure $f^*\mathcal{M}_X$.
 - (8) Let $\mathcal{P}$ be a property of (formal) schemes, including quasi-compact, quasi-separated, and locally noetherian. We say that a log (formal) scheme $(X, \mathcal{M}_X)$ satisfies $\mathcal{P}$ if the underlying (formal) scheme X satisfies $\mathcal{P}$.
 - (9) Let $\mathcal{P}$ be a property of morphisms of (formal) schemes. We say that a morphism of log (formal) schemes $(X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y)$ satisfies $\mathcal{P}$ if the morphism of the underlying (formal) schemes $X \rightarrow Y$ satisfies $\mathcal{P}$.
 - (10) A chart on a log (formal) scheme $(X, \mathcal{M}_X)$ is a monoid P and a map $P \rightarrow \mathcal{M}_X$ of sheaves of monoids, where P is regarded as the associated constant sheaf, such that the induced map of log structures $P^a \rightarrow \mathcal{M}_X$ is an isomorphism. A chart $P \rightarrow \mathcal{M}_X$ is called *neat* at $x \in X$ if $P \rightarrow \overline{\mathcal{M}}_{X, \bar{x}}$ is an isomorphism.
 - (11) For a site $\mathcal{C}$, the associated topos with $\mathcal{C}$ is denoted by $\mathrm{Shv}(\mathcal{C})$.

1.3. Quasi-canonical liftings of K3 surfaces

In Chapter 2, we give an application of the deformation theory of p -divisible groups to liftings of $K3$ surfaces.

The notion of quasi-canonical liftings of varieties over a perfect field k of positive characteristic is introduced by Nygaard–Ogus to show that Tate’s conjecture holds for $K3$ surfaces of finite height ([NO85, Definition 1.5]). Let X_0 be a $K3$ surface of finite height over a finite field k of characteristic p . They proved that, if there is a quasi-canonical lifting of X_0 , the $K3$ surface X_0 satisfies Tate’s conjecture [NO85, Theorem 2.1 and Remark 2.2.3]. Moreover, they constructed a quasi-canonical lifting of X_0 under the assumption that $p \geq 5$.

In Chapter 2, we prove the following theorem.

Theorem 1.3.1 (see Theorem 2.4.1). Let X_0 be a $K3$ surface of finite height $h < \infty$ over a finite field k of odd characteristic p . Then there exists a totally ramified finite extension $V/W(k)$ of degree h and a quasi-canonical lifting X/V of X_0/k .

This result is previously obtained by Nygaard–Ogus when $p \geq 5$ (see [NO85, Theorem 5.6]). We give an alternative proof when $p \geq 5$. Our result seems new when $p = 3$. While Nygaard–Ogus used crystalline deformation theory, we use display deformation theory in our proof. For details, see Section 2.1.

1.4. Slope filtrations of log p -divisible groups

In Chapter 3, we prove some extension properties of log p -divisible groups over a log regular base and a log version of the results of Oort–Zink.

First, we briefly recall the results of Oort–Zink for p -divisible groups. Zink proved that a p -divisible group over a regular $\mathbb{F}_p$ -scheme with constant Newton polygon is isogenous to a p -divisible group admitting a slope filtration ([Zin01a]), and Oort–Zink generalized

this result to the case where a base scheme is normal ([OZ02]). [OZ02] states that, for a locally noetherian normal scheme X over $\mathbb{F}_p$ and a p -divisible group G over X with constant Newton polygon, there exists a completely slope divisible p -divisible group H over X and an isogeny $G \rightarrow H$. Here, a p -divisible group H over X is called *completely slope divisible* if H admits a filtration of p -divisible subgroups

$$0 = H_0 \subset H_1 \subset \cdots \subset H_m = H,$$

and if there are integers

$$s \geq r_1 > r_2 > \cdots > r_m \geq 0$$

such that the following conditions are satisfied :

- (1) The quasi-isogeny $p^{-r_i} F^s: H_i \rightarrow H_i^{(p^s)}$ is an isogeny for $i = 1, \dots, m$.
- (2) The p -divisible group H_i/H_{i-1} is isoclinic of slope r_i/s for $i = 1, \dots, m$.

This theorem plays an important role in the theory of foliations in the moduli space of abelian varieties in characteristic p (see [Oor04]).

The goal of Chapter 3 is to prove a log version of the theorem of Oort-Zink. Kato introduced the notion of *log p -divisible groups* in [Kat23] (for the definition, Definition 3.2.28). Log p -divisible groups are degenerating objects of p -divisible groups in the framework of logarithmic geometry. In [KKN08b], Kajiwara-Kato-Nakayama introduced the notion of *log abelian varieties*, which are degenerating objects of abelian varieties in the framework of logarithmic geometry. Their motivation is to realize the toroidal compactification of the moduli space of abelian varieties constructed by Faltings-Chai as the fine moduli of log abelian varieties, and this is indeed achieved in [KKN21]. In the usual way, one can associate a log p -divisible group with a log abelian variety (see Example 3.2.30).

The following result is the main theorem of Chapter 3.

Theorem 1.4.1 (see Theorem 3.5.4). Let $(X, \mathcal{M}_X)$ be a log regular fs log scheme over $\mathbb{F}_p$ and U be a dense open subset of X . Let G be a log p -divisible group over $(X, \mathcal{M}_X)$. Suppose that we are given a completely slope divisible log p -divisible group H_U over $(U, \mathcal{M}_U)$ and an isogeny $G|_U \rightarrow H_U$. Then there exist a completely slope divisible log p -divisible group H over $(X, \mathcal{M}_X)$ and an isogeny $G \rightarrow H$ which restricts to the given isogeny $G|_U \rightarrow H_U$.

We expect that this result is helpful to study the structure of the toroidal compactification of the moduli space of abelian varieties in characteristic p .

1.5. Log p -divisible groups associated with semi-abelian schemes

In Chapter 4, we prove that, when an abelian scheme degenerates to a semi-abelian scheme along a normal crossings divisor on a regular scheme, the corresponding p -divisible group degenerates to a log p -divisible group.

It is an important to understand degeneration of abelian varieties. Among such degenerating objects, semi-abelian degeneration of abelian varieties is of particular importance, as is suggested by the semi-stable reduction theorem ([SGA7-I]) and the theory of toroidal compactifications of moduli spaces of abelian varieties ([FC90]). In Chapter 4, we study the behavior of torsion subgroups of abelian varieties degenerating to semi-abelian schemes.

We begin with a simple situation. Let K be a complete discrete valuation field of characteristic 0 with valuation ring $\mathcal{O}_K$ whose residue field k is of characteristic $p > 0$. Let A be a semi-abelian scheme over $\mathcal{O}_K$ with $A_K := A \otimes_{\mathcal{O}_K} K$ being an abelian variety. The associated p -divisible group $A_K[p^\infty]$ over K degenerates to a system of quasi-finite flat

group schemes $A[p^\infty] := \{A[p^n] := \text{Ker}(\times p^n: A \rightarrow A)\}_{n \geq 1}$ over $\mathcal{O}_K$. However, this object does not behave well. The height of $A[p^\infty]$ drops under the specialization, and so $A[p^\infty]$ does not have enough information.

The theory of log p -divisible groups improves this situation. Log p -divisible groups behave like usual p -divisible groups, as, for example, suggested by a log version of Dieudonné theory (cf. [Kat23] and Chapter 6). From this perspective, Kato and Zhao proved that the p -divisible group $A_K[p^\infty]$ over K uniquely extends to a log p -divisible group over $\mathcal{O}_K$ equipped with the standard log structure in [Zha21, Theorem 5.2] and [Kat23, §4.3].

The goal of Chapter 4 is to prove a higher dimensional version of the results of Kato and Zhao. The precise statement is the following.

Theorem 1.5.1 (Theorem 4.4.4). Let $(X, \mathcal{M}_X)$ be an fs log scheme defined by a locally noetherian regular scheme X with a normal crossings divisor D , and A be a semi-abelian scheme over X . Let $U := X \setminus D$. Suppose that A_U is an abelian scheme over U . Then the p -divisible group $A_U[p^\infty]$ over U uniquely extends to a log p -divisible group $A[p^\infty]^{\log}$ over $(X, \mathcal{M}_X)$.

For the relationship with the theory of log abelian varieties, see Section 4.1.

1.6. Log prismatic F -crystals and realization functors

In chapter 5, we construct realization functors for log prismatic F -crystals and study some properties of them. The results in this chapter are used in Chapter 6 to study log p -divisible groups.

The theory of prismatic cohomology is introduced in [BS22]. This cohomology theory involves coefficient objects called *prismatic F -crystals*. We let $\text{Vect}^\varphi(\mathfrak{X}_\Delta)$ denote the category of prismatic F -crystals on a bounded p -adic formal scheme $\mathfrak{X}$. As a striking feature, the prismatic cohomology theory specializes to other p -adic cohomology theories. To be precise, the works [GR24, IKY25] give the following: for a p -adic formal scheme which is smooth over $\mathcal{O}_K$, an *étale realization functor*

$$T_{\text{ét}}: \text{Vect}^\varphi(\mathfrak{X}_\Delta) \rightarrow \text{Loc}_{\mathbb{Z}_p}(\mathfrak{X}_\eta)$$

and a *crystalline realization functor*

$$T_{\text{fisc}}: \text{Vect}^\varphi(\mathfrak{X}_\Delta) \rightarrow \text{FillSoc}^\varphi(\mathfrak{X})$$

are constructed, where $\text{Loc}_{\mathbb{Z}_p}(\mathfrak{X}_\eta)$ is the category of $\mathbb{Z}_p$ -local systems on the generic fiber $\mathfrak{X}_\eta$ and $\text{FillSoc}^\varphi(\mathfrak{X})$ is the category of filtered F -isocrystals on $\mathfrak{X}$. Moreover, for a prismatic F -crystal $\mathcal{E}$, the coefficient objects $T_{\text{ét}}(\mathcal{E})$ and $T_{\text{fisc}}(\mathcal{E})$ are associated in the sense of Faltings.

A log version of prismatic cohomology theory is established in [Kos22, KY25]. Log prismatic cohomology theory has coefficient objects called *log prismatic F -crystals*. We let $\text{Vect}^\varphi((\mathfrak{X}, \mathcal{M}_\mathfrak{X})_\Delta)$ denote the category of log prismatic F -crystals on a bounded p -adic log formal scheme. Log prismatic cohomology theory enables us to treat p -adic Hodge theory on a smooth variety over a p -adic field K admitting an open immersion to a proper semi-stable scheme over $\mathcal{O}_K$ while (non-log) prismatic cohomology theory behaves well only for a proper smooth variety over K with good reduction. We make a precise setup.

Definition 1.6.1. Let k be a perfect field, K a totally ramified finite extension of $W(k)[1/p]$, and $\mathcal{O}_K$ a valuation ring of K . We fix a uniformizer π of $\mathcal{O}_K$. Consider the prelog ring $(R^0, \mathbb{N}^r)$ defined as follows. Let $l, m, n \in \mathbb{Z}_{\geq 0}$, $r := m + n$. We define R^0 as

follows:

$$R^0 := \begin{cases} \mathcal{O}_K \langle x_1^{\pm 1}, \dots, x_l^{\pm 1}, y_1, \dots, y_m \rangle & (n = 0) \\ \mathcal{O}_K \langle x_1^{\pm 1}, \dots, x_l^{\pm 1}, y_1, \dots, y_m, z_1, \dots, z_n \rangle / (\prod_{k=1}^n z_k - \pi) & (n \geq 1). \end{cases}$$

We define a prelog structure $\mathbb{N}^r \rightarrow R^0$ by

$$e_i \mapsto \begin{cases} y_i & (1 \leq i \leq m) \\ z_{i-m} & (m+1 \leq i \leq r). \end{cases}$$

A p -adic log formal scheme $(\mathrm{Spf}(R), \mathcal{M}_R)$ over $\mathcal{O}_K$ is called *small affine* if there exists a strict étale map $(\mathrm{Spf}(R), \mathcal{M}_R) \rightarrow (\mathrm{Spf}(R^0), \mathbb{N}^r)^a$ for some l, m, n . Such a strict étale map is called a *framing*. A p -adic log formal scheme $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ over $\mathcal{O}_K$ is called *semi-stable* if $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ has an étale covering consisting of small affine log formal schemes.

The first goal of Chapter 5 is to generalize the construction of realization functors and the relationship between them to log prismatic F -crystals on semi-stable log formal schemes over $\mathcal{O}_K$.

Theorem 1.6.2 (see [Theorem 5.6.18](#)). For a semi-stable log formal scheme $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ over $\mathcal{O}_K$, there exist an étale realization functor

$$T_{\text{ét}}: \mathrm{Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \mathrm{Loc}_{\mathbb{Z}_p}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta})$$

and a crystalline realization functor

$$T_{\text{fisc}}: \mathrm{Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \mathrm{FillSoc}^{\varphi}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}),$$

where $\mathrm{Loc}_{\mathbb{Z}_p}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta})$ denotes the category of Kummer étale $\mathbb{Z}_p$ -local systems on the generic fiber $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta}$ and $\mathrm{FillSoc}^{\varphi}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ denotes the category of filtered F -isocrystals on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Moreover, for a log prismatic F -crystal $\mathcal{E}$, coefficient objects $T_{\text{ét}}(\mathcal{E})$ and $T_{\text{fisc}}(\mathcal{E})$ are associated in the sense of Definition 5.4.18 (2) (cf. [\[Fal89, AI12, Tsu\]](#)).

The next goal is to prove the fully faithfulness and the bi-exactness of étale realization functors.

Theorem 1.6.3 (see [Theorem 5.6.27](#) and [Theorem 5.6.32](#)). Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a semi-stable log formal scheme over $\mathcal{O}_K$. Consider the étale realization functor

$$T_{\text{ét}}: \mathrm{Vect}^{\text{an}, \varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \mathrm{Loc}_{\mathbb{Z}_p}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta}).$$

Then the following statements are true.

- (1) The functor $T_{\text{ét}}$ is fully faithful.
- (2) When $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ admits framings with $n \leq 1$ étale locally, $T_{\text{ét}}$ gives a bi-exact equivalence to the essential image. In other words, a sequence of analytic log prismatic F -crystals

$$0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_2 \rightarrow \mathcal{E}_3 \rightarrow 0$$

is exact if and only if the associated sequence of Kummer étale local systems

$$0 \rightarrow T_{\text{ét}}(\mathcal{E}_1) \rightarrow T_{\text{ét}}(\mathcal{E}_2) \rightarrow T_{\text{ét}}(\mathcal{E}_3) \rightarrow 0$$

is exact.

For related previous work, see Section 5.1.

1.7. Log prismatic Dieudonné theory and its application to Shimura varieties

In chapter 6, we construct prismatic F -crystals associated with log p -divisible groups and apply it to Shimura varieties.

First, we recall *prismatic Dieudonné theory* established by Anschütz-Le Bras in [ALB23]. Let R be a quasi-syntomic p -complete ring. Anschütz-Le Bras constructed a fully faithful functor (called a *prismatic Dieudonné functor*) from the category of p -divisible groups over R to the category of prismatic Dieudonné crystals over R (for the definition, see Definition 5.5.3). Moreover, the image of the prismatic Dieudonné functor is specified, and, specifically, this functor gives an equivalence when there is a quasi-syntomic cover $R \rightarrow S$ with S being perfectoid.

One of the goals in Chapter 6 is to prove a log version of prismatic Dieudonné theory (which we call *log prismatic Dieudonné theory*) over semi-stable log formal schemes. The precise statement is the following.

Theorem 1.7.1 (see Theorem 6.5.12). Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a semi-stable log formal scheme over $\mathcal{O}_K$. Then there exists an exact anti-equivalence

$$\mathcal{M}_{\Delta}: \mathrm{BT}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow \mathrm{DM}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$$

(which is called a *log prismatic Dieudonné functor*), where $\mathrm{BT}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is the category of log p -divisible groups over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ and $\mathrm{DM}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ is the category of log prismatic Dieudonné crystals on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ (for the definition, see Definition 5.5.9).

As an application of log prismatic Dieudonné functors, we give the p -adic comparison isomorphism on (not necessarily compact) Shimura varieties, which is conjectured in [Lov17, 3.6.2]. For the precise statement, see Section 6.1.

CHAPTER 2

Quasi-canonical liftings of K3 surfaces of finite height in odd characteristic

2.1. Introduction

The goal of this chapter is to prove the following theorem.

Theorem 2.1.1 (see Theorem 2.4.1). Let X_0 be a K3 surface of finite height $h < \infty$ over a finite field k of odd characteristic p . Then there exists a totally ramified finite extension $V/W(k)$ of degree h and a quasi-canonical lifting X/V of X_0/k .

Recall that Nygaard–Ogus established a one-to-one correspondence between deformations of X_0 over $k[t]/(t^n)$ and deformations of the crystalline cohomology $H_{\text{crys}}^2(X_0/W(k))$ over $k[t]/(t^n)$ as crystals with additional structure (Frobenius, pairing, and Hodge filtration) under the assumption that $p \geq 5$. We need the assumption that $p \geq 5$ to justify some calculation for divided power (see [NO85, Lemma 4.6]). Furthermore, this correspondence is proved only over the base $k[t]/(t^n)$ because crystals behave well only over a base whose PD envelope is p -torsion free (see the proof of [NO85, Theorem 4.5]).

In Chapter 2, we use the display-theoretic deformation theory of K3 surfaces developed by Langer–Zink and Lau ([LZ19, Lau21]). When $p \geq 3$, the crystalline cohomology of a K3 surface is naturally equipped with a display structure, and there is a one-to-one correspondence between deformations of X_0 over an arbitrary artinian local ring R with residue field k and deformations of the crystalline cohomology $H_{\text{crys}}^2(X_0/W(k))$ over R as displays with additional structure.

Remark 2.1.2. From [NO85, Theorem 2.1] and Theorem 2.1.1, it follows that Tate’s conjecture holds for K3 surfaces of finite height over finite fields of characteristic $p \geq 3$. Note that Tate’s conjecture for K3 surfaces is previously proved by Madapusi Pera [Mad15, Theorem 1], [KM16, Theorem A.1]. (See also [Mad20], [IIK21, Section 6.4].)

Remark 2.1.3. If an extension of the base field k is allowed, Theorem 2.1.1 is previously obtained by Ito–Ito–Koshikawa [IIK21, Corollary 9.11] using the étaleness of the Kuga–Satake morphism. However, it seems difficult to control the extension degree by the method of [IIK21].

Remark 2.1.4. In characteristic 2, it is not known whether the crystalline cohomology of a K3 surface is equipped with a display structure. This is why we assume $p \geq 3$ throughout this chapter.

The key step to prove Theorem 2.1.1 is to study a relation between the display associated to the crystalline cohomology of a K3 surface and the Dieudonné display of the enlarged formal Brauer group. We note that the following Theorem is a display-theoretic analogue of [NO85, Theorem 3.20] (for the notation on displays, see Section 2.2).

Theorem 2.1.5 (see Corollary 2.3.4). Let k be a perfect field of characteristic $p \geq 3$, R be an Artin local ring with residue field k , and X be a K3 surface of finite height over

R (i.e. a proper flat scheme over R whose closed fiber is a $K3$ surface of finite height). Let $\widehat{\mathrm{Br}}_{X/R}$ (resp. $\psi_{X/R}$) be the formal Brauer group (resp. the enlarged formal Brauer group) associated to X/R . Then there exists the following exact sequence of displays over the small Witt frame $\widehat{W}(R)$:

$$0 \rightarrow \underline{D}(\psi_{X/R}) \rightarrow \underline{H}_{\mathrm{crys}}^2(X/\widehat{W}(R)) \rightarrow \underline{D}(\widehat{\mathrm{Br}}_{X/R}^*)(-1) \rightarrow 0.$$

Moreover, this sequence is compatible with base change with respect to R .

The organization of this chapter is as follows. In Section 2.2, we review the display-theoretic deformation theory of p -divisible groups and $K3$ -surfaces. In Section 2.3, we study the relation between the Dieudonné display of the enlarged formal Brauer group and the display of a $K3$ surface. In Section 2.4, we prove the main theorem of this chapter.

2.2. Frames and displays

In this section, we review some results about displays following [Lau21] (see also [LZ07, LZ19]).

2.2.1. Frames.

Definition 2.2.1 ([Lau21, Definition 2.0.1]). A *frame* $\underline{S} = (S, \sigma, \tau)$ consists of a commutative $\mathbb{Z}$ -graded ring

$$S = \bigoplus_{n \in \mathbb{Z}} S_n$$

and ring homomorphisms $\sigma, \tau : S \rightarrow S_0$ satisfying the following conditions:

- (1) $\tau_0 : S_0 \rightarrow S_0$ is the identity, and $\tau_{-n} : S_{-n} \rightarrow S_0$ is bijective for $n \geq 1$. Let $t \in S_{-1}$ be the unique element such that $\tau_{-1}(t) = 1$.
- (2) $\sigma_0 : S_0 \rightarrow S_0$ is a Frobenius lift (i.e. a ring homomorphism satisfying $\sigma_0(x) - x \in pS_0$ for all $x \in S_0$), and $\sigma_{-1}(t) = p$.
- (3) $p \in \mathrm{Rad}(S_0)$. Here $\mathrm{Rad}(S_0)$ is the Jacobson radical of S_0 .

A *morphism of frames* is a morphism of graded rings that commutes with σ and τ .

Remark 2.2.2. A frame $\underline{S}$ is uniquely determined by the graded ring $S_{\geq 0} = \bigoplus_{n \geq 0} S_n$ together with the ring homomorphism $\sigma : S_{\geq 0} \rightarrow S_0$ and the homomorphism of graded $S_{\geq 0}$ -modules $t : S_{\geq 1} \rightarrow S_{\geq 0}$ satisfying the following conditions:

- (1) $\sigma_0 : S_0 \rightarrow S_0$ is a Frobenius lift, and $\sigma(t(a)) = p\sigma(a)$ for all $a \in S_{\geq 1}$.
- (2) $p \in \mathrm{Rad}(S_0)$.

Example 2.2.3. Let A be a p -torsion free ring with $p \in \mathrm{Rad}(A)$, and $\sigma_0 : A \rightarrow A$ be a Frobenius lift. There is a unique frame $\underline{A} = (A[t], \sigma, \tau)$ ($\deg(t) = -1$) such that $\sigma_{-1}(t) = p$ and $\tau_{-1}(t) = 1$. This frame is called the *tautological frame* associated to A .

Example 2.2.4. Let R be an Artin local ring with perfect residue field of characteristic $p \geq 3$. Let $\widehat{W}(R)$ be the small Witt ring defined in [Zin01b, Section 2]. We obtain a frame $(\widehat{W}(R), \sigma, \tau)$ as follows. We set $\widehat{W}(R)_0 = \widehat{W}(R)$ and $\widehat{W}(R)_n = \hat{I}(R) = V(\widehat{W}(R))$ as a $\widehat{W}(R)_0$ -module for $n \geq 1$. For $n, m \geq 1$, a multiplication map $\widehat{W}(R)_n \times \widehat{W}(R)_m \rightarrow \widehat{W}(R)_{n+m}$ is given by

$$\hat{I}(R) \times \hat{I}(R) \rightarrow \hat{I}(R), \quad (V(a), V(b)) \mapsto V(ab).$$

Let $\sigma_0 : \widehat{W}(R) \rightarrow \widehat{W}(R)$ be the Witt vector Frobenius, and we put $\sigma_n(V(a)) = a$ for $a \in \widehat{W}(R)$ and $n \geq 1$. The map $t : \widehat{W}(R)_1 \rightarrow \widehat{W}(R)_0$ is the inclusion map, and $t : \widehat{W}(R)_{n+1} \rightarrow \widehat{W}(R)_n$ is the multiplication map $p : \hat{I}(R) \rightarrow \hat{I}(R)$ for $n \geq 1$. This determines a frame $(\widehat{W}(R), \sigma, \tau)$ uniquely by Remark 2.2.2. This frame is denoted simply by $\widehat{W}(R)$ and called the *small Witt frame* associated to R .

Remark 2.2.5. Let A be as in Example 2.2.3, R be as in Example 2.2.4, and $f : A \rightarrow \widehat{W}(R)$ be a ring homomorphism. Then f induces the unique morphism of graded rings $A[t] \rightarrow \widehat{W}(R)$ which commutes with τ . This map commutes with σ , and induces a morphism of frames $\underline{A} \rightarrow \widehat{W}(R)$.

2.2.2. Displays.

Definition 2.2.6 ([Lau21, Definition 3.2.1]). Let $\underline{S} = (S, \sigma, \tau)$ be a frame. Let $\pi : S \rightarrow S_0/(tS_1)$ be the composite of the projection $S \rightarrow S_0$ and $S_0 \rightarrow S_0/(tS_1)$.

- (1) An *$\underline{S}$ -display* (or simply *display*) $\underline{M} = (M, F)$ consists of a finite projective graded S -module M and a σ -linear map $F : M \rightarrow M^\tau$ which induces an isomorphism of S_0 -modules $M^\sigma \xrightarrow{\sim} M^\tau$. Here, for a ring homomorphism $f : A \rightarrow B$ and an A -module N , we put $N^f := N \otimes_{A,f} B$.
- (2) A *morphism of displays* is a morphism of graded S -modules that commutes with F .
- (3) We call a display (M, F) an *effective display* if $M^\pi := M \otimes_{S,\pi} S_0/tS_1$ concentrates on non-negative degrees.
- (4) An effective display (M, F) is called a *d -display* if $(M^\pi)_i = 0$ for $i \notin [0, d]$. Here $(M^\pi)_i$ denotes the degree i part of the graded S_0/tS_1 -module M^π .

Example 2.2.7. Let $\underline{S} = (S, \sigma, \tau)$ be a frame and $n \in \mathbb{Z}$. Then $(S(n), \sigma)$ is an $\underline{S}$ -display. Here we put $S(n)_i := S_{n+i}$. This display is denoted by $\underline{S}(n)$.

Definition 2.2.8. Let $\underline{M} = (M, F)$ and $\underline{M}' = (M', F')$ be $\underline{S}$ -displays.

- The *direct sum* of them is the $\underline{S}$ -display $\underline{M} \oplus \underline{M}' = (M \oplus M', F \oplus F')$.
- The *tensor product* of them is the $\underline{S}$ -display $\underline{M} \otimes_{\underline{S}} \underline{M}' = (M \otimes_S M', F \otimes_S F')$.
- The *dual* of $\underline{M}$ is the $\underline{S}$ -display $\underline{M}^* = (M^*, (F^*)^{-1})$. Here M^* is the dual of the S -module M .

Definition 2.2.9. Let $\underline{M} = (M, F)$, $\underline{M}' = (M', F')$, and $\underline{M}'' = (M'', F'')$ be $\underline{S}$ -displays. A sequence of morphisms of displays

$$0 \rightarrow \underline{M} \rightarrow \underline{M}' \rightarrow \underline{M}'' \rightarrow 0$$

is an *exact sequence of displays* if the underlying sequence of S -modules

$$0 \rightarrow M \rightarrow M' \rightarrow M'' \rightarrow 0$$

is exact.

Definition 2.2.10. Let $\underline{S} \rightarrow \underline{S}'$ be a morphism of frames, and $\underline{M} = (M, F)$ be an $\underline{S}$ -display. The *base change* of $\underline{M}$ by $\underline{S} \rightarrow \underline{S}'$ is the $\underline{S}'$ -display $\underline{M} \otimes_{\underline{S}} \underline{S}' = (M \otimes_S S', F \otimes_S S')$.

Remark 2.2.11. Let A be a p -adic ring (i.e. complete and separated in the p -adic topology). Assume that A is p -torsion free. Let $\sigma : A \rightarrow A$ be a Frobenius lift. An $\underline{A}$ -display is equivalent to a finite projective A -module M with a descending finite filtration by direct summands $(F^i M)_{i \in \mathbb{Z}}$ together with σ -linear maps $F_i : F^i M \rightarrow M$ satisfying the following conditions ([Lau21, Example 3.6.2]):

- $pF_{i+1} = F_i|_{F^{i+1}M}$ for all $i \in \mathbb{Z}$.
- M is generated by the union of all $F_i(F^i M)$.

In this equivalence, a d -display corresponds to a triple

$$(M, (F^i M)_{i \in \mathbb{Z}}, F_i)$$

such that $F^i M = 0$ for $i \geq d + 1$ and $F^i M = M$ for $i \leq 0$.

Remark 2.2.12. Let R be as in Example 2.2.4. An effective $\widehat{W}(R)$ -display is equivalent to a display $(P_i, \iota_i, \alpha_i, F_i)_{i \geq 0}$ over the small Witt frame in [LZ19, Definition 3] (see also [Lau21, Example 3.6.3 (1)]). In this equivalence, a d -display corresponds to $(P_i, \iota_i, \alpha_i, F_i)_{i \geq 0}$ such that $\alpha_i : \hat{I}(R) \otimes P_i \rightarrow P_{i+1}$ is surjective for $i \geq d$ (equivalently, $(P_i, \iota_i, \alpha_i, F_i)_{i \geq 0}$ has a *normal decomposition* $(L_i)_{i \geq 0}$ in the sense of [LZ19, Definition 3] such that $L_i = 0$ for $i \geq d + 1$). In particular, a 1-display over $\widehat{W}(R)$ is equivalent to a quadruple

$$(P, Q, F, F_1),$$

where P is a finite projective $\widehat{W}(R)$ -module, Q is a submodule of P such that $\hat{I}(R)P \subset Q$, the quotient P/Q is projective over R , the maps $F : P \rightarrow P$ and $F_1 : Q \rightarrow P$ are σ -linear morphisms such that $F_1(Q)$ generates P as a $\widehat{W}(R)$ -module. We note that this is a *Dieudonné display* over R in the sense of [Lau14, Definition 2.6].

2.2.3. Displays and p -divisible groups over p -torsion free bases with Frobenius lifts. Let A , p , and σ be as in Remark 2.2.11. Moreover, we assume that $p \geq 3$. Let G be a p -divisible group over A . We have a covariant Dieudonné crystal $\mathbb{D}(G)$ which is a crystal on the nilpotent crystalline site $\text{NilCrys}(\text{Spf}(A)/\text{Spf}(\mathbb{Z}_p))$ ([Mes72, Chapter IV]). We set

$$\begin{aligned} D(G) &:= \mathbb{D}(G)_A \\ F^1 D(G) &:= \text{Ker}(\mathbb{D}(G)_A \twoheadrightarrow \text{Lie}(G)) = \text{Lie}(G^*)^*. \end{aligned}$$

Let $G_0 := G \otimes A/pA$. The ring morphism σ , the Frobenius map $\text{Fr} : A/pA \rightarrow A/pA$, and the Verschiebung map $G_0^{(p)} \rightarrow G_0$ induce the following morphisms of A -modules

$$\begin{aligned} D(G)^\sigma &= (\mathbb{D}(G_0)_{(A \twoheadrightarrow A/pA)})^\sigma \\ &= (\text{Fr}_{\text{crys}}^* \mathbb{D}(G_0))_{(A \twoheadrightarrow A/pA)} \\ &= \mathbb{D}(G_0^{(p)})_{(A \twoheadrightarrow A/pA)} \\ &\rightarrow \mathbb{D}(G_0)_{(A \twoheadrightarrow A/pA)} \\ &= D(G). \end{aligned}$$

Note that the PD thickening $A \twoheadrightarrow A/pA$ is nilpotent because $p \geq 3$. The σ -linear morphism induced by the composite of the above morphisms is denoted by $F : D(G) \rightarrow D(G)$.

The following result is presumably well-known to experts (see [Lau14, Proposition 3.17]).

Proposition 2.2.13. The triple $(D(G) \supset F^1 D(G), F)$ is a 1-display over the tautological frame $\underline{A}$ (see Remark 2.2.11).

Proof. We give the proof for readers' convenience. It suffices to prove that $F(F^1 D(G))$ is contained in $pD(G)$, and $D(G)$ is generated by $F(D(G))$ and $p^{-1}F(F^1 D(G))$.

We consider the following A/pA -linear map

$$F_0 : (\mathbb{D}(G_0)_{A/pA})^{(p)} = D(G)^\sigma \otimes A/pA \rightarrow D(G) \otimes A/pA = \mathbb{D}(G_0)_{A/pA}.$$

We consider the following diagram.

$$\begin{array}{ccc} \mathrm{Lie}((G_0^{(p)})^*)^* & \xlongequal{\quad} & F^1 D(G)^\sigma \otimes A/pA \hookrightarrow D(G)^\sigma \otimes A/pA \\ \downarrow & & \downarrow F_0 \\ \mathrm{Lie}(G_0^*)^* & \xlongequal{\quad} & F^1 D(G) \otimes A/pA \hookrightarrow D(G) \otimes A/pA \end{array}$$

In this diagram, the left vertical map is zero because this map is induced by the relative Frobenius morphism $G_0^* \rightarrow (G_0^*)^{(p)} = (G_0^{(p)})^*$. Thus $F(F^1 D(G)) \subset pD(G)$.

Let us fix an A -submodule L of $D(G)$ such that $D(G) = L \oplus F^1 D(G)$. The σ -linear maps $F|_L : L \rightarrow D(G)$ and $p^{-1}F : F^1 D(G) \rightarrow D(G)$ induce a σ -linear map

$$\tilde{F} : D(G) = L \oplus F^1 D(G) \rightarrow D(G).$$

It suffices to show that $\tilde{F}$ induces a surjection of A -modules $D(G)^\sigma \rightarrow D(G)$. By Nakayama's lemma, it suffices to prove that, for an arbitrary field k and an arbitrary ring morphism $A/pA \rightarrow k$, the Frobenius-linear map $\tilde{F} \otimes_A k$ induces a surjective morphism of k -vector spaces $D(G)^\sigma \otimes_A k \rightarrow D(G) \otimes_A k$. We may assume that k is a perfect field. Since A is p -torsion free, the Frobenius lift $\sigma : A \rightarrow A$ induces a ring homomorphism $A \rightarrow W(A)$ by [Laz75, VII, Proposition 4.12]. Thus the composite of $A \rightarrow A/pA \rightarrow k$ induces a ring homomorphism $A \rightarrow W(k)$ commuting with Frobenius lifts. Taking base change by $A \rightarrow W(k)$, it suffices to consider the case where $A = W(k)$ and σ is the Witt vector Frobenius.

In this case, $D(G)$ is the covariant Dieudonné module of the p -divisible group G_0 over k . We have a σ^{-1} -linear map $V : D(G) \rightarrow D(G)$ satisfying $V(D(G)) = F^{-1}(pD(G))$. Therefore $pL \oplus F^1 D(G)$ is contained in $V(D(G))$. Since

$$\begin{aligned} \dim_k((pL \oplus F^1 D(G))/pD(G)) &= \dim_k(F^1 D(G)/pF^1 D(G)) \\ &= \dim_k((\mathrm{Lie}(G_0^*))^*) \\ &= \dim G^* \end{aligned}$$

and

$$\begin{aligned} \dim_k(V(D(G))/pD(G)) &= \dim_k(D(G)/F(D(G))) \\ &= \dim G^* \end{aligned}$$

(the last equality follows from [Dem86, Chapter III, Section 9]), we see that $pL \oplus F^1 D(G)$ coincides with $V(D(G))$. Therefore, we have

$$\begin{aligned} \mathrm{Im}(\tilde{F}) &= F(L) + (p^{-1}F)(F^1 D(G)) \\ &= (p^{-1}F)(pL \oplus F^1 D(G)) \\ &= (p^{-1}F)(V(D(G))) \\ &= D(G). \end{aligned}$$

□

Definition 2.2.14. The display defined in Proposition 2.2.13 is called the *Dieudonné display* associated to G over the tautological frame $\underline{A}$. It is denoted by $\underline{D}(G)$.

Proposition 2.2.15 (Berthelot–Breen–Messing). There exists a functorial isomorphism of displays

$$(\underline{D}(G))^*(-1) \xrightarrow{\sim} \underline{D}(G^*).$$

Proof. See [BBM82, Proposition 5.3.6]. $\square$

2.2.4. Displays and p -divisible groups over Artin local rings with perfect residue fields of odd characteristics. As in Example 2.2.4, let R be an Artin local ring with perfect residue field of characteristic $p \geq 3$.

Let G be a p -divisible group over R . We have a covariant Dieudonné crystal $\mathbb{D}(G)$ which is a crystal on the nilpotent crystalline site $\mathrm{NilCrys}(\mathrm{Spec}(R)/\mathrm{Spf}(\mathbb{Z}_p))$. We set

$$\begin{aligned} D(G) &:= \mathbb{D}(G)_{(\widehat{W}(R) \rightarrow R)} \\ F^1 D(G) &:= \mathrm{Ker}(\mathbb{D}(G)_{(\widehat{W}(R) \rightarrow R)} \rightarrow \mathbb{D}(G)_R \rightarrow \mathrm{Lie}(G)). \end{aligned}$$

The Witt vector Frobenius $\sigma : \widehat{W}(R) \rightarrow \widehat{W}(R)$, the Frobenius map $\mathrm{Fr} : R/pR \rightarrow R/pR$, and the Verschiebung map $(G \otimes R/pR)^{(p)} \rightarrow G \otimes R/pR$ induce the following morphisms of $\widehat{W}(R)$ -modules:

$$\begin{aligned} D(G)^\sigma &= (\mathbb{D}(G \otimes R/pR)_{(\widehat{W}(R) \rightarrow R/pR)})^\sigma \\ &= (\mathrm{Fr}_{\mathrm{crys}}^* \mathbb{D}(G \otimes R/pR))_{(\widehat{W}(R) \rightarrow R/pR)} \\ &= \mathbb{D}((G \otimes R/pR)^{(p)})_{(\widehat{W}(R) \rightarrow R/pR)} \\ &\rightarrow \mathbb{D}(G \otimes R/pR)_{(\widehat{W}(R) \rightarrow R/pR)} \\ &= D(G). \end{aligned}$$

The σ -linear map induced by the composite of the above morphisms is denoted by $F : D(G) \rightarrow D(G)$.

Proposition 2.2.16 (Lau). There is a unique σ -linear morphism $F_1 : F^1 D(G) \rightarrow D(G)$ which is functorial in R and G such that the quadruple

$$(D(G), F^1 D(G), F, F_1)$$

is a 1-display over the small Witt frame $\widehat{W}(R)$ (see Remark 2.2.12).

Proof. See [Lau14, Theorem 3.19]. $\square$

Definition 2.2.17. The display defined in Proposition 2.2.16 is called the *Dieudonné display* associated to G over the small Witt frame $\widehat{W}(R)$. It is denoted by $\underline{D}(G)$.

Remark 2.2.18. Let $\alpha : \widehat{W}(R) \rightarrow R$ be the graded ring homomorphism which is the composite of the two projection maps $\widehat{W}(R) \rightarrow \widehat{W}(R)$ and $\widehat{W}(R) \rightarrow R$. Let h (resp. d) be the height (resp. dimension) of G . We can take a normal decomposition $D(G) = L \oplus T$ with $F^1 D(G) = L \oplus \widehat{I}(R)T$ as in [Lau14, Section 2, 2A]. Then $\underline{D}(G)$ is isomorphic to $(T \oplus L(-1)) \otimes_{\widehat{W}(R)} \widehat{W}(R)$ as a graded $\widehat{W}(R)$ -module. Since we have the isomorphisms

$$T/\widehat{I}(R)T \cong D(G)/F^1 D(G) \cong \mathrm{Lie}(G),$$

the rank of T equals to d . It follows that the rank of L equals to $h - d$ from the fact that the rank of $D(G)$ equals to h (for example, [BBM82, Théorème 3.3.10]). Therefore we get the isomorphisms

$$\underline{D}(G)^\alpha \cong (T \oplus L(-1)) \otimes_{\widehat{W}(R)} R \cong R^{\oplus d} \oplus R(-1)^{\oplus h-d}$$

as graded R -modules. Here $\underline{D}(G)^\alpha$ denotes the graded R -module $\underline{D}(G) \otimes_{\widehat{W}(R), \alpha} R$.

Proposition 2.2.19. There exists a functorial isomorphism of $\widehat{W}(R)$ -displays

$$(\underline{D}(G))^*(-1) \xrightarrow{\sim} \underline{D}(G^*).$$

Proof. Let k be the residue field of R and $G_0 := G \otimes_R k$. We have the universal deformation of G_0

$$\mathcal{G} \rightarrow \mathrm{Spf}(W(k)[[t_1, \dots, t_d]]),$$

where $d := \dim(G_0) \cdot \dim(G_0^*)$ by [Ill85, Corollaire 4.8 (i)] (see also [Lau14, Proposition 3.11]). We put $A := W(k)[[t_1, \dots, t_d]]$ and $A_n := A/(t_1^n, \dots, t_d^n)$ for $n \geq 1$. Let $\sigma : A \rightarrow A$ be a Frobenius lift such that $\sigma(t_i) = t_i^p$ for all i and such that σ coincides with the Witt vector Frobenius on $W(k)$. The Frobenius lift σ defines tautological frames $\underline{A}$ and $\underline{A}_n$. We have the unique morphism of local $W(k)$ -algebras $A \rightarrow R$, $(t_i \mapsto x_i)$ corresponding to G/R . The morphism of local $W(k)$ -algebras $A \rightarrow \widehat{W}(R)$ which sends t_i to $[x_i]$ for all i induces a morphism of frames $\underline{A} \rightarrow \widehat{W}(R)$. Here, $[x_i]$ denotes the Teichmüller lift of x_i . Since R is an Artin local ring, the map $A \rightarrow R$ (resp. $\underline{A} \rightarrow \widehat{W}(R)$) factors as $A \twoheadrightarrow A_m \rightarrow R$ (resp. $\underline{A} \twoheadrightarrow \underline{A}_m \rightarrow \widehat{W}(R)$) for some $m \geq 1$. Then, by taking base change of the isomorphism of displays over $\underline{A}_m$ in Proposition 2.2.15

$$\underline{D}(\mathcal{G} \otimes_A A_m)^* \xrightarrow{\sim} \underline{D}((\mathcal{G} \otimes_A A_m)^*)$$

by $\underline{A}_m \rightarrow \widehat{W}(R)$, we get the desired isomorphism. $\square$

The following theorem is a main theorem of display-theoretic Dieudonné theory proved by Lau (see [Lau14] for details).

Theorem 2.2.20 (Lau). The functor

$$(p\text{-divisible groups over } R) \rightarrow (1\text{-displays over } \widehat{W}(R))$$

defined by $G \mapsto \underline{D}(G)$ gives an equivalence of categories.

Proof. See [Lau14, Theorem 3.19]. $\square$

2.2.5. Displays and p -adic formal schemes over p -torsion free bases with Frobenius lifts. Let A and $\sigma : A \rightarrow A$ be as in Remark 2.2.11. Moreover, we assume that A is Noetherian and $p \geq 3$. Let $\mathcal{X}$ be a p -adic proper smooth formal scheme over $\mathrm{Spf}(A)$ satisfying the following conditions:

- The Hodge-de Rham spectral sequence associated to $\mathcal{X}/A$

$$E_1^{i,j} = H^j(\mathcal{X}, \Omega_{\mathcal{X}/A}^i) \Rightarrow H_{\mathrm{dR}}^{i+j}(\mathcal{X}/A)$$

degenerates at E_1 -page.

- For $i, j \geq 0$, $H^j(\mathcal{X}, \Omega_{\mathcal{X}/A}^i)$ is a finite projective A -module. It follows from this that $H^j(\mathcal{X}, \Omega_{\mathcal{X}/A}^i)$ is compatible with base change with respect to A .

We put $X_0 := \mathcal{X} \otimes A/pA$.

When $\mathcal{X}$ is projective over A , the following result is proved by Langer–Zink (see [LZ07, Theorem 5.5]).

Proposition 2.2.21. Let

$$H_{\mathrm{dR}}^n(\mathcal{X}/A) \supset F^1 H_{\mathrm{dR}}^n(\mathcal{X}/A) \supset F^2 H_{\mathrm{dR}}^n(\mathcal{X}/A) \supset \dots$$

be the Hodge filtration. Let

$$F : H_{\mathrm{dR}}^n(\mathcal{X}/A) \rightarrow H_{\mathrm{dR}}^n(\mathcal{X}/A)$$

be a σ -linear map induced by σ and the identification $H_{\mathrm{dR}}^n(\mathcal{X}/A) \cong H_{\mathrm{crys}}^n(X_0/A)$. We assume $p > n$. Then the triple

$$(H_{\mathrm{dR}}^n(\mathcal{X}/A), F^\bullet H_{\mathrm{dR}}^n(\mathcal{X}/A), F)$$

is an n -display over the tautological frame $\underline{A}$ (see Remark 2.2.11).

Proof. It suffices to prove that $F(F^i H_{\mathrm{dR}}^n(\mathcal{X}/A))$ is contained in $p^i H_{\mathrm{dR}}^n(\mathcal{X}/A)$ for $i \geq 1$, and $H_{\mathrm{dR}}^n(\mathcal{X}/A)$ is generated by the union of $(p^{-i} F)(F^i H_{\mathrm{dR}}^n(\mathcal{X}/A))$ for $i \geq 0$.

The former assertion is proved in [LZ07, Lemma 5.4] when $\mathcal{X}$ is projective over A . In the general case, we use the results of Berthelot–Ogus as follows. By [BO78, Theorem 8.16 and Theorem 8.20], for $i \in [0, n]$, we get the following commutative diagram in the derived category.

$$\begin{array}{ccccc} \Omega_{\mathcal{X}^\sigma/A}^{\geq i} & \hookrightarrow & \Omega_{\mathcal{X}^\sigma/A}^\epsilon & \xrightarrow{\quad\quad\quad} & \Omega_{\mathcal{X}^\sigma/A}^\bullet \\ \downarrow & & \downarrow & & \downarrow \\ L\eta\Omega_{\mathcal{X}/A}^\bullet & \longrightarrow & p^i A \otimes_A^L \Omega_{\mathcal{X}/A}^\bullet & \longrightarrow & \Omega_{\mathcal{X}/A}^\bullet \end{array}$$

Here $\epsilon : \mathbb{N} \rightarrow \mathbb{N}$ is defined by $\epsilon(j) = \max\{j - i, 0\}$, $\eta : \mathbb{N} \rightarrow \mathbb{N}$ is defined by $\eta(j) = \epsilon(j) + j$, and $\Omega_{\mathcal{X}^\sigma/A}^{\geq i}$ (resp. $\Omega_{\mathcal{X}^\sigma/A}^\epsilon$) is the subcomplex of $\Omega_{\mathcal{X}^\sigma/A}^\bullet$ whose degree j part is 0 for $j \leq i - 1$ and $\Omega_{\mathcal{X}^\sigma/A}^j$ for $j \geq i$ (resp. 0 for $j \leq -1$ and $\Omega_{\mathcal{X}^\sigma/A}^{\epsilon(j)}$ for $j \geq 0$). Taking hypercohomology, the first assertion is proved.

We prove the latter assertion. By the same argument as the proof of Proposition 2.2.13, it suffices to consider the case that $A = W(k)$ where k is a perfect field and σ is the Witt vector Frobenius. Then the latter assertion follows from the strong divisibility of the crystalline cohomology $H_{\mathrm{crys}}^n(X_0/W(k))$ since $p > n$ (see [Fon83, p.91, Section 1, Proposition (i)]). $\square$

The n -display given by Proposition 2.2.21 is denoted by $\underline{H}_{\mathrm{dR}}^n(\mathcal{X}/A)$.

2.2.6. Displays and K3 surfaces over Artin local rings. Let R be an Artin local ring with perfect residue field k of characteristic $p \geq 3$.

Definition 2.2.22. ([Lau21, Definition 8.0.1]) A *K3 display* $(\underline{M}, (,))$ consists of a $\widehat{W}(R)$ -display $\underline{M} = (M, F)$ and a symmetric perfect pairing

$$(\cdot, \cdot) : \underline{M} \times \underline{M} \rightarrow \widehat{W}(R)(-2)$$

(i.e. a morphism of $\widehat{W}(R)$ -displays $\underline{M} \otimes \underline{M} \rightarrow \widehat{W}(R)(-2)$ which is a symmetric perfect pairing of graded $\widehat{W}(R)$ -modules) such that

$$M^\alpha \cong R \oplus R(-1)^{\oplus n} \oplus R(-2)$$

as graded R -modules for some $n \geq 0$. Here, $\alpha : \widehat{W}(R) \rightarrow R$ is the map of graded rings defined in 2.2.18.

The following result is proved by Langer–Zink [LZ19, Proposition 19] (see also [Lau21, Section 8.1]).

Proposition 2.2.23 (Langer–Zink). Let X be a $K3$ surface over R . Then $H_{\text{crys}}^2(X/\widehat{W}(R))$ is canonically equipped with $K3$ display structure. To be more precise, there exists a canonical $K3$ display

$$(\underline{H}_{\text{crys}}^2(X/\widehat{W}(R)), (\cdot, \cdot))$$

satisfying the following conditions:

- (1) The degree 0 part $(\underline{H}_{\text{crys}}^2(X/\widehat{W}(R)))_0$ coincides with $H_{\text{crys}}^2(X/\widehat{W}(R))$, and

$$(\cdot, \cdot)_0 : H_{\text{crys}}^2(X/\widehat{W}(R)) \times H_{\text{crys}}^2(X/\widehat{W}(R)) \rightarrow \widehat{W}(R)$$

is the pairing defined by Poincaré duality.

- (2) For $i = 1, 2$, let M^i be the image of the composite of the following maps:

$$(\underline{H}_{\text{crys}}^2(X/\widehat{W}(R)))_i \xrightarrow{\times t^i} H_{\text{crys}}^2(X/\widehat{W}(R)) \twoheadrightarrow H_{\text{dR}}^2(X/R).$$

Then

$$H_{\text{dR}}^2(X/R) \supset M^1 \supset M^2$$

is the Hodge filtration.

- (3) The $K3$ display $(\underline{H}_{\text{crys}}^2(X/\widehat{W}(R)), (\cdot, \cdot))$ is compatible with base change with respect to R .

Proof. We briefly give a sketch of the proof for readers' convenience. Let $X_0 := X \otimes_R k$. Let $\mathcal{X} \rightarrow \text{Spf}(W(k)[[t_1, \dots, t_{20}]])$ be the universal deformation of X_0 . We put $A := W(k)[[t_1, \dots, t_{20}]]$ and $A_n := A/(t_1^n, \dots, t_{20}^n)$. Let σ be a Frobenius lift such that $\sigma(t_i) = t_i^p$ for all i and such that σ coincides with the Witt vector Frobenius on $W(k)$. The Frobenius lift σ defines the tautological frames $\underline{A}$ and $\underline{A}_n$. We have the unique morphism of local $W(k)$ -algebras $A \rightarrow R, (t_i \mapsto x_i)$ corresponding to X/R . The morphism of local $W(k)$ -algebras $A \rightarrow \widehat{W}(R)$ which sends t_i to $[x_i]$ for all i induces a morphism of frames $\underline{A} \rightarrow \widehat{W}(R)$. Here, $[x_i]$ denotes the Teichmüller lift of x_i . Since R is an Artin local ring, the ring morphism $A \rightarrow R$ (resp. the morphism of frames $\underline{A} \rightarrow \widehat{W}(R)$) factors as $A \twoheadrightarrow A_m \rightarrow R$ (resp. $\underline{A} \rightarrow \underline{A}_m \rightarrow \widehat{W}(R)$) for some $m \geq 1$. Then, by taking base change of the $\underline{A}_m$ -display $\underline{H}_{\text{dR}}^2((\mathcal{X} \otimes_A A_m)/A_m)$ (defined in Subsection 2.2.5) and a symmetric perfect pairing

$$\underline{H}_{\text{dR}}^2((\mathcal{X} \otimes_A A_m)/A_m) \times \underline{H}_{\text{dR}}^2((\mathcal{X} \otimes_A A_m)/A_m) \rightarrow \underline{A}_m(-2)$$

by the morphism of frames $\underline{A}_m \rightarrow \widehat{W}(R)$, we get the desired $K3$ display. $\square$

The following theorem is a display-theoretic deformation theory for $K3$ surfaces proved by Lau [Lau21]. (When X is ordinary, it is proved by Langer–Zink [LZ19, Theorem 36].)

Theorem 2.2.24 (Langer–Zink, Lau). Let $S \twoheadrightarrow R$ be a surjection of Artin local rings with perfect residue field of characteristic $p \geq 3$ and X be a $K3$ surface over R . Then

$$X'/S \mapsto \underline{H}_{\text{crys}}^2(X'/\widehat{W}(S))$$

gives an equivalence of categories between deformations of X to $K3$ surfaces over S and deformations of $\underline{H}_{\text{crys}}^2(X/\widehat{W}(R))$ to $K3$ displays over $\widehat{W}(S)$.

Proof. See [Lau21, Theorem 8.1.1]. $\square$

2.3. Crystalline cohomology and the enlarged formal Brauer group

First, we recall some results in [NO85, Section 3]. Let p be a prime number, k be a perfect field of characteristic p , R be an Artin local ring over $W(k)$ with residue field k , and X/R be a $K3$ surface of finite height (i.e. a proper flat scheme over R whose closed fiber X_0 is a $K3$ surface of finite height). To this, we can associate a formal p -divisible group $\widehat{\mathrm{Br}}_{X/R}$ which is called the *formal Brauer group* of X/R and a p -divisible group $\psi_{X/R}$ which is called the *enlarged formal Brauer group*. We note that $\widehat{\mathrm{Br}}_{X/R}$ is the connected component of the identity of $\psi_{X/R}$. For a p -divisible group G over R , let $\mathbb{D}(G)$ be the (covariant) Dieudonné crystal on $\mathrm{NilCrys}(\mathrm{Spec}(R)/\mathrm{Spf}(\mathbb{Z}_p))$. Let $\mathbb{H}_{\mathrm{crys}}^2(X/R)$ be the crystal on $\mathrm{Crys}(\mathrm{Spec}(R)/\mathrm{Spf}(\mathbb{Z}_p))$ such that

$$\mathbb{H}_{\mathrm{crys}}^2(X/R)(S \twoheadrightarrow R) = H_{\mathrm{crys}}^2(X/S)$$

for any $(S \twoheadrightarrow R) \in \mathrm{Crys}(\mathrm{Spec}(R)/\mathrm{Spf}(\mathbb{Z}_p))$. By [NO85, Theorem 3.16], there is a natural morphism of F -crystals

$$\rho : \mathbb{D}(\psi_{X/R}) \rightarrow \mathbb{H}_{\mathrm{crys}}^2(X/R).$$

By the composition with the inclusion $\mathbb{D}(\widehat{\mathrm{Br}}_{X/R}) \hookrightarrow \mathbb{D}(\psi_{X/R})$, we get a morphism $\mathbb{D}(\widehat{\mathrm{Br}}_{X/R}) \rightarrow \mathbb{H}_{\mathrm{crys}}^2(X/R)$. Taking the dual, we get a morphism of F -crystals

$$\mathbb{H}_{\mathrm{crys}}^2(X/R) \rightarrow \mathbb{D}(\widehat{\mathrm{Br}}_{X/R}^*)(-1)$$

because $\mathbb{H}_{\mathrm{crys}}^2(X/R)^* = \mathbb{H}_{\mathrm{crys}}^2(X/R)(2)$ and $\mathbb{D}(\widehat{\mathrm{Br}}_{X/R})^* = \mathbb{D}(\widehat{\mathrm{Br}}_{X/R}^*)(1)$.

Theorem 2.3.1 (Nygaard–Ogus). There is the following exact sequence of F -crystals:

$$0 \rightarrow \mathbb{D}(\psi_{X/R}) \xrightarrow{\rho} \mathbb{H}_{\mathrm{crys}}^2(X/R) \rightarrow \mathbb{D}(\widehat{\mathrm{Br}}_{X/R}^*)(-1) \rightarrow 0.$$

Moreover, by evaluating on the trivial PD thickening $R \xrightarrow{\mathrm{id}} R$, we get the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & F^2 H_{\mathrm{dR}}^2(X/R) & \longrightarrow & F^1 \mathbb{D}(\widehat{\mathrm{Br}}_{X/R}^*)(-1)_R \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F^1 \mathbb{D}(\psi_{X/R})_R & \longrightarrow & F^1 H_{\mathrm{dR}}^2(X/R) & \longrightarrow & \mathbb{D}(\widehat{\mathrm{Br}}_{X/R}^*)(-1)_R \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathbb{D}(\psi_{X/R})_R & \xrightarrow{\rho} & H_{\mathrm{dR}}^2(X/R) & \longrightarrow & \mathbb{D}(\widehat{\mathrm{Br}}_{X/R}^*)(-1)_R \longrightarrow 0 \end{array}$$

In this diagram, all horizontal rows are exact.

Proof. See [NO85, Theorem 3.20]. □

In this section, we prove a display-theoretic analogue of Theorem 2.3.1. By evaluating on the PD thickening $\widehat{W}(R) \twoheadrightarrow R$, we get a morphism of $\widehat{W}(R)$ -modules

$$\hat{\rho} : \mathbb{D}(\psi_{X/R})_{(\widehat{W}(R) \twoheadrightarrow R)} \rightarrow H_{\mathrm{crys}}^2(X/\widehat{W}(R)).$$

We want to construct a morphism of $\widehat{W}(R)$ -displays $\underline{D}(\psi_{X/R}) \rightarrow \underline{H}_{\mathrm{crys}}^2(X/\widehat{W}(R))$ which induces $\hat{\rho}$ on the underlying $\widehat{W}(R)$ -modules. In order to do this, we work on p -torsion free bases.

In the following, we assume $p \geq 3$. Let $\mathcal{X} \rightarrow \mathrm{Spf} W(k)[[t_1, \dots, t_{20}]]$ be the universal deformation of X_0 . We put $A := W(k)[[t_1, \dots, t_{20}]]$, $A_n := A/(t_1^n, \dots, t_{20}^n)$, and $\mathcal{X}_n := \mathcal{X} \otimes_A A_n$. Let $\sigma : A \rightarrow A$ be the endomorphism such that $\sigma(t_i) = t_i^p$ and such that σ is the Frobenius on $W(k)$. By evaluating ρ on the trivial PD thickening $A_n \xrightarrow{\mathrm{id}} A_n$, we get a morphism of A_n -modules

$$\rho_{A_n} : \mathbb{D}(\psi_{\mathcal{X}_n/A_n})_{A_n} \rightarrow H_{\mathrm{dR}}^2(X_n/A_n).$$

Here $\mathbb{D}(\psi_{\mathcal{X}_n/A_n})_{A_n}$ is the underlying A_n -module of the Dieudonné display associated to $\psi_{\mathcal{X}_n/A_n}$, and $H_{\mathrm{dR}}^2(\mathcal{X}_n/A_n)$ is the underlying A_n -module of the $K3$ display associated to the $K3$ surface $\mathcal{X}_n/A_n$.

Proposition 2.3.2. The map ρ_{A_n} defines a morphism of A_n -displays

$$\underline{\rho}_{A_n} : \underline{D}(\psi_{\mathcal{X}_n/A_n}) \rightarrow \underline{H}_{\mathrm{dR}}^2(\mathcal{X}_n/A_n).$$

Proof. It suffices to prove that ρ_{A_n} commutes with F and preserves filtrations. The former is clear because ρ is a morphism of F -crystals. We prove the latter assertion. Since $\underline{D}(\psi_{\mathcal{X}_n/A_n})$ is a 1-display, it suffices to prove that $\rho_{A_n}(F^1 \underline{D}(\psi_{\mathcal{X}_n/A_n}))$ is contained in $F^1 \underline{H}_{\mathrm{dR}}^2(\mathcal{X}_n/A_n)$. This assertion follows from the commutative diagram in Theorem 2.3.1. $\square$

Composing ρ_{A_n} with $\underline{D}(\widehat{\mathrm{Br}}_{\mathcal{X}_n/A_n}) \hookrightarrow \underline{D}(\psi_{\mathcal{X}_n/A_n})$, we get a morphism of A_n -displays $\underline{D}(\widehat{\mathrm{Br}}_{\mathcal{X}_n/A_n}) \rightarrow \underline{H}_{\mathrm{dR}}^2(\mathcal{X}_n/A_n)$. By taking the dual of this map, we get a morphism of A_n -displays

$$\underline{H}_{\mathrm{dR}}^2(\mathcal{X}_n/A_n) \rightarrow \underline{D}(\widehat{\mathrm{Br}}_{\mathcal{X}_n/A_n}^*)(-1)$$

because $\underline{D}(\widehat{\mathrm{Br}}_{\mathcal{X}_n/A_n})^* \cong \underline{D}(\widehat{\mathrm{Br}}_{\mathcal{X}_n/A_n}^*)(1)$ and $(\underline{H}_{\mathrm{dR}}^2(\mathcal{X}_n/A_n))^* \cong \underline{H}_{\mathrm{dR}}^2(\mathcal{X}_n/A_n)(2)$.

Theorem 2.3.3. There is the following exact sequence of A_n -displays:

$$0 \rightarrow \underline{D}(\psi_{\mathcal{X}_n/A_n}) \xrightarrow{\rho_{A_n}} \underline{H}_{\mathrm{dR}}^2(\mathcal{X}_n/A_n) \rightarrow \underline{D}(\widehat{\mathrm{Br}}_{\mathcal{X}_n/A_n}^*)(-1) \rightarrow 0$$

Proof. It suffices to show that the following three sequences of A_n -modules are exact.

$$\begin{aligned} 0 &\rightarrow D(\psi_{\mathcal{X}_n/A_n}) \xrightarrow{\rho_{A_n}} H_{\mathrm{dR}}^2(\mathcal{X}_n/A_n) \rightarrow D(\widehat{\mathrm{Br}}_{\mathcal{X}_n/A_n}^*) \rightarrow 0 \\ 0 &\rightarrow F^1 D(\psi_{\mathcal{X}_n/A_n}) \xrightarrow{\rho_{A_n}} F^1 H_{\mathrm{dR}}^2(\mathcal{X}_n/A_n) \rightarrow D(\widehat{\mathrm{Br}}_{\mathcal{X}_n/A_n}^*) \rightarrow 0 \\ 0 &\rightarrow F^2 D(\psi_{\mathcal{X}_n/A_n}) \xrightarrow{\rho_{A_n}} F^2 H_{\mathrm{dR}}^2(\mathcal{X}_n/A_n) \rightarrow F^1 D(\widehat{\mathrm{Br}}_{\mathcal{X}_n/A_n}^*) \rightarrow 0 \end{aligned}$$

The exactness of all sequences follows from Theorem 2.3.1. $\square$

Corollary 2.3.4. Let X/R , $\psi_{X/R}$, and $\widehat{\mathrm{Br}}_{X/R}$ be the same as the beginning of this section. Then there is the following exact sequence of $\widehat{W}(R)$ -displays:

$$0 \rightarrow \underline{D}(\psi_{X/R}) \rightarrow \underline{H}_{\mathrm{crys}}^2(X/\widehat{W}(R)) \rightarrow \underline{D}(\widehat{\mathrm{Br}}_{X/R}^*)(-1) \rightarrow 0.$$

Moreover, this sequence is compatible with base change with respect to R .

Proof. Let $A \rightarrow R, (t_i \mapsto x_i)$ be the morphism which corresponds to the deformation X/R . We define $\phi : A \rightarrow \widehat{W}(R)$ by $\phi(t_i) = [x_i]$, where $[x_i]$ denotes the Teichmüller lift of x_i . Then, for a sufficiently large n , the map ϕ induces a morphism of frames $A_n \rightarrow \widehat{W}(R)$. Taking the base change of the sequence in Theorem 2.3.3 by this morphism, we get the desired exact sequence. $\square$

2.4. A construction of quasi-canonical liftings

In this section, we prove the main result of this chapter.

Theorem 2.4.1. Let X_0 be a $K3$ surface of finite height $h < \infty$ over a finite field k of characteristic $p \geq 3$. Then there exist a totally ramified finite extension $V/W(k)$ of degree h and a quasi-canonical lifting X/V of X_0/k .

Proof. We follow the strategy of Nygaard–Ogus [NO85]. Instead of the crystalline deformation theory used in [NO85], we use the display-theoretic deformation theory of $K3$ surfaces.

We put $k = \mathbb{F}_{p^m}$. Let $\widehat{\mathrm{Br}}_{X_0/k}$ (resp. $\psi_{X_0/k}$) be the formal Brauer group (resp. the enlarged formal Brauer group) of X_0/k . We note that $\mathrm{ht}(\widehat{\mathrm{Br}}_{X_0/k}) = h$, $\mathrm{ht}(\psi_{X_0/k}) = 22 - h$, and $\dim(\widehat{\mathrm{Br}}_{X_0/k}) = \dim(\psi_{X_0/k}) = 1$. Let $\mathrm{Fr} : \widehat{\mathrm{Br}}_{X_0/k} \rightarrow \widehat{\mathrm{Br}}_{X_0/k}^{(p)}$ be the relative Frobenius morphism. By [Haz12, Theorem 24.2.6, Theorem 24.3.4], there exists a totally ramified finite extension $V/W(k)$ of degree h , a lifting G of $\widehat{\mathrm{Br}}_{X_0/k}$ over V , and an endomorphism $F : G \rightarrow G$ which lifts $\mathrm{Fr}^m : \widehat{\mathrm{Br}}_{X_0/k} \rightarrow \widehat{\mathrm{Br}}_{X_0/k}$. Let π be a prime element of V .

For each $n \geq 0$, we put $R_n := V/\pi^{n+1}$ and $G_n := G \otimes_V R_n$. Let H_n denote the unique lifting of the étale part of $\psi_{X_0/k}$ over R_n . We define a $\widehat{W}(R_n)$ -display $\underline{K}(G_n)$ as

$$\underline{K}(G_n) := \underline{D}(G_n) \oplus \underline{D}(H_n) \oplus \underline{D}(G_n^*)(-1).$$

Taking base change by $\widehat{W}(R_n) \rightarrow W(k)$, the right hand side gives the slope decomposition of $\underline{H}_{\mathrm{crys}}^2(X_0/W(k))$. Let $\alpha : \widehat{W}(R_n) \rightarrow R_n$ be as in Remark 2.2.18. Since we have the following isomorphisms

$$\begin{aligned} \underline{D}(G_n)^\alpha &\cong R_n \oplus R_n(-1)^{\oplus h-1} \\ \underline{D}(H_n)^\alpha &\cong R_n(-1)^{\oplus 22-2h} \\ \underline{D}(G_n^*)^\alpha &\cong R_n^{\oplus h-1} \oplus R_n(-1) \end{aligned}$$

by Remark 2.2.18, $\underline{K}(G_n)^\alpha$ is isomorphic to

$$R_n \oplus R_n(-1)^{\oplus 20} \oplus R_n(-2).$$

We have a canonical perfect pairing

$$\underline{D}(G_n) \times \underline{D}(G_n^*)(-1) \rightarrow \widehat{W}(R_n)(-2)$$

and a perfect pairing

$$\underline{D}(H_n) \times \underline{D}(H_n) \rightarrow \widehat{W}(R_n)(-2)$$

which is the unique lifting of the pairing on the slope one part of $\underline{H}_{\mathrm{crys}}^2(X_0/W(k))$ ([Zin01b, Lemma 42]). These pairings define a $K3$ display structure on $\underline{K}(G_n)$.

Let $\mathcal{X}$ be a formal lifting which corresponds to the inverse system of $K3$ displays $\{\underline{K}(G_n)\}$ (Theorem 2.2.24). Let $X_n := \mathcal{X} \otimes_V R_n$. Then X_n is a $K3$ surface over R_n corresponding to the $K3$ display $\underline{K}(G_n)$. We consider the following morphisms of $\widehat{W}(R_n)$ -displays:

$$\underline{D}(G_n^*)(-1) \hookrightarrow \underline{K}(G_n) = \underline{H}_{\mathrm{crys}}^2(X_n/\widehat{W}(R_n)) \twoheadrightarrow \underline{D}(\widehat{\mathrm{Br}}_{X_n/R_n}^*)(-1),$$

$$\begin{array}{ccccccc}
0 & \longrightarrow & \underline{D}(\psi_{X_n/R_n}) & \longrightarrow & \underline{H}_{\text{crys}}^2(X_n/\widehat{W}(R_n)) & \longrightarrow & \underline{D}(\widehat{\text{Br}}_{X_n/R_n}^*(-1)) \longrightarrow 0 \\
& & \downarrow \text{dotted} & & \parallel & & \downarrow \simeq \\
0 & \longrightarrow & \underline{D}(G_n) \oplus \underline{D}(H_n) & \longrightarrow & \underline{K}(G_n) & \longrightarrow & \underline{D}(G_n^*(-1)) \longrightarrow 0
\end{array}$$
$$\underline{D}(\psi_{X_n/R_n}) \xrightarrow{\sim} \underline{D}(G_n) \oplus \underline{D}(H_n) = \underline{D}(G_n \oplus H_n).$$
$$\psi_{X_n/R_n} \xrightarrow{\sim} G_n \oplus H_n$$

By [NO85, Definition 1.5 and Theorem 1.9], in order to prove that X/V is a quasi-canonical lifting of X_0/k , it suffices to prove that there exists $\gamma \in W_{\text{crys}}(\overline{K})$ of non-zero degree such that the action of γ on $H_{\text{dR}}^2(X/V) \otimes_V \overline{K}$ preserves the Hodge filtration. Let H denote the unique lifting of the étale part of $\psi_{X_0/k}$. By the definition of $\underline{K}(G_n)$,

$$\begin{aligned} H_{\mathrm{dR}}^2(X/V) &= \mathbb{D}(G)_V \oplus \mathbb{D}(H)_V \oplus \mathbb{D}(G^*)(-1)_V \\ &\supset F^1 \mathbb{D}(G)_V \oplus \mathbb{D}(H)_V \oplus \mathbb{D}(G^*)(-1)_V \\ &\supset F^1 \mathbb{D}(G^*)(-1)_V \end{aligned}$$

$$\begin{aligned} H_{\mathrm{dR}}^2(X/V) \otimes \overline{K} &= (\mathbb{D}(G)_V \oplus \mathbb{D}(H)_V \oplus \mathbb{D}(G^*)(-1)_V) \otimes_V \overline{K} \\ &\cong (\mathbb{D}(\widehat{\mathrm{Br}}_{X_0/k})_{W(k)} \oplus \mathbb{D}(H_0)_{W(k)} \oplus \mathbb{D}(\widehat{\mathrm{Br}}_{X_0/k}^*(-1)_{W(k)}) \otimes_{W(k)} \overline{K} \\ &\cong H_{\mathrm{crys}}^2(X_0/W(k)) \otimes_{W(k)} \overline{K}. \end{aligned}$$
$$\mathbb{D}(G^*)_V \otimes_V \overline{K} \cong \mathbb{D}(G_0^*)_{W(k)} \otimes_{W(k)} \overline{K} = \mathbb{D}(\widehat{\mathrm{Br}}_{X_0/k}^*)_{W(k)} \otimes_{W(k)} \overline{K}$$

is functorial in G , an element $\gamma \in W_{\text{crys}}(\overline{K})$ with $\deg(\gamma) = m$ acts on $\mathbb{D}(G^*)_V \otimes_V \overline{K}$ as $\mathbb{D}(F^*) \otimes \gamma$. Therefore, the action of such $\gamma \in W_{\text{crys}}(\overline{K})$ on $H_{\text{dR}}^2(X/V) \otimes_V \overline{K}$ preserves the Hodge filtration. This completes the proof. $\square$

CHAPTER 3

Slope filtration of log p -divisible groups

3.1. Introduction

The following is the first goal of this chapter.

Theorem 3.1.1 (see [Corollary 3.3.7](#)). Let $(X, \mathcal{M}_X)$ be a locally noetherian normal fs log scheme over $\mathbb{F}_p$. Then, for a log p -divisible group G over $(X, \mathcal{M}_X)$ with constant Newton polygon, there exists a completely slope divisible log p -divisible group H and an isogeny $G \rightarrow H$.

This result is deduced from the corresponding result of Oort-Zink by using the fact that a completely slope divisible log p -divisible group is written as an extension of a classical étale p -divisible group by a classical completely slope divisible formal Lie group (see [Proposition 3.3.4](#)).

Next, we consider a stronger statement. Oort-Zink proved the above result by extending an isogeny generically defined. We prove that we can make the same procedure for log p -divisible groups.

Theorem 3.1.2 (see [Theorem 3.5.4](#)). Let $(X, \mathcal{M}_X)$ be a log regular fs log scheme over $\mathbb{F}_p$ and U be a dense open subset of X . Let G be a log p -divisible group over $(X, \mathcal{M}_X)$. Suppose that we are given a completely slope divisible log p -divisible group H_U over $(U, \mathcal{M}_U)$ and an isogeny $G|_U \rightarrow H_U$. Then there exist a completely slope divisible log p -divisible group H over $(X, \mathcal{M}_X)$ and an isogeny $G \rightarrow H$ which restricts to the given isogeny $G|_U \rightarrow H_U$.

When X is regular, this theorem follows from a result on the extendability of complete slope divisible log p -divisible groups (see [Proposition 3.5.3](#)). In general cases, we reduce the problem to the above case by using resolutions of toric singularities in [\[Niz06\]](#).

Note that we need the assumption of log regularity. [Theorem 3.1.2](#) does not hold under the assumption of [Theorem 3.1.1](#) (for a counterexample, see [Remark 3.4.7](#)).

The outline of this chapter is as follows. In [Section 3.2](#), we review the definitions of kfl topology, kfl quasi-coherent sheaves, log finite group schemes, and log p -divisible groups, and we prove some fundamental properties of them. In [Section 3.3](#), we introduce the notion of completely slope divisible log p -divisible groups and Newton polygons for log p -divisible groups. In [Section 3.4](#), we review the definition and properties of log regular fs log scheme, and we study log finite group schemes and kfl vector bundles on log regular fs log schemes for the proof of [Theorem 3.5.4](#). In [Section 3.5](#), we shall prove [Theorem 3.5.4](#).

Convention. Throughout this chapter, fiber products of saturated log schemes are taken in the category of saturated log schemes unless otherwise specified. For morphisms of saturated log schemes $(X, \mathcal{M}_X) \rightarrow (Z, \mathcal{M}_Z)$ and $(Y, \mathcal{M}_Y) \rightarrow (Z, \mathcal{M}_Z)$, the saturated fiber product of $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$ over $(Z, \mathcal{M}_Z)$ is simply denoted by

$$(X, \mathcal{M}_X) \times_{(Z, \mathcal{M}_Z)} (Y, \mathcal{M}_Y).$$

3.2. Preliminaries on log schemes and log p -divisible groups

Throughout this section, let $(X, \mathcal{M}_X)$ denote an fs log scheme.

3.2.1. Kummer log flat topologies.

We recall some basic results on *Kummer log flat topology* (which is called *kfl topology*, for short) introduced by Kato in [Kat21, Definition 2.3].

Definition 3.2.1 (Kummer maps of monoids). A monoid map $f: M \rightarrow N$ is called *Kummer* if f is injective and, for every $q \in N$, there exist an integer $n \geq 1$ and $p \in M$ such that $f(p) = q^n$.

Definition 3.2.2 ([Kat21, (1.10) and Definition 2.2]). Let $f: (X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y)$ be a morphism of fs log schemes.

- (1) The morphism f is *log flat* (resp. *log étale*) if, fppf (resp. étale) locally on X and Y , there exists a chart $P \rightarrow Q$ of f , where P and Q are finitely generated, such that the following conditions are satisfied:
 - the induced map $P^{\text{gp}} \rightarrow Q^{\text{gp}}$ is injective (resp. injective and its cokernel is a finite abelian group with an order invertible on X);
 - the induced morphism $(X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y) \otimes_{(\mathbb{Z}[P], P)^a} (\mathbb{Z}[Q], Q)^a$ is strict flat (resp. strict étale).
- (2) The morphism f is *Kummer* if, for each $x \in X$, the natural map $\mathcal{M}_{Y, \bar{y}} / \mathcal{O}_{Y, \bar{y}}^\times \rightarrow \mathcal{M}_{X, \bar{x}} / \mathcal{O}_{X, \bar{x}}^\times$ is Kummer, where $y := f(x)$.

Lemma 3.2.3. Let $f: (X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y)$ be a Kummer morphism of fs log schemes. Suppose that X is quasi-compact, and that we are given an fs chart $P \rightarrow \mathcal{M}_Y$. We put $(Y_n, \mathcal{M}_{Y_n}) := (Y, \mathcal{M}_Y) \otimes_{(\mathbb{Z}[P], P)^a} (\mathbb{Z}[P^{1/n}], P^{1/n})^a$ for an integer $n \geq 1$. Then the natural morphism

$$(X, \mathcal{M}_X) \times_{(Y, \mathcal{M}_Y)} (Y_n, \mathcal{M}_{Y_n}) \rightarrow (Y_n, \mathcal{M}_{Y_n})$$

is strict for some $n \geq 1$.

Proof. See the proof of [Kat21, Proposition 2.7 (2)]. □

Definition 3.2.4. A family of morphisms $\{f_i: (U_i, \mathcal{M}_{U_i}) \rightarrow (X, \mathcal{M}_X)\}$ of fs log schemes is called a *Kummer log flat covering* (resp. a *Kummer log étale covering*) if the following conditions are satisfied.

- (1) Each f_i is log flat (resp. log étale) and Kummer, and its underlying morphism of schemes is locally of finite presentation.
- (2) The family is set-theoretically surjective, i.e. $X = \cup f_i(U_i)$.

For short, a Kummer log flat covering (resp. a Kummer log étale covering) is called a *kfl covering* (resp. a *két covering*).

We define a site $(X, \mathcal{M}_X)_{\text{kfl}}$ (resp. $(X, \mathcal{M}_X)_{\text{két}}$) as the category of fs log schemes over $(X, \mathcal{M}_X)$ equipped with the topology defined by kfl coverings (resp. két coverings) ([Kat21, Definition 2.3]). This topology is called *kfl topology* (resp. *két topology*), and the resulting site is called a *kfl site* (resp. a *két site*). The kfl topology is subcanonical ([Kat21, Theorem 3.1]). In other words, for an fs log scheme $(Z, \mathcal{M}_Z)$ over $(X, \mathcal{M}_X)$, the presheaf on $(X, \mathcal{M}_X)_{\text{kfl}}$ given by $(Y, \mathcal{M}_Y) \mapsto \text{Mor}_{(X, \mathcal{M}_X)}((Y, \mathcal{M}_Y), (Z, \mathcal{M}_Z))$ is a sheaf. In particular, we have a sheaf on $(X, \mathcal{M}_X)_{\text{kfl}}$ defined by $(Y, \mathcal{M}_Y) \mapsto \Gamma(Y, \mathcal{O}_Y)$, denoted by $\mathcal{O}_{(X, \mathcal{M}_X)}$. Furthermore, we define $\mathbb{G}_{m, \log}$ as the strict étale sheafification of the presheaf on $(X, \mathcal{M}_X)_{\text{kfl}}$ given by $(Y, \mathcal{M}_Y) \mapsto \Gamma(Y, \mathcal{M}_Y)^{\text{gp}}$. Then $\mathbb{G}_{m, \log}$ is a sheaf on $(X, \mathcal{M}_X)_{\text{kfl}}$ ([Kat21, Theorem 3.2]).

3.2.2. Kfl vector bundles. Let $(X, \mathcal{M}_X)$ be an fs log scheme.

Definition 3.2.5. A vector bundle on the ringed site $((X, \mathcal{M}_X)_{\text{kfl}}, \mathcal{O}_{(X, \mathcal{M}_X)})$ is called a *kfl vector bundle* on $(X, \mathcal{M}_X)$. The category of kfl vector bundles on $(X, \mathcal{M}_X)$ is denoted by $\text{Vect}_{\text{kfl}}(X, \mathcal{M}_X)$. The family of short exact sequences as sheaves on $(X, \mathcal{M}_X)_{\text{kfl}}$ makes $\text{Vect}_{\text{kfl}}(X, \mathcal{M}_X)$ an exact category.

For a vector bundle $\mathcal{E}$ on X , we define the $\mathcal{O}_{(X, \mathcal{M}_X)}$ -module $\iota(\mathcal{E})$ by $(Y, \mathcal{M}_Y) \mapsto \Gamma(Y, f^*\mathcal{E})$, where $f: Y \rightarrow X$ is the structure morphism. By [Niz08, Proposition 2.19], $\iota(\mathcal{E})$ is a sheaf on $(X, \mathcal{M}_X)_{\text{kfl}}$.

Lemma 3.2.6. The functor $\iota: \text{Vect}(X) \rightarrow \text{Vect}_{\text{kfl}}(X, \mathcal{M}_X)$ gives a bi-exact equivalence to the essential image, where $\text{Vect}(X)$ denotes the category of vector bundles on X .

Proof. Fully faithfulness and exactness follow from the definition. It suffices to prove that a sequence of vector bundles on X

$$(3.2.1) \quad 0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_2 \rightarrow \mathcal{E}_3 \rightarrow 0$$

is exact if the associated sequence

$$(3.2.2) \quad 0 \rightarrow \iota(\mathcal{E}_1) \rightarrow \iota(\mathcal{E}_2) \rightarrow \iota(\mathcal{E}_3) \rightarrow 0$$

is exact in $\text{Vect}_{\text{kfl}}(X, \mathcal{M}_X)$. Let U be an affine open subset of X . Since $H^1((U, \mathcal{M}_U), \iota(\mathcal{E}_1))$ vanishes by [Niz08, Proposition 3.27], applying $H^0((U, \mathcal{M}_U), -)$ to the exact sequence (3.2.2) gives an exact sequence

$$0 \rightarrow \mathcal{E}_1(U) \rightarrow \mathcal{E}_2(U) \rightarrow \mathcal{E}_3(U) \rightarrow 0.$$

Therefore, the sequence (3.2.1) is exact. $\square$

We regard $\text{Vect}(X)$ as a full subcategory of $\text{Vect}_{\text{kfl}}(X, \mathcal{M}_X)$ via the functor ι . When there is no possibility of confusion, we simply write $\mathcal{E}$ for $\iota(\mathcal{E})$. A kfl vector bundle $\mathcal{E}$ on $(X, \mathcal{M}_X)$ is called *classical* if $\mathcal{E}$ belongs to $\text{Vect}(X)$.

Lemma 3.2.7. Let $(X, \mathcal{M}_X)$ be a quasi-compact fs log scheme and $\mathcal{F}$ be a kfl quasi-coherent sheaf on $(X, \mathcal{M}_X)$. Suppose that we are given an fs chart $P \rightarrow \mathcal{M}_X$. Then the pullback of $\mathcal{F}$ by a kfl covering

$$(X, \mathcal{M}_X) \otimes_{(\mathbb{Z}[P], P)^a} (\mathbb{Z}[P^{1/n}], P^{1/n})^a \rightarrow (X, \mathcal{M}_X)$$

is classical for some $n \geq 1$.

Proof. Take a kfl covering $(Y, \mathcal{M}_Y) \rightarrow (X, \mathcal{M}_X)$ such that Y is quasi-compact and $\mathcal{F}|_{(Y, \mathcal{M}_Y)}$ is classical. For an integer $n \geq 1$, we set

$$\begin{aligned} (X_n, \mathcal{M}_{X_n}) &:= (X, \mathcal{M}_X) \otimes_{(\mathbb{Z}[P], P)^a} (\mathbb{Z}[P^{1/n}], P^{1/n})^a, \\ (Y_n, \mathcal{M}_{Y_n}) &:= (Y, \mathcal{M}_Y) \otimes_{(\mathbb{Z}[P], P)^a} (\mathbb{Z}[P^{1/n}], P^{1/n})^a. \end{aligned}$$

By Lemma 3.2.3, the natural morphism

$$(Y_m, \mathcal{M}_{Y_m}) \rightarrow (X_m, \mathcal{M}_{X_m})$$

is a strict fppf covering for some $m \geq 1$. Since $\mathcal{F}|_{(Y, \mathcal{M}_Y)}$ is classical, $\mathcal{F}|_{(Y_m, \mathcal{M}_{Y_m})}$ is classical. By fppf descent, $\mathcal{F}|_{(X_m, \mathcal{M}_{X_m})}$ is also classical. $\square$

Proposition 3.2.8. Let $\{(X_i, \mathcal{M}_{X_i})\}_{i \in I}$ be a cofiltered system of fs log schemes satisfying the condition $(*)$ in the beginning of Appendix 3.A. We put $(X, \mathcal{M}_X) := \varprojlim_{i \in I} (X_i, \mathcal{M}_{X_i})$.

- (1) Suppose that the cofiltered category I has a final object 0. Let $\mathcal{E}_0$ and $\mathcal{F}_0$ be kfl vector bundles on $(X_0, \mathcal{M}_{X_0})$. We put

$$\begin{aligned}\mathcal{E}_i &:= \mathcal{E}_0|_{(X_i, \mathcal{M}_{X_i})}, & \mathcal{E} &:= \mathcal{E}_0|_{(X, \mathcal{M}_X)}, \\ \mathcal{F}_i &:= \mathcal{F}_0|_{(X_i, \mathcal{M}_{X_i})}, & \mathcal{F} &:= \mathcal{F}_0|_{(X, \mathcal{M}_X)}.\end{aligned}$$

Then the natural map

$$\varinjlim_{i \in I} \mathrm{Hom}(\mathcal{E}_i, \mathcal{F}_i) \rightarrow \mathrm{Hom}(\mathcal{E}, \mathcal{F})$$

is an isomorphism.

- (2) For any kfl vector bundle $\mathcal{E}$ on $(X, \mathcal{M}_X)$, there exist $i \in I$ and a kfl vector bundle $\mathcal{E}_i$ on $(X_i, \mathcal{M}_{X_i})$ with an isomorphism $\mathcal{E}_i|_X \cong \mathcal{E}$.

Proof. (1) By working Kummer log flat locally on $(X_0, \mathcal{M}_{X_0})$, this statement follows from the limit argument for usual vector bundles on schemes.

(2) Take a qcqs kfl covering $(Y, \mathcal{M}_Y) \rightarrow (X, \mathcal{M}_X)$ such that $\mathcal{E}|_Y$ is classical. By Proposition 3.A.3 (2) and Proposition 3.A.5 (4), $(Y, \mathcal{M}_Y) \rightarrow (X, \mathcal{M}_X)$ comes from a kfl covering $(Y_i, \mathcal{M}_{Y_i}) \rightarrow (X_i, \mathcal{M}_{X_i})$ and $\mathcal{E}|_Y$ comes from a vector bundle $\mathcal{E}_{Y_i}$ over Y_i for some $i \in I$. By replacing i with a bigger object of I , we may assume that the descent datum of $\mathcal{E}|_Y$ over $(Y, \mathcal{M}_Y) \times_{(X, \mathcal{M}_X)} (Y, \mathcal{M}_Y)$ comes from the descent datum of $\mathcal{E}_{Y_i}$ over $(Y_i, \mathcal{M}_{Y_i}) \times_{(X_i, \mathcal{M}_{X_i})} (Y_i, \mathcal{M}_{Y_i})$, which defines a kfl vector bundle $\mathcal{E}_i$ on $(X_i, \mathcal{M}_{X_i})$ with an isomorphism $\mathcal{E}_i|_X \cong \mathcal{E}$. $\square$

Proposition 3.2.9 (cf. [Kat21, Theorem 6.2]). Let $\mathcal{E}$ be a kfl vector bundle on $(X, \mathcal{M}_X)$. Suppose that the pullback of $\mathcal{E}$ to a point $x \in X$ is classical. Then there exists an open neighborhood $U \subset X$ of x such that $\mathcal{E}|_{(U, \mathcal{M}_U)}$ is classical. In particular, if the pullback of $\mathcal{E}$ to every point $x \in X$ is classical, then $\mathcal{E}$ is also classical.

Proof. By the limit argument (Proposition 3.2.8), we may assume that X is the spectrum of a noetherian strict local ring with the unique closed point x . In this case, the assertion follows from [Kat21, Theorem 6.2]. $\square$

3.2.3. Log finite group schemes.

In this subsection, we review the definition of log finite group schemes and prove some fundamental properties.

Definition 3.2.10 ([Kat23], Section 1).

- (1) A sheaf of abelian groups G on $(X, \mathcal{M}_X)_{\mathrm{kfl}}$ is called a *weak log finite group scheme* if there exists a kfl covering $\{(U_i, \mathcal{M}_{U_i}) \rightarrow (X, \mathcal{M}_X)\}$ such that the pullback $G|_{(U_i, \mathcal{M}_{U_i})}$ is representable by a finite locally free group scheme over U_i . The category of weak log finite group schemes over $(X, \mathcal{M}_X)$ is denoted by $\mathrm{wFin}(X, \mathcal{M}_X)$.
- (2) For a weak log finite group scheme G over $(X, \mathcal{M}_X)$, the sheaf $G^* := \mathcal{H}om(G, \mathbb{G}_m)$ on $(X, \mathcal{M}_X)_{\mathrm{kfl}}$ is called the *Cartier dual* of G . The sheaf G^* is also a weak log finite group scheme over $(X, \mathcal{M}_X)$.
- (3) A weak log finite group scheme G over $(X, \mathcal{M}_X)$ is called a *log finite group scheme* if G and G^* are representable by a finite Kummer log flat fs log scheme of finite presentation over $(X, \mathcal{M}_X)$. The full subcategory of $\mathrm{wFin}(X, \mathcal{M}_X)$ consisting of log finite group schemes is denoted by $\mathrm{Fin}(X, \mathcal{M}_X)$.

We have the inclusion relation of categories

$$\mathrm{Fin}(X) \subset \mathrm{Fin}(X, \mathcal{M}_X) \subset \mathrm{wFin}(X, \mathcal{M}_X),$$

where $\text{Fin}(X)$ denotes the category of finite locally free group schemes over X . A weak log finite group scheme G over $(X, \mathcal{M}_X)$ is called *classical* if G belongs to $\text{Fin}(X)$.

Definition 3.2.11. Let $(X, \mathcal{M}_X)$ be an fs log scheme. A weak log finite group scheme G over $(X, \mathcal{M}_X)$ is called of *order* n if there exists a kfl covering $\{(U_i, \mathcal{M}_{U_i}) \rightarrow (X, \mathcal{M}_X)\}$ such that $G|_{(U_i, \mathcal{M}_{U_i})}$ is classical and of order n for every i .

Example 3.2.12. For an fs log scheme $(X, \mathcal{M}_X)$ with $\mathcal{M}_X$ being trivial, every log finite group scheme over $(X, \mathcal{M}_X)$ is classical. This follows from the fact that every Kummer morphism $(Y, \mathcal{M}_Y) \rightarrow (X, \mathcal{M}_X)$ of fs log schemes is strict.

Example 3.2.13. Let $n \geq 1$. A log abelian variety A over an fs log scheme $(X, \mathcal{M}_X)$ (for its definition, see [KKN08b, Definition 4.1]) is a sheaf on $(X, \mathcal{M}_X)_{\text{kfl}}$ by [Zha17, Theorem 2.1(1)]. Then $A[n] := \text{Ker}(\times n: A \rightarrow A)$ is a log finite group scheme of order $n^{2\dim(A)}$ over $(X, \mathcal{M}_X)$ by [KKN15, Proposition 18.1 (2)], Proposition 3.2.24 below, and [Zha17, Proposition 3.4 (3)].

Example 3.2.14. Let $(X, \mathcal{M}_X)$ be an fs log scheme. For $n \geq 1$, we have a Kummer sequence ([Kat21, Proposition 4.2])

$$0 \rightarrow \mu_n \rightarrow \mathbb{G}_{m, \log} \xrightarrow{(-)^n} \mathbb{G}_{m, \log} \rightarrow 0.$$

From this sequence, we get a connecting homomorphism

$$\mathbb{G}_{m, \log}(X, \mathcal{M}_X) = \Gamma(X, \mathcal{M}_X^{\text{gp}}) \rightarrow H^1((X, \mathcal{M}_X)_{\text{kfl}}, \mu_n) = \text{Ext}_{(X, \mathcal{M}_X)_{\text{kfl}}}^1(\mathbb{Z}/n\mathbb{Z}, \mu_n),$$

which associates to any $q \in \Gamma(X, \mathcal{M}_X^{\text{gp}})$ an extension of sheaves on $(X, \mathcal{M}_X)_{\text{kfl}}$

$$0 \rightarrow \mu_n \rightarrow G_q \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow 0.$$

The sheaf G_q is a log finite group scheme by [Kat23, Proposition 2.3].

Lemma 3.2.15. Let G be a weak log finite group scheme over a quasi-compact fs log scheme $(X, \mathcal{M}_X)$ with an fs chart $P \rightarrow \mathcal{M}_X$. Then the pullback of G by a kfl covering

$$(X, \mathcal{M}_X) \otimes_{(\mathbb{Z}[P], P)^a} (\mathbb{Z}[P^{1/n}], P^{1/n})^a \rightarrow (X, \mathcal{M}_X)$$

is classical for some $n \geq 1$.

Proof. This follows from the same argument as Lemma 3.2.7. $\square$

Proposition 3.2.16. Let $\{(X_i, \mathcal{M}_{X_i})\}_{i \in I}$ be a cofiltered system of fs log schemes satisfying the condition $(*)$ in the beginning of Appendix 3.A. We put $(X, \mathcal{M}_X) := \varprojlim_{i \in I} (X_i, \mathcal{M}_{X_i})$.

- (1) Suppose that the cofiltered category I has a final object 0. Let G_0 and H_0 be weak log finite group schemes over $(X_0, \mathcal{M}_{X_0})$. We put

$$\begin{aligned} G_i &:= G_0|_{(X_i, \mathcal{M}_{X_i})}, & G &:= G_0|_{(X, \mathcal{M}_X)}, \\ H_i &:= H_0|_{(X_i, \mathcal{M}_{X_i})}, & H &:= H_0|_{(X, \mathcal{M}_X)}. \end{aligned}$$

Then the natural map

$$\varinjlim_{i \in I} \text{Hom}(G_i, H_i) \rightarrow \text{Hom}(G, H)$$

is bijective.

- (2) For a weak log finite group scheme G over $(X, \mathcal{M}_X)$, there exist $i \in I$ and a weak log finite group scheme G_i over $(X_i, \mathcal{M}_{X_i})$ with an isomorphism $G_i|_{(X, \mathcal{M}_X)} \cong G$.

- (3) For a log finite group scheme G over $(X, \mathcal{M}_X)$, there exist $i \in I$ and a log finite group scheme G_i over $(X_i, \mathcal{M}_{X_i})$ with an isomorphism $G_i|_{(X, \mathcal{M}_X)} \cong G$.

Proof. (1) By working Kummer log flat locally on $(X_0, \mathcal{M}_{X_0})$, this statement follows from the limit argument for usual finite locally free group schemes over schemes.

(2) This follows from the same argument as Proposition 3.2.8 (2).

(3) This follows from (1) and the limit argument for finite Kummer log flat fs log schemes of finite presentation (Proposition 3.A.3 and Proposition 3.A.5 (4)). $\square$

Remark 3.2.17. The theory of log finite group schemes is introduced and developed in [Kat23] under the assumption that the base log scheme is noetherian. Since an fs affine noetherian log scheme can be written as a limit of a cofiltered system of fs affine log schemes satisfying the condition $(*)$ in the beginning of Appendix 3.A by Lemma 3.A.2, Proposition 3.2.16 enables us to generalize several results in *loc. cit.* to the non-noetherian case.

Let G be a weak log finite group scheme over $(X, \mathcal{M}_X)$. We let $\mathcal{A}_G$ denote the sheaf of morphisms $\mathcal{M}or(G, \mathcal{O}_{(X, \mathcal{M}_X)})$ on $(X, \mathcal{M}_X)_{\text{kfl}}$. The $\mathcal{O}_{(X, \mathcal{M}_X)}$ -algebra structure on $\mathcal{O}_{(X, \mathcal{M}_X)}$ makes $\mathcal{A}_G$ an $\mathcal{O}_{(X, \mathcal{M}_X)}$ -algebra, and the abelian group structure on G defines a commutative Hopf

$$\mathcal{O}_{(X, \mathcal{M}_X)}$$

-algebra structure on $\mathcal{A}_G$. When G is classical, $\mathcal{A}_G$ coincides with the usual coordinate ring of G . Hence, by working kfl locally on $(X, \mathcal{M}_X)$, we see that the underlying $\mathcal{O}_{(X, \mathcal{M}_X)}$ -module of $\mathcal{A}_G$ is a kfl vector bundle. Therefore, $\mathcal{A}_G$ is a commutative Hopf object of the unitary symmetric monoidal category $\text{Vect}_{\text{kfl}}(X, \mathcal{M}_X)$.

Conversely, for a commutative Hopf object $\mathcal{A}$ of $\text{Vect}_{\text{kfl}}(X, \mathcal{M}_X)$, we let $G_{\mathcal{A}}$ denote the sheaf of homomorphisms of $\mathcal{O}_{(X, \mathcal{M}_X)}$ -algebras $\mathcal{H}om(\mathcal{A}, \mathcal{O}_{(X, \mathcal{M}_X)})$ on $(X, \mathcal{M}_X)_{\text{kfl}}$. The commutative Hopf structure on $\mathcal{A}$ defines an abelian group structure $G_{\mathcal{A}}$. By working kfl locally on $(X, \mathcal{M}_X)$, we see that $G_{\mathcal{A}}$ is a weak log finite group scheme over $(X, \mathcal{M}_X)$.

Proposition 3.2.18 (cf. [Kat23, Proposition 2.15]). The assignments $G \mapsto \mathcal{A}_G$ and $\mathcal{A} \mapsto G_{\mathcal{A}}$ give a contravariant equivalence between the category $\text{wFin}(X, \mathcal{M}_X)$ and the category of commutative Hopf objects of the unitary symmetric monoidal category $\text{Vect}_{\text{kfl}}(X, \mathcal{M}_X)$. Moreover, a weak log finite group scheme G over $(X, \mathcal{M}_X)$ is classical if and only if $\mathcal{A}_G$ is classical.

Proof. The former assertion follows from the definition, and the latter assertion follows from the former one. $\square$

Proposition 3.2.19. Let $H \subset G$ be weak log finite group schemes over $(X, \mathcal{M}_X)$. Suppose that G is classical. Then H and G/H are also classical.

Proof. By working kfl locally on $(X, \mathcal{M}_X)$, we see that the inclusion morphism $H \hookrightarrow G$ induces a surjection $\mathcal{A}_G \twoheadrightarrow \mathcal{A}_H$. Since $\mathcal{A}_G$ is classical, $\mathcal{A}_H$ is also classical by [Niz08, Proposition 3.29]. Hence, Proposition 3.2.18 implies that H is classical. Therefore, G/H is also classical. $\square$

Construction 3.2.20 (Connected-étale sequences, cf. [Kat23, (2.6)]).

Suppose that $X = \text{Spec} R$ for a henselian local ring R , and let G be a weak log finite group scheme over $(X, \mathcal{M}_X)$. We shall construct a unique exact sequence of objects of weak log finite group schemes

$$(3.2.3) \quad 0 \rightarrow G^0 \rightarrow G \rightarrow G^{\text{ét}} \rightarrow 0$$

such that, for every finite Kummer log flat cover $(Y, \mathcal{M}_Y) \rightarrow (X, \mathcal{M}_X)$ with $G|_{(Y, \mathcal{M}_Y)}$ being classical, the pullback of the sequence (3.2.3) is the connected-étale sequence for $G|_{(Y, \mathcal{M}_Y)}$. This sequence is called the *connected-étale sequence* for G . The uniqueness follows from the uniqueness of connected-étale sequences in non-log cases.

First, we assume that R is strict local. Take a chart $P \rightarrow \mathcal{M}_X$. By Lemma 3.2.15, there exists an integer $n \geq 1$ such that the pullback $G|_{(Y, \mathcal{M}_Y)}$ is classical, where we set $(Y, \mathcal{M}_Y) := (X, \mathcal{M}_X) \otimes_{(\mathbb{Z}[P], P)^a} (\mathbb{Z}[P^{1/n}], P^{1/n})^a$. Since Y is the spectrum of the finite product of strict local rings, the connected-étale sequence for $G|_{(Y, \mathcal{M}_Y)}$ descends to the sequence (3.2.1) via kfl descent by the uniqueness of connected-étale sequences in non-log cases.

In general, what we proved in the previous paragraph and the limit argument (Proposition 3.2.16) allow us to take a strict finite étale cover $(X', \mathcal{M}_{X'}) \rightarrow (X, \mathcal{M}_X)$ such that $G|_{(X', \mathcal{M}_{X'})}$ admits a connected-étale sequence. By the uniqueness of connected-étale sequences, this descends to the desired sequence (3.2.1).

Lemma 3.2.21. Suppose that X is the spectrum of a henselian local ring. Let $H \subset G$ be weak log finite group schemes over $(X, \mathcal{M}_X)$. Then there are natural isomorphisms

$$H^0 \cong H \cap G^0, \quad H^{\text{ét}} \cong \text{Im}(H \rightarrow G^{\text{ét}}), \quad (G/H)^0 \cong G^0/H^0, \quad (G/H)^{\text{ét}} \cong G^{\text{ét}}/H^{\text{ét}}.$$

Proof. The analogous statements for non-log finite locally free group schemes are well-known, to which one can reduce the original statement by kfl descent. $\square$

Proposition 3.2.22 (cf. [Kat23, Proposition 2.7 (3)]). Suppose that $X = \text{Spec} R$ for a henselian local ring R with residue characteristic $p > 0$. Let G be a weak log finite group scheme over $(X, \mathcal{M}_X)$ killed by some power of p . Then G is a log finite group scheme if and only if both of G^0 and $G^{\text{ét}}$ are classical. Moreover, in this case, G^0 (resp. $G^{\text{ét}}$) is a connected finite locally free group scheme (resp. a finite étale group scheme) over X .

Proof. This is proved in [Kat23, Proposition 2.7 (3)] in the case where R is noetherian. In general, one can reduce the problem to this case by the limit argument (Proposition 3.2.16). $\square$

Lemma 3.2.23. Let G be a log finite group scheme over $(X, \mathcal{M}_X)$. Suppose that p is locally nilpotent on X and that the order of $(G|_x)^{\text{ét}}$ is constant for every $x \in X$. Then there exists a unique exact sequence

$$0 \rightarrow G^0 \rightarrow G \rightarrow G^{\text{ét}} \rightarrow 0,$$

where G^0 is a finite locally free radiciel group scheme over X and $G^{\text{ét}}$ is a finite étale group scheme over X .

Proof. Since the exact sequence as in the statement is unique if it exists, we may work étale locally on X . Take $x \in X$. By applying the limit argument (Proposition 3.2.16) to the connected-étale sequence for the pullback of G to the strict localization of X at $\bar{x}$, one can take an étale neighborhood U of X at $\bar{x}$ and an exact sequence

$$0 \rightarrow G_U^0 \rightarrow G|_U \rightarrow G_U^{\text{ét}} \rightarrow 0,$$

where G_U^0 is a finite locally free group scheme over U and $G_U^{\text{ét}}$ is a finite étale group scheme over U by Proposition 3.2.22. For every $t \in U$, we have the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & (G|_t)^0 & \longrightarrow & G|_t & \longrightarrow & (G|_t)^{\text{ét}} \longrightarrow 0 \\ & & \downarrow \text{ } & & \downarrow \sim & & \downarrow \text{ } \\ 0 & \longrightarrow & G_U^0|_t & \longrightarrow & (G|_U)|_t & \longrightarrow & G_U^{\text{ét}}|_t \longrightarrow 0. \end{array}$$

By the assumption, the right vertical homomorphism is a homomorphism between finite étale group schemes of the same order, which is an isomorphism. Hence, the left vertical morphism is also an isomorphism. In particular, G_U^0 is radiciel over U . Since we may work étale locally on X , this proves the statement. $\square$

Proposition 3.2.24 (cf. [Kat23, Proposition 2.16]). Let G be a weak log finite group scheme over $(X, \mathcal{M}_X)$. Suppose that the pullback of G to a point $x \in X$ is a log finite group scheme (resp. classical). Then there exists an open neighborhood $U \subset X$ of x such that $G|_{(U, \mathcal{M}_U)}$ is a log finite group scheme (resp. classical). In particular, if the pullback of G to every point $x \in X$ is a log finite group scheme (resp. classical), then G is also a log finite group scheme (resp. classical).

Proof. By the limit argument (Proposition 3.2.16), we may assume that X is noetherian. In this case, the assertion follows from [Kat23, Proposition 2.16]. $\square$

Lemma 3.2.25. Let G be a log finite group scheme over $(X, \mathcal{M}_X)$ of order p . Suppose that p is locally nilpotent over X . Then G is classical.

Proof. Due to Proposition 3.2.24, we may assume that the X is the spectrum of a field. Since the order of G is equal to the product of the orders of G^0 and $G^{\text{ét}}$, the log finite group scheme G is isomorphic to either G^0 or $G^{\text{ét}}$. Hence, the assertion follows from [Kat23, Proposition 2.7 (3)]. $\square$

Lemma 3.2.26. Suppose that $X = \text{Spec} R$ for a reduced henselian local ring R with residue characteristic $p > 0$. Let G be a log finite group scheme over $(X, \mathcal{M}_X)$ killed by some power of p , and let H_1, H_2 be log finite subgroup schemes of G . Suppose that H_1^0 is contained in H_2^0 , $H_1^{\text{ét}}$ is contained in $H_2^{\text{ét}}$, and G^0 is radiciel over X . Then H_1 is contained in H_2 .

Proof. By Lemma 3.2.21, we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_1^0 & \longrightarrow & H_1 & \longrightarrow & H_1^{\text{ét}} \longrightarrow 0 \\ & & \downarrow 0 & & \downarrow & & \downarrow 0 \\ 0 & \longrightarrow & G^0/H_2^0 & \longrightarrow & G/H_2 & \longrightarrow & G^{\text{ét}}/H_2^{\text{ét}} \longrightarrow 0. \end{array}$$

By the above diagram, the canonical morphism $H_1 \rightarrow G/H_2$ factors as $H_1 \rightarrow H_1^{\text{ét}} \rightarrow G^0/H_2^0 \rightarrow G/H_2$. Since $H_1^{\text{ét}}$ is reduced and G^0/H_2^0 is radiciel over X , we get

$$\text{Hom}(H_1^{\text{ét}}, G^0/H_2^0) = 0.$$

Therefore, H_1 is contained in H_2 . $\square$

Proposition 3.2.27. Let $H \subset G$ be weak log finite group schemes over $(X, \mathcal{M}_X)$. Suppose that G is a log finite group scheme. Then H and G/H are also log finite group schemes.

Proof. Proposition 3.2.24 allows us to assume that $X = \text{Spec} k$ for a field k of characteristic $p \geq 0$. By [Kat23, Proposition 2.1], it is enough to treat the case where $p > 0$ and G is killed by some power of p .

Note that G^0 and $G^{\text{ét}}$ are classical by Proposition 3.2.22. There are inclusion relations $H^0 \subset G^0$ and $H^{\text{ét}} \subset G^{\text{ét}}$ by Lemma 3.2.21. Hence, Proposition 3.2.19 implies that H^0 and $H^{\text{ét}}$ are also classical. By using Proposition 3.2.22 again, we see that H is a log finite group scheme. By applying this result to the subobject $(G/H)^* \subset G^*$, we conclude that G/H is also a log finite group scheme. $\square$

3.2.4. Log p -divisible groups.

Definition 3.2.28 ([Kat23], Section 4). Let G be a sheaf of abelian groups on $(X, \mathcal{M}_X)_{\text{kfl}}$. We say that G is a *weak log p -divisible group* over $(X, \mathcal{M}_X)$ if the following conditions are satisfied.

- (1) The map $\times p: G \rightarrow G$ is surjective.
- (2) For each $n \geq 1$, the sheaf $G[p^n] := \text{Ker}(\times p^n: G \rightarrow G)$ is a weak log finite group scheme over $(X, \mathcal{M}_X)$.
- (3) $G = \bigcup_{n \geq 1} G[p^n]$.

The category of weak log p -divisible groups over $(X, \mathcal{M}_X)$ is denoted by $\text{wBT}(X, \mathcal{M}_X)$. A weak log p -divisible group G over $(X, \mathcal{M}_X)$ is called a *log p -divisible group* if $G[p^n]$ is a log finite group scheme for each $n \geq 1$. The category of log p -divisible groups over $(X, \mathcal{M}_X)$ is denoted by $\text{BT}(X, \mathcal{M}_X)$. The category $\text{BT}(X)$ of p -divisible groups over X is regarded as the full subcategory of $\text{wBT}(X, \mathcal{M}_X)$. A weak log p -divisible group G over $(X, \mathcal{M}_X)$ is called *classical* if G belongs to $\text{BT}(X)$. Clearly, G is classical if and only if $G[p^n]$ is classical for each $n \geq 1$.

Definition 3.2.29. For a weak log p -divisible group G over $(X, \mathcal{M}_X)$, we say that G is of *height h* if the order of $G[p^n]$ is p^{nh} for each $n \geq 1$ (equivalently, the order of $G[p]$ is p^h).

Example 3.2.30. Let $n \geq 1$. For any log abelian variety A over $(X, \mathcal{M}_X)$ with $\dim(A) = d$, $A[p^\infty] := \bigcup_{n \geq 1} A[p^n]$ is a log p -divisible group of height $2d$ over $(X, \mathcal{M}_X)$ because $A[p^n]$ is a log finite group scheme of order p^{2nd} by Example 3.2.13 and $\times p: A \rightarrow A$ is surjective from [KKN15, Subsection 18.6].

Lemma 3.2.31. Suppose that X is the spectrum of a henselian local ring. Let G be a weak log p -divisible group over $(X, \mathcal{M}_X)$. Then

$$G^0 := \bigcup_{n \geq 1} G[p^n]^0 \quad \text{and} \quad G^{\text{ét}} := \bigcup_{n \geq 1} G[p^n]^{\text{ét}}$$

are weak log p -divisible groups over $(X, \mathcal{M}_X)$, and there exists an exact sequence

$$0 \rightarrow G^0 \rightarrow G \rightarrow G^{\text{ét}} \rightarrow 0.$$

Proof. Let $\star \in \{0, \text{ét}\}$ be a subscript and $n, m \geq 1$ be integers. By Lemma 3.2.21, the homomorphism $\times p^n: G[p^{n+m}]^\star \rightarrow G[p^{n+m}]^\star$ factors as

$$G[p^{n+m}]^\star \rightarrow G[p^m]^\star \hookrightarrow G[p^{n+m}]^\star.$$

We also simply write $\times p^n$ for the homomorphism $G[p^{n+m}]^\star \rightarrow G[p^m]^\star$ defined above. Then, by applying Lemma 3.2.21 again, we see that the sequence

$$0 \rightarrow G[p^n]^\star \hookrightarrow G[p^{n+m}]^\star \xrightarrow{\times p^n} G[p^m]^\star \rightarrow 0$$

is exact, which implies that G^* is a weak log p -divisible group over $(X, \mathcal{M}_X)$. The remaining assertion is clear. $\square$

Lemma 3.2.32. Let G be a log p -divisible group over $(X, \mathcal{M}_X)$. Suppose that p is locally nilpotent on X and that the height of $(G|_x)^{\text{ét}}$ is constant for every $x \in X$. Then there exists a unique exact sequence

$$0 \rightarrow G^0 \rightarrow G \rightarrow G^{\text{ét}} \rightarrow 0,$$

where G^0 is a formal Lie group over X and $G^{\text{ét}}$ is an étale p -divisible group over X .

Proof. This immediately follows from Lemma 3.2.23. $\square$

Definition 3.2.33. Let G_1 and G_2 be log p -divisible groups over $(X, \mathcal{M}_X)$. A homomorphism $f: G_1 \rightarrow G_2$ is called an *isogeny* if f is surjective and $\text{Ker}(f)$ is a weak log finite group scheme over $(X, \mathcal{M}_X)$.

Let $\text{QIsog}(G_1, G_2)$ be the subset of $\text{Hom}(G_1, G_2)[1/p]$ consisting of elements which can be written as g/p^n for an integer $n \geq 1$ and an isogeny $g: G_1 \rightarrow G_2$. A *quasi-isogeny* from G_1 to G_2 is an element of the global section of the Zariski sheaf on X associated with the presheaf given by $U \mapsto \text{QIsog}(G_1|_U, G_2|_U)$ for an open subset $U \subset X$. Note that, when X is quasi-compact, the set of quasi-isogenies from G_1 to G_2 is naturally in bijection with the set $\text{QIsog}(G_1, G_2)$.

Lemma 3.2.34. Let G be a log p -divisible group and H be a log finite subgroup scheme of G . Then the sheaf G/H is a log p -divisible group, and the canonical map $G \rightarrow G/H$ is an isogeny. Moreover, this gives a one-to-one correspondence between log finite subgroup schemes of G and isogenies from G .

Proof. By working Zariski locally on X , we may assume that H is killed by p^d for an integer $d \geq 1$. For an integer $n \geq d$, applying the snake lemma to the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & G/H \longrightarrow 0 \\ & & \downarrow 0 & & \downarrow \times p^n & & \downarrow \times p^n \\ 0 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & G/H \longrightarrow 0 \end{array}$$

gives an exact sequence

$$0 \rightarrow G[p^n]/H \rightarrow (G/H)[p^n] \rightarrow H \rightarrow 0.$$

Hence, it follows from [Kat23, Proposition 2.3] that $(G/H)[p^n]$ is a log finite group scheme, and so G/H is a log p -divisible group.

Conversely, for an isogeny $f: G_1 \rightarrow G_2$, the weak log finite group scheme $\text{Ker}(f)$ is a log finite group scheme by Proposition 3.2.27. This proves the assertion. $\square$

Proposition 3.2.35. Let $(X, \mathcal{M}_X)$ be a locally noetherian fs log scheme. Let $f: G_1 \rightarrow G_2$ be a homomorphism of log p -divisible groups over $(X, \mathcal{M}_X)$. Suppose that f is surjective and, Zariski locally, $\text{Ker}(f)$ is killed by p^d for an integer $d \geq 1$. Then f is an isogeny.

Proof. It is enough to prove that $\text{Ker}(f)$ is a log finite group scheme over $(X, \mathcal{M}_X)$. By working Zariski locally on X , we may assume that $\text{Ker}(f) \subset G_1[p^d]$ for an integer $d \geq 1$. The sheaf $\text{Ker}(f)$ is representable by an fs log scheme of finite presentation over $(X, \mathcal{M}_X)$ as it coincides with the kernel of the homomorphism of log finite group schemes $G_1[p^d] \rightarrow G_2[p^d]$ induced from f . Hence, by the limit argument (Proposition 3.2.16 and Proposition 3.A.3), we may assume that X is the spectrum of a strict local ring.

Since $\text{Ker}(f)$ is killed by p^d and f is surjective, there exists a unique homomorphism $g: G_2 \rightarrow G_1$ such that both of fg and gf are equal to the multiplication by p^d . The homomorphism f uniquely induces homomorphisms $f^0: G_1^0 \rightarrow G_2^0$ and $f^{\text{ét}}: G_1^{\text{ét}} \rightarrow G_2^{\text{ét}}$ fitting into the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & G_1^0 & \longrightarrow & G_1 & \longrightarrow & G_1^{\text{ét}} \longrightarrow 0 \\ & & \downarrow f^0 & & \downarrow f & & \downarrow f^{\text{ét}} \\ 0 & \longrightarrow & G_2^0 & \longrightarrow & G_2 & \longrightarrow & G_2^{\text{ét}} \longrightarrow 0. \end{array}$$

Similarly, the homomorphism g uniquely induces homomorphisms $g^0: G_2^0 \rightarrow G_1^0$ and $g^{\text{ét}}: G_2^{\text{ét}} \rightarrow G_1^{\text{ét}}$. By the uniqueness of the induced morphism on the connected part and the étale part, all of the homomorphisms $f^0 g^0$, $f^{\text{ét}} g^{\text{ét}}$, $g^0 f^0$, and $g^{\text{ét}} f^{\text{ét}}$ are the multiplication by p^d . It follows from [CCO14, Proposition 3.3.8] that all of the homomorphisms f^0 , $f^{\text{ét}}$, g^0 , and $g^{\text{ét}}$ are isogenies. By applying the snake lemma to the above diagram, we get an exact sequence

$$0 \rightarrow \text{Ker}(f^0) \rightarrow \text{Ker}(f) \rightarrow \text{Ker}(f^{\text{ét}}) \rightarrow 0.$$

Since $\text{Ker}(f^0)$ and $\text{Ker}(f^{\text{ét}})$ are finite locally free group schemes over X , the sheaf $\text{Ker}(f)$ is a log finite group scheme over $(X, \mathcal{M}_X)$ by [Kat23, Proposition 2.3]. $\square$

3.3. Completely slope divisible log p -divisible groups

Throughout this section, let $(X, \mathcal{M}_X)$ be an fs log scheme over $\mathbb{F}_p$.

Definition 3.3.1. Let G be a sheaf of abelian groups on $(X, \mathcal{M}_X)_{\text{kfl}}$ (the examples of primary interest are objects of weak log finite group schemes or weak log p -divisible groups. Let $G^{(p^n)}$ denote the pullback of G by the n -th power Frobenius morphism on $(X, \mathcal{M}_X)$.

In the usual way, Frobenius morphisms on log schemes over $\mathbb{F}_p$ induce a homomorphism $F: G \rightarrow G^{(p)}$, called the *Frobenius homomorphism*. When G belongs to $\text{Fin}(X)$ or $\text{BT}(X)$, the homomorphism F is nothing but the usual Frobenius homomorphism. For an integer $n \geq 1$, the composite of homomorphisms

$$G \xrightarrow{F} G^{(p)} \xrightarrow{F^{(p)}} G^{(p^2)} \xrightarrow{F^{(p^2)}} \dots \xrightarrow{F^{(p^{n-1})}} G^{(p^n)}$$

are simply denoted by F^n , where $F^{(p^k)}$ denotes the base change of F by the k -th power Frobenius morphism on $(X, \mathcal{M}_X)$ for an integer $k \geq 1$.

Lemma 3.3.2. Let G be a log finite group scheme over $(X, \mathcal{M}_X)$. Suppose that there is an integer $n \geq 1$ such that the homomorphism $F^n: G \rightarrow G^{(p^n)}$ is an isomorphism (resp. a zero map). Then G is classical. In particular, G is a finite étale group scheme (resp. a finite radiciel locally free group scheme) over X .

Proof. We shall prove the assertion just in the case where F^n is an isomorphism because the same argument also works in the other case. By Proposition 3.2.24, we may assume that $X = \text{Spec} k$ for a field k . In this case, we have the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & G^0 & \longrightarrow & G & \longrightarrow & G^{\text{ét}} \longrightarrow 0 \\ & & \downarrow F^n & & \sim \downarrow F^n & & \sim \downarrow F^n \\ 0 & \longrightarrow & (G^0)^{(p^n)} & \longrightarrow & G^{(p^n)} & \longrightarrow & (G^{\text{ét}})^{(p^n)} \longrightarrow 0. \end{array}$$

Since $G^{\text{ét}}$ is a finite étale group scheme over X , the right vertical map in this diagram is an isomorphism. Hence, the left vertical map is also an isomorphism. On the other hand, the m -th power of the Frobenius homomorphism $G^0 \rightarrow (G^0)^{(p^m)}$ is a zero map for some integer $m \geq 1$ because G^0 is a connected finite group scheme over the field k . Therefore, G^0 is trivial, and so $G \cong G^{\text{ét}}$. In particular, G is classical. $\square$

Definition 3.3.3. A log p -divisible group G over $(X, \mathcal{M}_X)$ over $\mathbb{F}_p$ is *completely slope divisible* if G admits a filtration of log p -divisible subgroups

$$0 = G_0 \subset G_1 \subset \cdots \subset G_m = G,$$

and if there are integers

$$s \geq r_1 > r_2 > \cdots > r_m \geq 0$$

such that the following conditions are satisfied :

- (1) The quasi-isogeny $p^{-r_i}F^s$ from G_i to $G_i^{(p^s)}$ is an isogeny for every $1 \leq i \leq m$.
- (2) The morphism $G_i/G_{i-1} \rightarrow (G_i/G_{i-1})^{(p^s)}$ induced from $p^{-r_i}F^s: G_i \rightarrow G_i^{(p^s)}$ is an isomorphism for every $1 \leq i \leq m$.

Proposition 3.3.4. Let G be a completely slope divisible log p -divisible group over $(X, \mathcal{M}_X)$. Then there exists a unique exact sequence

$$0 \rightarrow G^0 \rightarrow G \rightarrow G^{\text{ét}} \rightarrow 0,$$

where G^0 is a completely slope divisible formal Lie group over X and $G^{\text{ét}}$ is an étale p -divisible group over X .

Proof. We use the notation in Definition 3.3.3. We may assume that $r_m = 0$. Let $n \geq 1$ be an integer. By the definition, the Frobenius homomorphism $F: (G/G_{m-1})[p^n] \rightarrow (G/G_{m-1})[p^n]^{(p)}$ is an isomorphism, and the homomorphism $F^{sn}: G_{m-1}[p^n] \rightarrow G_{m-1}[p^n]^{(p^{sn})}$ is a zero map. Hence, by Lemma 3.3.2, G/G_{m-1} is an étale p -divisible group over X , and G_{m-1} is a formal Lie group over X . Therefore, the exact sequence

$$0 \rightarrow G_{m-1} \rightarrow G \rightarrow G/G_{m-1} \rightarrow 0$$

is the desired one. $\square$

We generalize the notion of Newton polygons to log p -divisible groups.

Definition 3.3.5. Suppose that X is the spectrum of a field of characteristic $p > 0$. Let G be a log p -divisible group over $(X, \mathcal{M}_X)$. The *Newton polygon* of G is the Newton polygon of the classical p -divisible group $G^0 \oplus G^{\text{ét}}$.

The notion of Newton polygons for log p -divisible groups enables us to formulate a log version of [OZ02, Proposition 2.3] as follows.

Corollary 3.3.6. Let G be a log p -divisible group over $(X, \mathcal{M}_X)$ with constant Newton polygon. Suppose that X is integral, and that the pullback $G|_{\eta}$ is completely slope divisible, where η is the generic point. Then G is completely slope divisible.

Proof. We take G^0 and $G^{\text{ét}}$ as in Lemma 3.2.32. The formal Lie group G^0 has a constant Newton polygon, and its generic fiber $G^0|_{\eta}$ is completely slope divisible. Hence, [OZ02, Proposition 2.3] implies that G^0 is completely slope divisible. Therefore, G is also completely slope divisible. $\square$

The following corollary is a log version of [OZ02, Theorem 2.1].

Corollary 3.3.7. Let G be a log p -divisible group over $(X, \mathcal{M}_X)$ with constant Newton polygon. Suppose that X is normal. Then there exist a completely slope divisible log p -divisible group H over $(X, \mathcal{M}_X)$ and an isogeny $G \rightarrow H$.

Proof. We take G^0 and $G^{\text{ét}}$ as in Lemma 3.2.32. Since G^0 has a constant Newton polygon, there exist a completely slope divisible p -divisible group H' over X and an isogeny $\phi: G^0 \rightarrow H'$ by [OZ02, Theorem 2.1]. Then there is an exact sequence

$$0 \rightarrow H' \rightarrow G/\text{Ker}(\phi) \rightarrow G^{\text{ét}} \rightarrow 0,$$

and so $G/\text{Ker}(\phi)$ is a completely slope divisible log p -divisible group. Hence, the canonical surjection $G \rightarrow G/\text{Ker}(\phi)$ is the desired isogeny. $\square$

The following lemma is a log version of [OZ02, Lemma 2.5], which we use in the proof of the main theorem.

Lemma 3.3.8. Suppose that X is the spectrum of an algebraically closed field. Let G be a log p -divisible group over $(X, \mathcal{M}_X)$ and $d \geq 1$ be an integer. Then there are only finitely many isomorphism classes of isogenies $G \rightarrow H$ of degree p^d in which H is a completely slope divisible log p -divisible group.

Proof. Let K be a log finite subgroup scheme of G of order p^d . Since G/K is completely slope divisible if and only if $(G/K)^0 = G^0/K^0$ and $(G/K)^{\text{ét}} = G^{\text{ét}}/K^{\text{ét}}$, there are only finitely many possibilities of K^0 and $K^{\text{ét}}$ such that G/K is completely slope divisible by [OZ02, Lemma 2.5]. Lemma 3.2.26 implies that K is determined from K^0 and $K^{\text{ét}}$ uniquely. This proves the statement. $\square$

3.4. Some extension properties on log regular schemes

In this section, we study log finite groups and log p -divisible groups over log regular bases.

3.4.1. Log regular schemes. In this subsection, we recall the definition of log regularity and some properties of log regular log schemes.

Definition 3.4.1 ([Kat94, Niz06]). Let $(X, \mathcal{M}_X)$ be a locally noetherian fs log scheme. For $x \in X$, let $\bar{x}$ denote a geometric point on x . Let $I(\bar{x})$ be the ideal of $\mathcal{O}_{X, \bar{x}}$ generated by the image of the map $\mathcal{M}_{X, \bar{x}} \setminus \mathcal{O}_{X, \bar{x}}^\times \rightarrow \mathcal{O}_{X, \bar{x}}$. We say that $(X, \mathcal{M}_X)$ is *log regular* at x if the following conditions are satisfied:

- (1) $\mathcal{O}_{X, \bar{x}}/I(\bar{x})$ is a regular local ring.
- (2) $\dim(\mathcal{O}_{X, \bar{x}}) = \dim(\mathcal{O}_{X, \bar{x}}/I(\bar{x})) + \text{rk}(\mathcal{M}_{X, \bar{x}}^{\text{gp}}/\mathcal{O}_{X, \bar{x}}^\times)$.

The log scheme $(X, \mathcal{M}_X)$ is called *log regular* if it is log regular at every point $x \in X$. For example, an fs log scheme $(X, \mathcal{M}_X)$ defined by a locally noetherian regular scheme X and a normal crossings divisor D is log regular. Conversely, for a log regular log scheme $(X, \mathcal{M}_X)$ whose underlying scheme X is regular, the log structure $\mathcal{M}_X$ is defined by a normal crossings divisor by [Kat94, Theorem 11.6] and [Ogu18, Chapter III, Theorem 1.11.6].

For a log regular log scheme $(X, \mathcal{M}_X)$, the condition (2) implies that the largest open subset U on which the log structure $\mathcal{M}_X$ is trivial is dense. Such an open subset U is called the *interior* of $(X, \mathcal{M}_X)$.

Proposition 3.4.2 (Kato). Let $(X, \mathcal{M}_X)$ be a locally noetherian fs log scheme.

- (1) The subset $\{x \in X \mid (X, \mathcal{M}_X) \text{ is log regular at } x\} \subset X$ is stable under generalization.
- (2) If $(X, \mathcal{M}_X)$ is log regular at $x \in X$, the scheme X is normal at x .
- (3) Suppose that $(X, \mathcal{M}_X)$ is log regular. Let U be the interior of $(X, \mathcal{M}_X)$. Then $\mathcal{M}_X$ is the subsheaf of $\mathcal{O}_X$ consisting of functions invertible on U .

Proof. (1) See [Kat94, Proposition 7.1].

(2) See [Kat94, Theorem 4.1].

(3) See [Kat94, Theorem 11.6]. □

Lemma 3.4.3. Let $(X, \mathcal{M}_X)$ be a log regular log scheme such that X is the spectrum of a noetherian strict local ring. Let x be the unique closed point of X . Fix a chart $P \rightarrow \mathcal{M}_X$ which is neat at x . Then, for a sharp fs monoid Q with a Kummer map $P \rightarrow Q$, the fs log scheme $(Y, \mathcal{M}_Y) := (X, \mathcal{M}_X) \otimes_{(\mathbb{Z}[P], P)^a} (\mathbb{Z}[Q], Q)^a$ is also log regular.

Proof. First, we claim that $R \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$ is a noetherian strict local ring. Since Y is finite over X , it suffices to prove that the special fiber of $Y \rightarrow X$ consists of a single point. Let k be the residue field of R . Since $P \rightarrow Q$ is local, sending $Q \setminus \{1\}$ to 0 defines a ring map $k \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q] \rightarrow k$. This ring map induces a ring map $(k \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q])_{\text{red}} \rightarrow k$, which is the inverse of a natural map $k \rightarrow (k \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q])_{\text{red}}$. Hence, we get $k \cong (k \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q])_{\text{red}}$, and so the special fiber of $Y \rightarrow X$ consists of a single point.

By Proposition 3.4.2 (1), it suffices to prove that $(Y, \mathcal{M}_Y)$ is log regular at the unique closed point y . Since $R \rightarrow R \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$ is finite and injective, we have the equation $\dim(R) = \dim(R \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q])$. The Kummerness of $P \rightarrow Q$ implies

$$\text{rk}(\mathcal{M}_{X,x}^{\text{gp}} / \mathcal{O}_{X,x}^\times) = \text{rk}(P^{\text{gp}}) = \text{rk}(Q^{\text{gp}}) = \text{rk}(\mathcal{M}_{Y,y}^{\text{gp}} / \mathcal{O}_{Y,y}^\times).$$

Since $(X, \mathcal{M}_X)$ is log regular, it is enough to show the equation $\dim(R/I(x)) = \dim((R \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q])/I(y))$. The ideal $I(x) \subset R$ (resp. $I(y) \subset R \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$) is generated by the image of $P \setminus \{1\}$ (resp. $Q \setminus \{1\}$). Sending $Q \setminus \{1\}$ to 0 defines a ring morphism $(R \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q])/I(y) \rightarrow R/I(x)$, and it can be checked directly that this map is the inverse of the natural map $R/I(x) \rightarrow (R \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q])/I(y)$. In particular, we have the equation $\dim(R/I(x)) = \dim((R \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q])/I(y))$. □

Lemma 3.4.4. Let $(X, \mathcal{M}_X)$ be a quasi-compact log regular fs log scheme with the interior $U \subset X$. Suppose that we are given a finitely generated monoid P and a chart $\alpha: P \rightarrow \mathcal{M}_X$. Then a natural monoid map $P^{\text{gp}} \oplus \mathcal{M}_X(X) \rightarrow \mathcal{O}_X(U)^\times$ is surjective.

Proof. Take a generator $\{p_1, \dots, p_m\}$ of P , and let $p := \prod_{i=1}^m p_i$. Then the vanishing locus of $\alpha(p) \in \mathcal{M}_X(X) \subset \mathcal{O}_X(X)$ coincides with $X \setminus U$. Let $f \in \mathcal{O}_U(U)^\times$. For the generic point η of each irreducible component E of $X \setminus U$ with $\text{codim}_X(E) = 1$, the local ring $\mathcal{O}_{X,\eta}$ is a discrete valuation ring by Proposition 3.4.2 (2). Take a sufficiently large integer $N \geq 1$ such that the valuation of $\alpha(p)^N f$ defined by the discrete valuation ring $\mathcal{O}_{X,\eta}$ is non-negative for every η . Then $\alpha(p)^N f \in \mathcal{O}_X(X)$, and so $\alpha(p)^N f \in \mathcal{M}_X(X)$ by Proposition 3.4.2 (3). This proves the assertion. □

3.4.2. One dimensional cases.

Lemma 3.4.5. Let $(X, \mathcal{M}_X)$ be a log regular fs log scheme such that $X = \text{Spec} R$ for a strict local discrete valuation ring R . Then the log structure $\mathcal{M}_X$ is either of the trivial one or the standard one (i.e. the log structure defined by the unique closed point).

Proof. Let x denote the unique closed point of X . The condition $\mathrm{rk}((\overline{\mathcal{M}_{X,x}})^{\mathrm{gp}}) = 1$ implies the monoid $\overline{\mathcal{M}_{X,x}}$ is isomorphic to either of 0 or $\mathbb{N}$. In the former case, the log structure $\mathcal{M}_X$ is trivial. In the latter case, we can take a chart $\mathbb{N} \rightarrow \mathcal{M}_{X,x}$ which is neat at x . Since $(X, \mathcal{M}_X)$ is log regular, $R/I(x)$ is a field. Hence, the monoid map $\mathbb{N} \rightarrow \mathcal{M}_{X,x} \rightarrow R$ has to map $1 \in \mathbb{N}$ to a uniformizer of R . Therefore, the log structure $\mathcal{M}_X$ is the standard log structure. $\square$

Proposition 3.4.6. Let $(X, \mathcal{M}_X)$ be a log regular fs log scheme such that $X = \mathrm{Spec} R$ for a henselian discrete valuation ring R . Let K be a fraction field of R . Suppose that G is a log finite group scheme and H_K is a log finite subgroup scheme of G_K . Then H_K uniquely extends to a log finite subgroup scheme of G .

Proof. By the limit argument and Galois descent, we may assume that R is strictly local. By Lemma 3.4.5, $\mathcal{M}_X$ is either of the trivial log structure or the standard log structure. In the former case, G is classical, and so the scheme theoretic closure of H_K in G is the unique extension of H_K .

We consider the latter case. We fix a uniformizer $\pi \in R$. By Lemma 3.2.15, there exists an integer $n \geq 1$ such that $G|_{(Y, \mathcal{M}_Y)}$ is classical, where $Y = \mathrm{Spec} R[\pi^{1/n}]$ and $\mathcal{M}_Y$ is the standard log structure. By the argument in the previous paragraph, $H|_{K(\pi^{1/n})}$ extends to a finite subgroup scheme H_Y of $G|_{(Y, \mathcal{M}_Y)}$ uniquely. The pullbacks of H_Y by two projections

$$(Y', \mathcal{M}_{Y'}) := (Y, \mathcal{M}_Y) \times_{(X, \mathcal{M}_X)} (Y, \mathcal{M}_Y) \rightarrow (Y, \mathcal{M}_Y)$$

coincide on $\mathrm{Spec}(K(\pi^{1/n}) \otimes_K K(\pi^{1/n}))$, which is dense open in

$$Y' \cong \mathrm{Spec}(R[\pi^{1/n}][X]/(X^n - 1)).$$

Hence, they coincide on the whole of Y' . Hence, H_Y descends to a weak log finite subgroup scheme H of G , which is the unique extension of $H|_K$. By Proposition 3.2.19, H is a log finite group scheme. $\square$

Remark 3.4.7. Proposition 3.4.6 does not hold without the assumption that $(X, \mathcal{M}_X)$ is log regular. Let R be a henselian discrete valuation ring over $\mathbb{F}_p$ with a uniformizer π and a fraction field K . Consider the prelog ring $(R, \mathbb{N})$ given by $\mathbb{N} \rightarrow R$ mapping 1 to π^p . Let $(X, \mathcal{M}_X)$ be its associated log scheme (which is not log regular). Let η be the generic point of X and x be the closed point of X . By abuse of notation, we simply write x for the log scheme $(\mathrm{Spec} k(x), \mathcal{M}_{\mathrm{Spec} k(x)})$. As in Example 3.2.14, we can associate to $q := \pi^p \in \mathbb{G}_{m, \log}(X, \mathcal{M}_X)$ an exact sequence of log finite group schemes over $(X, \mathcal{M}_X)$

$$(3.4.1) \quad 0 \rightarrow \mu_p \rightarrow G_q \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow 0.$$

The Kummer sequence

$$0 \rightarrow \mu_p \rightarrow \mathbb{G}_{m, \log} \rightarrow \mathbb{G}_{m, \log} \rightarrow 0$$

induces the following commutative diagram in which both rows are exact:

$$\begin{array}{ccccc} \pi^{p\mathbb{Z}} R^\times & \xrightarrow{(-)^p} & \pi^{p\mathbb{Z}} R^\times & \longrightarrow & \mathrm{Ext}_{(X, \mathcal{M}_X)_{\mathrm{kff}}}^1(\mathbb{Z}/p\mathbb{Z}, \mu_p) \\ \downarrow & & \downarrow & \searrow 0 & \downarrow \\ K^\times & \xrightarrow{(-)^p} & K^\times & \longrightarrow & \mathrm{Ext}_{\eta_{\mathrm{fppf}}}^1(\mathbb{Z}/p\mathbb{Z}, \mu_p). \end{array}$$

By this diagram, we see that the generic fiber of the exact sequence (3.4.1) splits. We claim that the unique section $(\mathbb{Z}/p\mathbb{Z})_\eta \hookrightarrow (G_q)_\eta$ does not extend to a log finite subgroup scheme of G_q . Assume that there exists an extension $H \subset G_q$. Since the order of H

is a prime number p , H is classical by Lemma 3.2.25. Since the surjection $G_q \twoheadrightarrow \mathbb{Z}/p\mathbb{Z}$ admits no section, the composite $H \hookrightarrow G_q \twoheadrightarrow \mathbb{Z}/p\mathbb{Z}$ is not an isomorphism. Hence, the composite $H|_x \hookrightarrow G_q|_x \twoheadrightarrow (\mathbb{Z}/p\mathbb{Z})_x$ is not an isomorphism, and so it is trivial. Therefore, we obtain $H|_x \cong \mu_{p,x}$ by (3.4.1). In particular, the multiplicative rank of H jumps under the specialization. This is contradiction.

3.4.3. Extension properties on log regular schemes.

Proposition 3.4.8. Let $(X, \mathcal{M}_X)$ be a log regular fs log scheme and U be a dense open subset of X . Then the restriction functors

$$\begin{aligned} \mathrm{Vect}_{\mathrm{kfl}}(X, \mathcal{M}_X) &\rightarrow \mathrm{Vect}_{\mathrm{kfl}}(U, \mathcal{M}_U) \\ \mathrm{wFin}(X, \mathcal{M}_X) &\rightarrow \mathrm{wFin}(U, \mathcal{M}_U) \end{aligned}$$

are faithful.

Proof. Due to Proposition 3.2.18, it is enough to prove the faithfulness of the former functor. Let $\mathcal{E}_1, \mathcal{E}_2$ be kfl vector bundles on $(X, \mathcal{M}_X)$. It suffices to show that the map

$$\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2) \rightarrow \mathrm{Hom}(\mathcal{E}_1|_U, \mathcal{E}_2|_U)$$

is bijective. By the limit argument (Proposition 3.2.8), we may assume that X is the spectrum of a strict local ring. Let x be the unique closed point of X . Fix a chart $P \rightarrow \mathcal{M}_X$ which is neat at x .

By Lemma 3.2.7, there exist an integer $n \geq 1$ such that $\mathcal{E}_1|_{(Y, \mathcal{M}_Y)}$ and $\mathcal{E}_2|_{(Y, \mathcal{M}_Y)}$ are classical, where we put $(Y, \mathcal{M}_Y) := (X, \mathcal{M}_X) \otimes_{(\mathbb{Z}[P], P)^a} (\mathbb{Z}[P^{1/n}], P^{1/n})^a$. Since $(Y, \mathcal{M}_Y)$ is log regular by Lemma 3.4.3, it follows from Lemma 3.4.2 (2) that Y is normal. Let V be the preimage of U by $Y \rightarrow X$. Since $Y \rightarrow X$ is a finite dominant morphism of normal schemes, the going down property holds for $Y \rightarrow X$ by [SP25, Proposition 00H8]. Hence, V is dense in Y , and so the restriction map

$$\mathrm{Hom}(\mathcal{E}_1|_{(Y, \mathcal{M}_Y)}, \mathcal{E}_2|_{(Y, \mathcal{M}_Y)}) \rightarrow \mathrm{Hom}(\mathcal{E}_1|_{(V, \mathcal{M}_V)}, \mathcal{E}_2|_{(V, \mathcal{M}_V)})$$

is injective. Therefore, the restriction map

$$\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2) \rightarrow \mathrm{Hom}(\mathcal{E}_1|_U, \mathcal{E}_2|_U)$$

is injective by kfl descent. □

Lemma 3.4.9. Let $f: X \rightarrow Y$ be a flat morphism of from a (not necessarily locally noetherian) scheme X to a locally noetherian normal scheme Y . Let U be a dense open subset of Y containing all points of codimension 1. Then the restriction functors

$$\begin{aligned} \mathrm{Vect}(X) &\rightarrow \mathrm{Vect}(f^{-1}(U)) \\ \mathrm{Fin}(X) &\rightarrow \mathrm{Fin}(f^{-1}(U)) \end{aligned}$$

are fully faithful.

Proof. It is enough to prove the fully faithfulness of the former functor. By taking internal homomorphisms, the problem is reduced to showing that, for a vector bundle $\mathcal{E}$ on X , the restriction map $\Gamma(X, \mathcal{E}) \rightarrow \Gamma(f^{-1}(U), \mathcal{E})$ is an isomorphism. Let $i: f^{-1}(U) \hookrightarrow X$ and $j: U \hookrightarrow Y$ be natural open immersions. Then we have isomorphisms of $\mathcal{O}_X$ -modules

$$i_* i^* \mathcal{E} \cong \mathcal{E} \otimes i_* \mathcal{O}_{f^{-1}(U)} \cong \mathcal{E} \otimes f^* j_* \mathcal{O}_U \cong \mathcal{E} \otimes f^* \mathcal{O}_Y \cong \mathcal{E},$$

where the first isomorphism is the projection formula, the second one is the flat base change, and the third one follows from the assumption that U is an open subset of a

locally noetherian normal scheme Y containing all points of codimension 1. Taking global sections on both sides, we obtain the statement. $\square$

Proposition 3.4.10. Let $f: (X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y)$ be a strict flat morphism from a (not necessarily locally noetherian) fs log scheme $(X, \mathcal{M}_X)$ to a log regular log scheme $(Y, \mathcal{M}_Y)$. Let U be a dense open subset of Y containing all points of codimension 1. Then the restriction functors

$$\begin{aligned} \mathrm{Vect}_{\mathrm{kfl}}(X, \mathcal{M}_X) &\rightarrow \mathrm{Vect}_{\mathrm{kfl}}(f^{-1}(U), \mathcal{M}_{f^{-1}(U)}) \\ \mathrm{wFin}(X, \mathcal{M}_X) &\rightarrow \mathrm{wFin}(f^{-1}(U), \mathcal{M}_{f^{-1}(U)}) \end{aligned}$$

are fully faithful.

Proof. Due to Proposition 3.2.18, it is enough to prove that the fully faithfulness of the former functor. Let $\mathcal{E}_1, \mathcal{E}_2$ be kfl vector bundles on $(X, \mathcal{M}_X)$. We shall prove that the restriction map

$$\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2) \rightarrow \mathrm{Hom}(\mathcal{E}_1|_{f^{-1}(U)}, \mathcal{E}_2|_{f^{-1}(U)})$$

is an isomorphism. By the limit argument (Proposition 3.2.8), we may assume that Y is the spectrum of a strict local ring. Let $y \in Y$ be a unique closed point. Take a chart $P \rightarrow \mathcal{M}_Y$ which is neat at y . By Lemma 3.2.7, we can take an integer $n \geq 1$ such that the pullback of $\mathcal{E}_i$ to $(X', \mathcal{M}_{X'}) := (X, \mathcal{M}_X) \otimes_{(\mathbb{Z}[P], P)^a} (\mathbb{Z}[P^{1/n}], P^{1/n})^a$ is classical for $i = 1, 2$. Let $(Y', \mathcal{M}_{Y'}) := (Y, \mathcal{M}_Y) \otimes_{(\mathbb{Z}[P], P)^a} (\mathbb{Z}[P^{1/n}], P^{1/n})^a$. By Lemma 3.4.3, $(Y', \mathcal{M}_{Y'})$ is log regular, and so Y' is normal by Proposition 3.4.2 (2). Let $(X'', \mathcal{M}_{X''})$ denote the self-fiber product of $(X', \mathcal{M}_{X'})$ over $(X, \mathcal{M}_X)$. Let V' (resp. V'') (resp. U') be the preimage of U in X' (resp. X'') (resp. Y'). Since $Y' \rightarrow Y$ is a finite dominant morphism of normal schemes, the going down property holds for $Y' \rightarrow Y$ by [SP25, Proposition 00H8]. Hence, U' is a dense open subset of Y' containing all points of codimension 1. Applying Lemma 3.4.9 to flat morphisms of schemes $X' \rightarrow Y'$ and $X'' \rightarrow Y'$ and the open subset $U' \subset Y'$, we conclude that the restriction maps

$$\begin{aligned} \mathrm{Hom}(\mathcal{E}_1|_{(X', \mathcal{M}_{X'})}, \mathcal{E}_2|_{(X', \mathcal{M}_{X'})}) &\rightarrow \mathrm{Hom}(\mathcal{E}_1|_{(V', \mathcal{M}_{V'})}, \mathcal{E}_2|_{(V', \mathcal{M}_{V'})}), \\ \mathrm{Hom}(\mathcal{E}_1|_{(X'', \mathcal{M}_{X''})}, \mathcal{E}_2|_{(X'', \mathcal{M}_{X''})}) &\rightarrow \mathrm{Hom}(\mathcal{E}_1|_{(V'', \mathcal{M}_{V''})}, \mathcal{E}_2|_{(V'', \mathcal{M}_{V''})}) \end{aligned}$$

are isomorphisms. Therefore, the claim follows from kfl descent. $\square$

The following proposition is a log version of the theorem of Tate and de Jong ([Tat67, Theorem 4] and [dJ98, Corollary 1.2]). This is proved in the case where X is the spectrum of a discrete valuation ring by Bertapelle-Wang-Zhao in [BWZ24, Theorem 5.19]. We prove the following proposition by using the reduction to 1-dimensional cases.

Proposition 3.4.11. Let $(X, \mathcal{M}_X)$ be a log regular fs log scheme and U be a dense open subset of X . Then the restriction functor

$$\mathrm{BT}(X, \mathcal{M}_X) \rightarrow \mathrm{BT}(U, \mathcal{M}_U)$$

is fully faithful.

Proof. Let G and H be log p -divisible groups over $(X, \mathcal{M}_X)$. We shall prove the natural map

$$\mathrm{Hom}(G, H) \rightarrow \mathrm{Hom}(G|_U, H|_U)$$

is an isomorphism. The injectivity follows from Lemma 3.4.8. To show the surjectivity, we take a homomorphism $f_U: G|_U \rightarrow H|_U$. Fix an integer $n \geq 1$. By [BWZ24, Theorem 5.19] and Lemma 3.4.5, there exists an open subset V of S containing U and all points

of codimension 1 such that the homomorphism $f_{U,n}: G[p^n]|_U \rightarrow H[p^n]|_U$ induced from f_U extends to a homomorphism $f_{V,n}: G[p^n]|_V \rightarrow H[p^n]|_V$. Furthermore, $f_{V,n}$ extends to a homomorphism $f_n: G[p^n] \rightarrow H[p^n]$ by Proposition 3.4.10. By Lemma 3.4.8, f_n is the unique extension of $f_{U,n}$, and so, when n varies, the system $\{f_n\}_{n \geq 1}$ gives a homomorphism $f: G \rightarrow H$ restricting to f_U . $\square$

Proposition 3.4.12. Let $(X, \mathcal{M}_X)$ be a log regular fs log scheme and U be a dense open subset of X . Let $f: G_1 \rightarrow G_2$ be a homomorphism of log p -divisible groups over $(X, \mathcal{M}_X)$. Suppose that $f|_U: G_1|_U \rightarrow G_2|_U$ is an isogeny. Then f itself is an isogeny.

Proof. We may assume that X is quasi-compact. Take an integer $d \geq 1$ such that $\text{Ker}(f|_U)$ is killed by p^d . Since $f|_U$ is surjective, there exists a unique homomorphism $g_U: G_2|_U \rightarrow G_1|_U$ such that both of $f|_U \circ g_U$ and $g_U \circ f|_U$ are the multiplication by p^d . By Proposition 3.4.11, g_U uniquely extends to a homomorphism $g: G_2 \rightarrow G_1$, and both of fg and gf are the multiplication by p^d . In particular, $\text{Ker}(f)$ is killed by p^d . Therefore, it follows from Proposition 3.2.35 that f is an isogeny. $\square$

3.5. Slope filtrations of log p -divisible groups

The goal of this section is to prove Theorem 3.5.4. We start with some technical lemmas.

Lemma 3.5.1. Let $f: X \rightarrow Y$ be a faithfully flat morphism of locally noetherian normal schemes. Let U be a dense open subset of Y containing all points on Y of codimension 1, and $V := f^{-1}(U)$. Let G_U be a finite locally free group scheme over U . Suppose that $G_V := G_U|_V$ extends to a finite locally free group scheme G_X over X . Then G_U itself uniquely extends to a finite locally free group scheme over Y .

Proof. Let $j: U \hookrightarrow Y$ and $j': V \hookrightarrow X$ denote the natural open immersions. Let $f_U: V \rightarrow U$ be the restriction of f . For a finite locally free group scheme H over a scheme S , we write $\mathcal{A}_H$ for the Hopf algebra object in $\text{Vect}(S)$ corresponding to H (cf. Proposition 3.2.18). By the flat base change theorem, there are isomorphisms

$$(3.5.1) \quad f^* j_* \mathcal{A}_{G_U} \cong j'_* (f_U)^* \mathcal{A}_{G_U} \cong j'_* \mathcal{A}_{G_V}.$$

Since f has the going down property by the flatness of f , the open subset V contains all points on Y of codimension 1. Hence, there are isomorphisms

$$(3.5.2) \quad j'_* \mathcal{A}_{G_V} \cong j'_* j'^* \mathcal{A}_{G_X} \cong \mathcal{A}_{G_X},$$

by the normality of X . It follows from (3.5.1) and (3.5.2) that $f^* j_* \mathcal{A}_{G_U}$ is a vector bundle on X , and so $j_* \mathcal{A}_{G_U}$ is also a vector bundle on Y . Since the restriction functor $\text{Vect}(Y) \rightarrow \text{Vect}(U)$ is fully faithful, the Hopf algebra structure on $\mathcal{A}_{G_U}$ uniquely induces a Hopf algebra structure on $j_* \mathcal{A}_{G_U}$, which corresponds to the desired finite locally free group scheme over Y extending G_U . $\square$

Lemma 3.5.2. Let X be a normal scheme and U be a dense open subset of X containing all points of codimension 1. Let $G' \rightarrow G \rightarrow G''$ be a sequence of finite locally free group schemes over X . Suppose that G'' is finite étale over X , and that the sequence

$$0 \rightarrow G'|_U \rightarrow G|_U \rightarrow G''|_U \rightarrow 0$$

is exact. Then the sequence

$$0 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 0$$

is also exact.

Proof. Since G'' is finite étale over X , the morphism $G \rightarrow G''$ is finite flat. The image of $G \rightarrow G''$ is closed and contains a dense open subset $G''_U = G \times_X U$, and so $G \rightarrow G''$ is faithfully flat. What remains to be proved is that the composite $G' \rightarrow G \rightarrow G''$ is a zero map and that the induced homomorphism $G' \rightarrow \text{Ker}(G \rightarrow G'')$ is an isomorphism. Both of them follow from the fully faithfulness of the restriction functor $\text{Fin}(X) \rightarrow \text{Fin}(U)$ (cf. Proposition 3.4.10). $\square$

Proposition 3.5.3. Let $(X, \mathcal{M}_X)$ be a log regular fs log scheme over $\mathbb{F}_p$ with X being regular, and let U be a dense open subset of X containing all points of codimension 1. Let G_U be a completely slope divisible log p -divisible group over $(U, \mathcal{M}_U)$. Then G_U uniquely extends to a log p -divisible group over $(X, \mathcal{M}_X)$.

Proof. Since G_U is completely slope divisible, there exists an exact sequence

$$0 \rightarrow G_U^0 \rightarrow G_U \rightarrow G_U^{\text{ét}} \rightarrow 0$$

as in Proposition 3.3.4. By [Zin01a, Proposition 14], G_U^0 (resp. $G_U^{\text{ét}}$) extends to a formal Lie group H_1 (resp. an étale p -divisible group H_2) over X .

Note that $\mathcal{M}_{X,\bar{x}}/\mathcal{O}_{X,\bar{x}}^\times$ is a free monoid for all $x \in X$ by [Ogu18, Chapter III, Theorem 1.11.6]. It is enough to prove the following two claims:

- (1) $G_U[p^n]$ uniquely extends to a log finite group scheme G_n over $(X, \mathcal{M}_X)$ for each $n \geq 1$;
- (2) $G := \varinjlim_{n \geq 1} G_n$ is a log p -divisible group over $(X, \mathcal{M}_X)$.

Hence, the limit argument (Proposition 3.2.16) allows us to assume that $X = \text{Spec} R$ for a strict local ring R . Let $x \in X$ be the unique closed point. Take a chart $\mathbb{N}^r \rightarrow \mathcal{M}_{X,x}$ which is neat at x . Let $\alpha: \mathbb{N}^r \rightarrow R$ denote the composite map $\mathbb{N}^r \rightarrow \mathcal{M}_{X,x} \rightarrow R$. By the log regularity of $(X, \mathcal{M}_X)$, the sequence $(\alpha(e_1), \dots, \alpha(e_r))$ can be extended to a system of parameter of R , where e_i is the i -th standard basis of $\mathbb{N}^r$ for $1 \leq i \leq r$.

We shall prove the claim (1) for a fixed $n \geq 1$. By Lemma 3.2.15, there exists an integer $m \geq 1$ such that $G_U[p^n]|_V$ is classical, where we set $(Y, \mathcal{M}_Y) := (X, \mathcal{M}_X) \otimes_{(\mathbb{Z}[\mathbb{N}^r], \mathbb{N}^r)^a} (\mathbb{Z}[\frac{1}{m}\mathbb{N}^r], \frac{1}{m}\mathbb{N}^r)^a$ and V to be the preimage of U by $Y \rightarrow X$. Since the sequence $(\alpha(e_1), \dots, \alpha(e_r))$ can be extended to a system of parameter of R , the scheme Y is regular. We have an exact sequence of finite locally free group schemes over V

$$(3.5.3) \quad 0 \rightarrow H_1[p^n]|_V \rightarrow G_U[p^n]|_V \rightarrow H_2[p^n]|_V \rightarrow 0.$$

By the argument in [Zin01a, page 7] (or the fifth paragraph of the proof of [Zin01a, Proposition 14]), this sequence splits after taking the pullback by the k -th power of the Frobenius morphism $F_V^k: V \rightarrow V$ (which is a fpqc covering by the regularity of V) for some integer $k \geq 1$. In particular, the pullback of $G_U[p^n]|_V$ by F_V^k extends to a finite locally free group scheme over Y . Hence, by Lemma 3.5.1, $G_U[p^n]|_V$ itself also extends to a finite locally free group scheme $G_{n,Y}$ over Y . By Lemma 3.5.2, the exact sequence (3.5.3) uniquely extends to an exact sequence of finite locally free group schemes over Y

$$(3.5.4) \quad 0 \rightarrow H_1[p^n]|_{(Y, \mathcal{M}_Y)} \rightarrow G_{n,Y} \rightarrow H_2[p^n]|_{(Y, \mathcal{M}_Y)} \rightarrow 0.$$

Since projection morphisms $(Y, \mathcal{M}_Y) \times_{(X, \mathcal{M}_X)} (Y, \mathcal{M}_Y) \rightarrow (Y, \mathcal{M}_Y)$ are strict finite free, the descent datum on $G_U[p^n]|_V$ over $(V, \mathcal{M}_V) \times_{(U, \mathcal{M}_U)} (V, \mathcal{M}_V)$ uniquely extends to the descent datum on $G_{n,Y}$ over $(Y, \mathcal{M}_Y) \times_{(X, \mathcal{M}_X)} (Y, \mathcal{M}_Y)$ by Lemma 3.4.9. Therefore, $G_{n,Y}$ descends to a weak log finite group scheme G_n over $(X, \mathcal{M}_X)$ by kfl descent. Since the exact sequence (3.5.4) descends to an exact sequence

$$(3.5.5) \quad 0 \rightarrow H_1[p^n] \rightarrow G_n \rightarrow H_2[p^n] \rightarrow 0,$$

G_n is a log finite group scheme by [Kat23, Proposition 2.3]. This proves the claim (1).

By construction, G_n is killed by p^n for any integer $n \geq 1$. Due to Proposition 3.4.10, the sequence $G_U[p] \hookrightarrow G_U[p^2] \hookrightarrow \dots$ uniquely extends to a sequence

$$G_1 \rightarrow G_2 \rightarrow \dots$$

To prove the claim (2), it is enough to prove that the following claims are true:

- $G_n \rightarrow G_{n+1}$ is an injection for any integer $n \geq 1$;
- the sequence

$$0 \rightarrow G_n \rightarrow G_{n+m} \xrightarrow{\times p^n} G_m \rightarrow 0$$

is exact.

These claims follow from the exact sequence (3.5.5) and the fact that similar properties hold for systems $\{H_1[p^n]\}$ and $\{H_2[p^n]\}$. $\square$

Theorem 3.5.4. Let $(X, \mathcal{M}_X)$ be a locally noetherian log regular fs log scheme over $\mathbb{F}_p$ and U be a dense open subset of X . Let G be a log p -divisible group over $(X, \mathcal{M}_X)$. Suppose that we are given a completely slope divisible log p -divisible group H_U over $(U, \mathcal{M}_U)$ and an isogeny $f_U: G|_U \rightarrow H_U$. Then there exist a completely slope divisible log p -divisible group H over $(X, \mathcal{M}_X)$ and an isogeny $f: G \rightarrow H$ extending f_U .

Proof. By the limit argument (Proposition 3.2.16), we may assume that the X is the spectrum of a strict local ring with the unique closed point x . Fix a chart $P \rightarrow \mathcal{M}_X$ which is neat at x and an integer $d \geq 1$ with $\text{Ker}(f_U) \subset G[p^d]|_U$. By Lemma 3.2.15, there exists an integer $n \geq 1$ such that $G[p^d]|_{(Y, \mathcal{M}_Y)}$ is classical, where we set $(Y, \mathcal{M}_Y) := (X, \mathcal{M}_X) \times_{(\mathbb{Z}[P], P)^a} (\mathbb{Z}[P^{1/n}], P^{1/n})^a$. Let V be the preimage of U by $Y \rightarrow X$.

Then, Nizioł's desingularization theorem [Niz06, Theorem 5.10] gives a log regular fs log scheme $(Y', \mathcal{M}_{Y'})$ with Y' being regular and a proper birational morphism $\pi: (Y', \mathcal{M}_{Y'}) \rightarrow (Y, \mathcal{M}_Y)$. By Proposition 3.5.3, Proposition 3.4.11, and Proposition 3.4.12, there exists a unique isogeny $f': G|_{(Y', \mathcal{M}_{Y'})} \rightarrow H_{Y'}$ extending $f|_{\pi^{-1}(V)}$.

We shall prove that f' descends to $(Y, \mathcal{M}_Y)$ by using the method of Oort-Zink; see [OZ02, Proposition 2.7]. Let $\mathcal{M} \rightarrow Y$ be the moduli (non-log) scheme of isogenies from $G|_{(Y, \mathcal{M}_Y)}$ whose kernel is contained in $G[p^d]|_{(Y, \mathcal{M}_Y)}$. The isogeny f' induces a morphism $\varphi: Y' \rightarrow \mathcal{M}$ of schemes. It suffices to prove that φ factors as $Y' \rightarrow Y \rightarrow \mathcal{M}$. Since $\mathcal{O}_Y \cong \pi_* \mathcal{O}_{Y'}$ by the normality of Y , it is enough to show that, for $y \in Y$, the morphism φ maps $\pi^{-1}(y)$ to a single point. This follows from Lemma 3.3.8 and the connectivity of $\pi^{-1}(y)$.

As a result, we obtain a log p -divisible group H_Y over $(Y, \mathcal{M}_Y)$ with $(H_Y)|_{(Y', \mathcal{M}_{Y'})} \cong H_{Y'}$ and an isogeny $f_Y: G|_{(Y, \mathcal{M}_Y)} \rightarrow H_Y$ with $(f_Y)|_{(Y', \mathcal{M}_{Y'})} = f'$. Since π is birational, f_Y is the extension of $(f_U)|_{(V, \mathcal{M}_V)}$ by Lemma 3.4.8. Hence, by Lemma 3.4.9, the two pullbacks of $\text{Ker}(f_Y)$ by two projection morphisms $(Y, \mathcal{M}_Y) \times_{(X, \mathcal{M}_X)} (Y, \mathcal{M}_Y) \rightarrow (Y, \mathcal{M}_Y)$ are equal as subobjects of $G[p^d]|_{(Y, \mathcal{M}_Y) \times_{(X, \mathcal{M}_X)} (Y, \mathcal{M}_Y) \rightarrow (Y, \mathcal{M}_Y)}$. Therefore, by kfl descent, $\text{Ker}(f_Y)$ descends to a log finite subgroup of $G[p^d]$. Let $H := G/\text{Ker}(f_Y)$ and $f: G \rightarrow H$ be the natural surjection. Then the isogeny f restricts to the given isogeny f_U . By Corollary 3.3.6, H is completely slope divisible. This proves the statement. $\square$

3.A. Limit arguments for log schemes

In this appendix, we shall prove some fundamental results on limits of log schemes which are well-known to experts. We note that Lemma 3.A.2 for fine log schemes can be deduced

easily from the fact that Olsson's stack $\mathcal{L}og_S$ is of finite presentation ([Ols03, Theorem 1.1]). Here, we shall give a more direct proof.

Let $\{(X_i, \mathcal{M}_{X_i})\}_{i \in I}$ be a cofiltered system of coherent log schemes. In this section, we say that $\{(X_i, \mathcal{M}_{X_i})\}$ satisfies the *condition* $(*)$ if the following conditions are satisfied.

- $(*)$ $\left\{ \begin{array}{l} (1) \text{ The scheme } X_i \text{ is quasi-compact and quasi-separated for any } i \in I. \\ (2) \text{ Every transition morphism } (X_i, \mathcal{M}_{X_i}) \rightarrow (X_j, \mathcal{M}_{X_j}) \text{ is affine and strict.} \end{array} \right.$

When $\{(X_i, \mathcal{M}_{X_i})\}$ satisfies the condition $(*)$, there exists a coherent log scheme $(X, \mathcal{M}_X)$ which is a limit of $\{(X_i, \mathcal{M}_{X_i})\}$ in the category of log schemes, and one can construct $(X, \mathcal{M}_X)$ explicitly as follows. The underlying scheme X is a limit of schemes $\varprojlim X_i$. The log structure $\mathcal{M}_X$ is the pullback log structure of $\mathcal{M}_{X_i}$ by the natural projection morphism $X \rightarrow X_i$ for some i , which is independent of the choice of i by the assumption $(*)$. Obviously, when $(X_i, \mathcal{M}_{X_i})$ is fine (resp. fs) for each $i \in I$, the log scheme $(X, \mathcal{M}_X)$ is a limit of $\{(X_i, \mathcal{M}_{X_i})\}$ in the category of fine (resp. fs) log schemes.

Lemma 3.A.1. Let $\{(X_i, \mathcal{M}_{X_i})\}$ be a cofiltered system of coherent log schemes satisfying the condition $(*)$. We put $(X, \mathcal{M}_X) := \varprojlim_{i \in I} (X_i, \mathcal{M}_{X_i})$. Then the canonical morphism

$$\varinjlim_{i \in I} \Gamma(X_i, \mathcal{M}_{X_i}) \rightarrow \Gamma(X, \mathcal{M}_X)$$

is an isomorphism.

Proof. Let $p_i: X \rightarrow X_i$ be the projection morphism. We have the following diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \varinjlim p_i^{-1} \mathcal{O}_{X_i}^\times & \longrightarrow & \varinjlim p_i^{-1} \mathcal{M}_{X_i} & \longrightarrow & \varinjlim p_i^{-1} (\mathcal{M}_{X_i} / \mathcal{O}_{X_i}^\times) \\ & & \downarrow \simeq & & \downarrow & & \downarrow \simeq \\ 1 & \longrightarrow & \mathcal{O}_X^\times & \longrightarrow & \mathcal{M}_X & \longrightarrow & \mathcal{M}_X / \mathcal{O}_X^\times \end{array}$$

where the both rows are exact. Here, note that $p_i^{-1} \mathcal{M}_{X_i}$ is not the pullback log structure but the inverse image of $\mathcal{M}_{X_i}$ by p_i as a sheaf. Hence, the canonical morphism $\varinjlim p_i^{-1} \mathcal{M}_{X_i} \rightarrow \mathcal{M}_X$ is an isomorphism.

For a quasi-compact and quasi-separated étale morphism $U \rightarrow X$, we define a monoid $\mathcal{M}'_X(U)$ as follows. For some $j \in I$, there exists a quasi-compact and quasi-separated étale morphism $U_j \rightarrow X_j$ with an isomorphism $U \cong U_j \times_{X_j} X$ over X . Here, we set

$$\mathcal{M}'_X(U) := \varinjlim_{i \geq j} \mathcal{M}_{X_i}(U_j \times_{X_j} X_i).$$

Clearly, this definition is independent of the choice of j . Then $U \mapsto \mathcal{M}'_X(U)$ defines a presheaf $\mathcal{M}'_X$ on the category of quasi-compact and quasi-separated étale morphisms to X . Since $\mathcal{M}'_X$ satisfies the sheaf condition for quasi-compact and quasi-separated étale coverings, $\mathcal{M}'_X$ extends to an étale sheaf on X which is also denoted by $\mathcal{M}'_X$. By construction, $\mathcal{M}'_X$ satisfies the same universal mapping property as $\varinjlim p_i^{-1} \mathcal{M}_{X_i}$. Hence, the natural morphism $\varinjlim p_i^{-1} \mathcal{M}_{X_i} \rightarrow \mathcal{M}'_X$ is an isomorphism. The isomorphisms

$$\mathcal{M}_X \cong \varinjlim p_i^{-1} \mathcal{M}_{X_i} \cong \mathcal{M}'_X$$

give the desired isomorphism $\varinjlim \Gamma(X_i, \mathcal{M}_{X_i}) \cong \Gamma(X, \mathcal{M}_X)$. $\square$

Lemma 3.A.2. Let $\{X_i\}_{i \in I}$ be a cofiltered system of quasi-compact and quasi-separated schemes in which every transition morphism is affine. We put $X := \varprojlim_{i \in I} X_i$.

- (1) Suppose that the cofiltered category I has a final object 0. Let $\mathcal{M}_0, \mathcal{N}_0$ be log structures on X_0 . Let $\mathcal{M}_i$ (resp. $\mathcal{N}_i$) denote the pullback log structure of $\mathcal{M}_0$ (resp. $\mathcal{N}_0$) to X_i , and let $\mathcal{M}$ (resp. $\mathcal{N}$) denote the pullback log structure of $\mathcal{M}_0$ (resp. $\mathcal{N}_0$) to X . Suppose that $\mathcal{M}_0$ is coherent. Then the map

$$\varinjlim_{i \in I} \mathrm{Hom}_{X_i}(\mathcal{M}_i, \mathcal{N}_i) \rightarrow \mathrm{Hom}_X(\mathcal{M}, \mathcal{N})$$

is bijective, where $\mathrm{Hom}_{X_i}(\mathcal{M}_i, \mathcal{N}_i)$ (resp. $\mathrm{Hom}_X(\mathcal{M}, \mathcal{N})$) is the set of morphisms of log structures $\mathcal{M}_i \rightarrow \mathcal{N}_i$ (resp. $\mathcal{M} \rightarrow \mathcal{N}$).

- (2) Let $\mathcal{M}$ be a coherent (resp. fine) (resp. fs) log structure on X . Then there exist $i \in I$ and a coherent (resp. fine) (resp. fs) log structure $\mathcal{M}_i$ on X_i such that $\mathcal{M}$ is isomorphic to the pullback log structure of $\mathcal{M}_i$.

Proof. (1) By working étale locally on X_0 , we may assume that there are a finitely generated monoid P and a chart $P \rightarrow \mathcal{M}_0$. Then the map in the statement factors as follows:

$$\begin{aligned} \varinjlim \mathrm{Hom}_{X_i}(\mathcal{M}_i, \mathcal{N}_i) &\xrightarrow{\sim} \varinjlim \mathrm{Hom}(P, \Gamma(X_i, \mathcal{N}_i)) \\ &\rightarrow \mathrm{Hom}(P, \varinjlim \Gamma(X_i, \mathcal{N}_i)) \\ &\rightarrow \mathrm{Hom}(P, \Gamma(X, \mathcal{N})) \\ &\xrightarrow{\sim} \mathrm{Hom}_X(\mathcal{M}, \mathcal{N}). \end{aligned}$$

Here, the second morphism is bijective by the fact that P is finitely presented, and the third morphism is bijective by 3.A.1. This proves (1).

(2) We prove only the assertion for coherent log structures because other assertion follows from the same argument. First, we consider the case that $\mathcal{M}$ admits a chart $P \rightarrow \mathcal{M}$ for a finitely generated monoid P . Since P is finitely presented and $\varinjlim \Gamma(X_i, \mathcal{O}_{X_i})$ is isomorphic to $\Gamma(X, \mathcal{O}_X)$, the morphism $P \rightarrow \mathcal{M} \rightarrow \mathcal{O}_X$ descends to a morphism $P \rightarrow \mathcal{O}_{X_i}$ for some $i \in I$. Then the associated log structure to the morphism $P \rightarrow \mathcal{O}_{X_i}$ is a desired one.

We consider the general case. Take a quasi-compact and quasi-separated étale covering $\pi: U \rightarrow X$ such that $\mathcal{M}_U := \pi^* \mathcal{M}$ admits a chart. For some $i \in I$, the morphism π descends to an étale covering $\pi_i: U_i \rightarrow X_i$ and $\mathcal{M}_U$ descends to a coherent log structure $\mathcal{M}_{U_i}$ on U_i . Let $p_k: U \times_X U \rightarrow U$ and $p_{i,k}: U_i \times_{X_i} U_i \rightarrow U_i$ be natural projection morphisms for $k = 1, 2$. Due to the assertion (1), we may assume that the isomorphism $p_1^* \mathcal{M}_U \cong p_2^* \mathcal{M}_U$ descends to an isomorphism $p_{i,1}^* \mathcal{M}_{U_i} \cong p_{i,2}^* \mathcal{M}_{U_i}$ satisfying the cocycle condition over $U_i \times_{X_i} U_i \times_{X_i} U_i$ by replacing i with a bigger one. This descent datum gives a coherent log structure $\mathcal{M}_i$ on X_i such that the pullback log structure of $\mathcal{M}_i$ to X is isomorphic to $\mathcal{M}$. $\square$

Proposition 3.A.3. Let $\{(X_i, \mathcal{M}_{X_i})\}_{i \in I}$ be a cofiltered system of coherent log schemes satisfying the condition (*). We put $(X, \mathcal{M}_X) := \varprojlim_{i \in I} (X_i, \mathcal{M}_{X_i})$.

- (1) Suppose that the cofiltered category I has a final object 0. Let $(Y_0, \mathcal{M}_{Y_0})$ and $(Z_0, \mathcal{M}_{Z_0})$ be coherent log schemes over $(X_0, \mathcal{M}_{X_0})$. Assume that Z_0 is of finite

presentation over X_0 . We set

$$\begin{aligned} (Y_i, \mathcal{M}_{Y_i}) &:= (Y_0, \mathcal{M}_{Y_0}) \times_{(X_0, \mathcal{M}_{X_0})} (X_i, \mathcal{M}_{X_i}), \\ (Y, \mathcal{M}_Y) &:= (Y_0, \mathcal{M}_{Y_0}) \times_{(X_0, \mathcal{M}_{X_0})} (X, \mathcal{M}_X), \\ (Z_i, \mathcal{M}_{Z_i}) &:= (Z_0, \mathcal{M}_{Z_0}) \times_{(X_0, \mathcal{M}_{X_0})} (X_i, \mathcal{M}_{X_i}), \\ (Z, \mathcal{M}_Z) &:= (Z_0, \mathcal{M}_{Z_0}) \times_{(X_0, \mathcal{M}_{X_0})} (X, \mathcal{M}_X). \end{aligned}$$

Then the natural map

$$\varinjlim_{i \in I} \text{Mor}_{(X_i, \mathcal{M}_{X_i})}((Y_i, \mathcal{M}_{Y_i}), (Z_i, \mathcal{M}_{Z_i})) \rightarrow \text{Mor}_{(X, \mathcal{M}_X)}((Y, \mathcal{M}_Y), (Z, \mathcal{M}_Z))$$

is bijective.

- (2) Let $(Y, \mathcal{M}_Y)$ be a coherent (resp. fine) (resp. fs) log scheme of finite presentation over $(X, \mathcal{M}_X)$. Then there exist $i \in I$ and a coherent (resp. fine) (resp. fs) log scheme $(Y_i, \mathcal{M}_{Y_i})$ of finite presentation over $(X_i, \mathcal{M}_{X_i})$ with an isomorphism $(Y, \mathcal{M}_Y) \cong (Y_i, \mathcal{M}_{Y_i}) \times_{(X_i, \mathcal{M}_{X_i})} (X, \mathcal{M}_X)$.

Proof. These statements are direct consequences of the analogous statements for schemes and Lemma 3.A.2. $\square$

Lemma 3.A.4. Let $\{(X_i, \mathcal{M}_{X_i})\}_{i \in I}$ be a cofiltered system of coherent log schemes satisfying the condition (*). We put $(X, \mathcal{M}_X) := \varprojlim_{i \in I} (X_i, \mathcal{M}_{X_i})$. Suppose that we are given a finitely generated monoid P and a chart $P \rightarrow \mathcal{M}_X$. Then this chart descends to a chart $P \rightarrow \mathcal{M}_{X_i}$ for some $i \in I$.

Proof. Since P is finitely presented, the given chart descends to a homomorphism $P \rightarrow \mathcal{M}_{X_i}$ for some $i \in I$. Let $\mathcal{N}_i$ be the log structure associated to the homomorphism $P \rightarrow \mathcal{M}_{X_i} \rightarrow \mathcal{O}_{X_i}$ and $\phi_i: \mathcal{N}_i \rightarrow \mathcal{M}_{X_i}$ be the morphism of coherent log structures induced from $P \rightarrow \mathcal{M}_{X_i}$. Since the pullback of ϕ_i to X is an isomorphism, the pullback of ϕ_i to X_j is also an isomorphism for some $j \geq i$ by Lemma 3.A.2 (1). Then the homomorphism $P \rightarrow \mathcal{M}_{X_j}$ induced from $P \rightarrow \mathcal{M}_{X_i}$ is the desired chart. $\square$

Proposition 3.A.5. Let $\{(X_i, \mathcal{M}_{X_i})\}_{i \in I}$ be a cofiltered system of fs log schemes satisfying the condition (*). We put $(X, \mathcal{M}_X) := \varprojlim_{i \in I} (X_i, \mathcal{M}_{X_i})$. Suppose that the cofiltered category I has a final object 0. Let $(Y_0, \mathcal{M}_{Y_0})$ be an fs log scheme of finite presentation over $(X_0, \mathcal{M}_{X_0})$. We set

$$\begin{aligned} (Y_i, \mathcal{M}_{Y_i}) &:= (Y_0, \mathcal{M}_{Y_0}) \times_{(X_0, \mathcal{M}_{X_0})} (X_i, \mathcal{M}_{X_i}), \\ (Y, \mathcal{M}_Y) &:= (Y_0, \mathcal{M}_{Y_0}) \times_{(X_0, \mathcal{M}_{X_0})} (X, \mathcal{M}_X). \end{aligned}$$

Let $f_i: (Y_i, \mathcal{M}_{Y_i}) \rightarrow (X_i, \mathcal{M}_{X_i})$ and $f: (Y, \mathcal{M}_Y) \rightarrow (X, \mathcal{M}_X)$ denote the natural morphisms.

Let $\mathcal{P}$ be one of the following properties of morphisms of fs log schemes. Then, if f satisfies the property $\mathcal{P}$, f_i also satisfies the property $\mathcal{P}$ for some $i \in I$.

- (1) log flat
- (2) log smooth
- (3) log étale
- (4) Kummer log flat

Proof. These properties have a criterion using a chart of a morphism. For (1), this is just the definition. For (2) and (3), see [Kat89, Theorem 3.5]. For (4), see [INT13, Proposition 1.3]. Then the statement can be deduced from these criteria and Lemma 3.A.4. $\square$

CHAPTER 4

Log p -divisible groups associated with semi-abelian schemes

4.1. Introduction

The goal of this chapter is to prove the following theorem.

Theorem 4.1.1 (Theorem 4.4.4). Let $(X, \mathcal{M}_X)$ be an fs log scheme defined by a locally noetherian regular scheme X with a normal crossings divisor D , and A be a semi-abelian scheme over X . Let $U := X - D$. Suppose that A_U is an abelian scheme over U . Then the p -divisible group $A_U[p^\infty]$ over U uniquely extends to a log p -divisible group $A[p^\infty]^{\log}$ over $(X, \mathcal{M}_X)$.

We also prove an analogue of Theorem 4.1.1 for n -torsion points of A_U when n is invertible on the generic points of D . This assumption is necessary in order to ensure the uniqueness of the extension of $A_U[n]$.

Theorem 4.1.2 (Theorem 4.4.3). Let $n \geq 1$ be an integer. Let $(X, \mathcal{M}_X)$ be an fs log scheme defined by a locally noetherian regular scheme X with a normal crossings divisor D , and A be a semi-abelian scheme over X . Let $U := X \setminus D$. Suppose that A_U is an abelian scheme over U and that $D \otimes_{\mathbb{Z}} \mathbb{Z}[1/n]$ is dense in D . Then the finite flat group scheme $A_U[n]$ over U uniquely extends to a log finite group scheme $A[n]^{\log}$ over $(X, \mathcal{M}_X)$.

Let us explain the relationship between this result and the theory of log abelian varieties, which we do not use in this thesis. The conclusion of Theorem 1.5.1 is true when A is the semi-abelian part of a log abelian scheme over $(X, \mathcal{M}_X)$ in the sense of [KKN08b, 4.4] (cf. Example 3.2.30). It seems to be natural to ask the following question.

Question 4.1.3. Let $(X, \mathcal{M}_X)$ and A be as in Theorem 1.5.1. Then there is a unique log abelian scheme $A^{\log}$ over $(X, \mathcal{M}_X)$ whose semi-abelian part is isomorphic to A .

When Question 4.1.3 has an affirmative answer, Theorem 1.5.1 is directly deduced from it.

Remark 4.1.4. Question 4.1.3 is answered affirmatively in the following cases.

- (1) When $(X, \mathcal{M}_X)$ is the spectrum of a complete discrete valuation ring equipped with the standard log structure, Question 4.1.3 is resolved affirmatively in [KKN19, Corollary 4.5].
- (2) Let $n \geq 3$ be an integer. Let $\mathcal{A}_{g,n}$ denote the moduli space of principally polarized g -dimensional abelian varieties with level- n structures, and $\mathcal{A}_{g,n}^\Sigma$ denote the toroidal compactification of $\mathcal{A}_{g,n}$ associated with a fixed smooth cone decomposition Σ constructed by Faltings-Chai in [FC90]. We equip $\mathcal{A}_{g,n}^\Sigma$ with the log structure $\mathcal{M}_{\mathcal{A}_{g,n}^\Sigma}$ defined by the boundary divisor. Kajiwara-Kato-Nakayama studied fundamental properties of log abelian varieties and interpreted the log scheme $(\mathcal{A}_{g,n}^\Sigma, \mathcal{M}_{\mathcal{A}_{g,n}^\Sigma})$ as the moduli space of log abelian varieties in a series of papers [KKN08a, KKN08b, KKN15, KKN18, KKN19, KKN21, KKN22], . In particular, their results imply that there exists a universal log abelian scheme $A_{\text{univ}}^{\log}$

on $(\mathcal{A}_{g,n}^\Sigma, \mathcal{M}_{\mathcal{A}_{g,n}^\Sigma})$ such that its semi-abelian part is isomorphic to the universal semi-abelian scheme $A_{\text{univ}}^{\text{sab}}$.

Let $(X, \mathcal{M}_X)$ and A be as in Theorem 1.5.1. Suppose that there is a morphism $X \rightarrow \mathcal{A}_{g,n}^\Sigma$ such that A is obtained as the pullback of $A_{\text{univ}}^{\text{sab}}$ along this morphism. Then it follows that Question 4.1.3 has an affirmative answer from the functoriality of the semi-abelian parts of log abelian schemes. For example, this assumption holds for the toroidal compactification of the integral canonical model of a Shimura variety of Hodge type with hyperspecial level constructed by Lan and Madapusi in [Lan13, Mad19].

- (3) When X is a smooth variety over the complex number field $\mathbb{C}$ and A admits a polarization, Question 4.1.3 is resolved affirmatively in [KKN08a, Proposition 3.9.2] by using log Hodge theory.

Remark 4.1.5. Even if Question 4.1.3 has an affirmative answer, our method in Chapter 4 gives another simpler proof of Theorem 1.5.1 because we do not use the theory of log abelian varieties.

In what follows, we explain the strategy of our proof of Theorem 4.1.2. Theorem 4.1.1 is proved in a similar way. We may assume that $X = \text{Spec}(R)$ is affine. First, we consider the case where R is a complete regular local ring. In this case, Mumford's degeneration theory (cf. [FC90]) allows us to associate a log 1-motive with a semi-abelian degeneration A . Then the log finite group scheme associated with this log 1-motive constructed in Proposition 4.3.14) is the desired object. In this step, we prove the following.

Theorem 4.1.6 (Theorem 4.3.20). Let X be a spectrum of a complete regular local ring and U be an open subset of X whose complement is a normal crossings divisor. Then there are natural equivalences of categories:

$$\text{DEG}(X, U) \simeq \text{DD}(X, U) \simeq \text{DD}^{\log}(X, U).$$

Here, the categories $\text{DEG}(X, U)$, $\text{DD}(X, U)$, and $\text{DD}^{\log}(X, U)$ are defined in Definition 4.3.17.

While the first equivalence is already known in [FC90], the second one is new and allows us to interpret the degeneration theory in terms of the theory of log 1-motives. We believe that Theorem 4.1.6 is well-known to experts.

Next, we treat the case where R is a (not necessarily complete) discrete valuation ring. In this case, Theorem 4.1.2 follows from what we proved above and Beauville-Laszlo gluing (Proposition 4.2.2).

Finally, we consider general cases. By the results in the previous cases and the limit argument, we obtain an extension $A_V[n]^{\log}$ of $A_U[n]$ to an open subset $V \subset X$ containing U such that the codimension of the complement of V in X is at least 2. The purity for homomorphisms of log finite group schemes (Proposition 3.4.10) and the limit argument allow us to assume that R is local. Let $\widehat{R}$ be the completion of R . Consider a strict fpqc cover $(\text{Spec}(\widehat{R}), \mathcal{M}_{\widehat{R}}) \rightarrow (X, \mathcal{M}_X)$. What we proved above implies that there is a log finite group scheme $A_{\widehat{R}}[n]^{\log}$ over $(\text{Spec}(\widehat{R}), \mathcal{M}_{\widehat{R}})$ extending the pullback of $A_V[n]^{\log}$, and $A_{\widehat{R}}[n]^{\log}$ is naturally equipped with a descent datum over $(\text{Spec}(\widehat{R} \otimes_R \widehat{R}), \mathcal{M}_{\widehat{R} \otimes_R \widehat{R}})$ by the purity for homomorphisms of log finite group schemes (Proposition 3.4.10). Therefore, we obtain the desired extension by strict fpqc descent for log finite group schemes ([Kat21, Theorem 7.1 and Theorem 8.1]).

Let us explain the outline of this chapter. In Section 4.2, we recall some basic properties of log p -divisible groups. In Section 4.3, we give the reinterpretation of Mumford's degeneration theory of abelian schemes in terms of log 1-motives. In Section 4.4, we prove the main theorem (Theorem 4.4.3).

We refer readers to [Ogu18] for notation and terminologies concerning log schemes.

4.2. Beauville-Laszlo gluing for log p -divisible groups

Proposition 4.2.1 (Beauville-Laszlo gluing for kfl vector bundles). Let $(\mathrm{Spec}(R), \mathcal{M}_R)$ be a spectrum of a discrete valuation ring R equipped with the log structure defined by the unique closed point. Let K be the fraction field of R , $\widehat{R}$ be the completion of R , and $\widehat{K}$ be the fraction field of $\widehat{R}$. Then a natural functor

$$\mathrm{Vect}_{\mathrm{kfl}}(\mathrm{Spec}(R), \mathcal{M}_R) \rightarrow \mathrm{Vect}(K) \times_{\mathrm{Vect}(\widehat{K})} \mathrm{Vect}_{\mathrm{kfl}}(\mathrm{Spec}(\widehat{R}), \mathcal{M}_{\widehat{R}})$$

is an equivalence of categories.

Proof. Fix a uniformizer $\pi \in R$, and let $\alpha: \mathbb{N} \rightarrow \mathcal{M}_R$ be a chart defined by $1 \mapsto \pi$. For an integer $n \geq 1$, we set

$$(\mathrm{Spec}(R_n^{(0)}), \mathcal{M}_{R_n^{(0)}}) := (\mathrm{Spec}(R), \mathcal{M}_R) \otimes_{(\mathbb{Z}[\mathbb{N}], \mathbb{N})} (\mathbb{Z}[\frac{1}{n}\mathbb{N}], \frac{1}{n}\mathbb{N}),$$

and we let $(\mathrm{Spec}(R_n^{(m)}), \mathcal{M}_{R_n^{(m)}})$ denote the $(m+1)$ -fold self-product of $(\mathrm{Spec}(R_n^{(0)}), \mathcal{M}_{R_n^{(0)}})$ in the category of saturated log schemes over $(\mathrm{Spec}(R), \mathcal{M}_R)$ for $m \geq 0$. Let $K_n^{(m)} := R_n^{(m)}[1/\pi]$, and let $\widehat{R}_n^{(m)}$ denote the π -adic completion of $R_n^{(m)}$. Let $\widehat{K}_n^{(m)} := \widehat{R}_n^{(m)}[1/\pi]$. The ring $R_n^{(0)}$ is a discrete valuation ring, and $R_n^{(m)}$ is flat over $R_n^{(0)}$ for $m \geq 1$. Hence, $R_n^{(m)}$ is π -torsion free for $m \geq 0$. Beauville-Laszlo gluing ([BL95, Theorem and Remarques (1)]) gives equivalences

$$\mathrm{Vect}(R_n^{(m)}) \xrightarrow{\sim} \mathrm{Vect}(K_n^{(m)}) \times_{\mathrm{Vect}(\widehat{K}_n^{(m)})} \mathrm{Vect}(\widehat{R}_n^{(m)})$$

for $m \geq 0$. Therefore, by working kfl locally, we obtain an equivalence

$$\mathrm{Vect}_{\mathrm{kfl}, n}(\mathrm{Spec}(R), \mathcal{M}_R) \rightarrow \mathrm{Vect}(K) \times_{\mathrm{Vect}(\widehat{K})} \mathrm{Vect}_{\mathrm{kfl}, n}(\mathrm{Spec}(\widehat{R}), \mathcal{M}_{\widehat{R}}),$$

where $\mathrm{Vect}_{\mathrm{kfl}, n}(\mathrm{Spec}(R), \mathcal{M}_R)$ (resp. $\mathrm{Vect}_{\mathrm{kfl}, n}(\mathrm{Spec}(\widehat{R}), \mathcal{M}_{\widehat{R}})$) is the full subcategory of the category of kfl vector bundles on $(\mathrm{Spec}(R), \mathcal{M}_R)$ (resp. $(\mathrm{Spec}(\widehat{R}), \mathcal{M}_{\widehat{R}})$) consisting of objects which become classical after being pulled back to $(\mathrm{Spec}(R_n^{(0)}), \mathcal{M}_{R_n^{(0)}})$ (resp. $(\mathrm{Spec}(\widehat{R}_n^{(0)}), \mathcal{M}_{\widehat{R}_n^{(0)}})$). Taking the colimit with respect to $n \geq 1$, we obtain the equivalence in the assertion by Lemma 6.2.17. $\square$

In this subsection, we recall basic results on log finite group schemes and log p -divisible groups introduced by Kato in [Kat23].

For a scheme X , let $\mathrm{Fin}(X)$ (resp. $\mathrm{BT}(X)$) denote the category of finite and locally free group schemes (resp. p -divisible groups) over X . When $X = \mathrm{Spec}(R)$ is affine, we write $\mathrm{Fin}(R) = \mathrm{Fin}(X)$ and $\mathrm{BT}(R) = \mathrm{BT}(X)$.

Proposition 4.2.2 (Beauville-Laszlo gluing for log finite group schemes). Under the notation of Proposition 4.2.1, natural functors

$$\mathrm{wFin}(\mathrm{Spec}(R), \mathcal{M}_R) \rightarrow \mathrm{Fin}(K) \times_{\mathrm{Fin}(\widehat{K})} \mathrm{wFin}(\mathrm{Spec}(\widehat{R}), \mathcal{M}_{\widehat{R}}),$$

$$\mathrm{Fin}(\mathrm{Spec}(R), \mathcal{M}_R) \rightarrow \mathrm{Fin}(K) \times_{\mathrm{Fin}(\widehat{K})} \mathrm{Fin}(\mathrm{Spec}(\widehat{R}), \mathcal{M}_{\widehat{R}})$$

are equivalence of categories.

Proof. The equivalence of the former functor follows from Proposition 4.2.1 and Proposition 3.2.18. Then the former equivalence restricts to the latter equivalence thanks to strict fpqc descent for finite Kummer log flat log schemes ([Kat21, Theorem 7.1 and Theorem 8.1]). Although the statement in *loc. cit.* proves strict fppf descent, the same proof as in *loc. cit.* also shows strict fpqc descent. $\square$

Let $\mathrm{Lcf}((X, \mathcal{M}_X)_{\mathrm{k\acute{e}t}})$ denote the category of locally constant sheaves of finite abelian groups on $(X, \mathcal{M}_X)_{\mathrm{k\acute{e}t}}$. Then we have a natural fully faithful functor

$$\mathrm{Lcf}((X, \mathcal{M}_X)_{\mathrm{k\acute{e}t}}) \hookrightarrow \mathrm{wFin}(X, \mathcal{M}_X)$$

by [Kat21, Theorem 10.2 (2)].

Lemma 4.2.3. Let $n \geq 1$ be an integer that is invertible on X . Then the above functor induces equivalences

$$\mathrm{Lcf}((X, \mathcal{M}_X)_{\mathrm{k\acute{e}t}}, \mathbb{Z}/n) \xrightarrow{\sim} \mathrm{Fin}((X, \mathcal{M}_X), \mathbb{Z}/n) \xrightarrow{\sim} \mathrm{wFin}((X, \mathcal{M}_X), \mathbb{Z}/n),$$

where $\mathrm{Lcf}((X, \mathcal{M}_X)_{\mathrm{k\acute{e}t}}, \mathbb{Z}/n)$, $\mathrm{Fin}((X, \mathcal{M}_X), \mathbb{Z}/n)$, $\mathrm{wFin}((X, \mathcal{M}_X), \mathbb{Z}/n)$ are the full subcategories of $\mathrm{Lcf}((X, \mathcal{M}_X)_{\mathrm{k\acute{e}t}}, \mathbb{Z}/n)$, $\mathrm{Fin}(X, \mathcal{M}_X)$, $\mathrm{wFin}(X, \mathcal{M}_X)$ consisting of objects killed by n , respectively.

Proof. The equivalence of the second functor follows from [Kat23, Proposition 2.1].

Since a finite and locally free group scheme killed by an integer invertible on the base is an étale locally constant sheaf, $\mathrm{wFin}((X, \mathcal{M}_X), \mathbb{Z}/n)$ is nothing but the category of locally constant sheaves of finite $\mathbb{Z}/n$ -modules on $(X, \mathcal{M}_X)_{\mathrm{k\acute{e}t}}$. Hence, it follows from [Kat21, Theorem 10.2 (2)] that the composite of the functors in the statement is an equivalence. $\square$

4.3. Degeneration theory of abelian schemes

The goal of this section is to reinterpret the degeneration theory of abelian schemes established by Mumford, Falting-Chai, and Lan in [FC90, Lan13] in terms of log 1-motives (Proposition 4.3.20).

4.3.1. 1-motives. Let S be a base scheme. A commutative group scheme G of finite presentation over S is called a *semi-abelian scheme* if every geometric fiber of G is written as an extension of an abelian variety by a torus. We say that a semi-abelian scheme G is *split* if there is an exact sequence $0 \rightarrow T \rightarrow G \rightarrow A \rightarrow 0$, where T is a torus and A is an abelian scheme over S . This exact sequence is unique up to a unique isomorphism if it exists, and T (resp. A) is called the *torus part* (resp. *abelian part*) of G .

Definition 4.3.1 (1-motives, [Del74, Définition 10.1.2 and Variante 10.1.10]). A 1-motive over S is a morphism $\mathcal{Q} = (Y \xrightarrow{u} G)$ of étale sheaves, where Y is a locally constant sheaf of free $\mathbb{Z}$ -modules of finite rank on $S_{\mathrm{\acute{e}t}}$ and G is a split semi-abelian scheme over S .

Let $\mathcal{Q} = (Y \xrightarrow{u} G)$ be a 1-motive over S and T (resp. A) be the torus part (resp. the abelian part) of G . We let $c: Y \rightarrow A$ denote the composite of $Y \xrightarrow{u} G \rightarrow A$. Let X denote the character group sheaf of T . The extension class corresponding to G belongs to

$$\mathrm{Ext}_{S_{\mathrm{\acute{e}t}}}^1(A, T) \cong \mathrm{Hom}_{S_{\mathrm{\acute{e}t}}}(X, \mathcal{E}xt_{S_{\mathrm{\acute{e}t}}}^1(A, \mathbb{G}_m)) \cong \mathrm{Hom}_{S_{\mathrm{\acute{e}t}}}(X, A^\vee).$$

This gives a group homomorphism $c^\vee: X \rightarrow A^\vee$. Take $x \in X(S)$. Taking the pushout along $x: T \rightarrow \mathbb{G}_m$ and the pullback along $c: Y \rightarrow A$ for the exact sequence $0 \rightarrow T \rightarrow G \rightarrow A \rightarrow 0$ gives an exact sequence

$$0 \rightarrow \mathbb{G}_m \rightarrow (c \times c^\vee(x))^* \mathcal{P}_A \rightarrow Y \rightarrow 0,$$

where $\mathcal{P}_A$ is the Poincaré biextension over $A \times A^\vee$ and $c \times c^\vee(x)$ denotes a map $Y \rightarrow A \times A^\vee$ given by $y \mapsto c(y) \times c^\vee(x)$. Then u induces a section of this exact sequence. By varying $x \in X(S)$, sections defined in this way are totalized into a trivialization of a $\mathbb{G}_m$ -biextension over $Y \times X$

$$\tau: 1_{Y \times X} \xrightarrow{\sim} (c \times c^\vee)^* \mathcal{P}_A.$$

By construction, we can recover the 1-motive $\mathcal{Q}$ from the tuple $(X, Y, A, c, c^\vee, \tau)$. As a summary, we get the following lemma.

Lemma 4.3.2 (The description of 1-motives of a symmetric form). Consider the category of tuples $(X, Y, A, c, c^\vee, \tau)$ consisting of the following objects:

- X and Y are locally constant sheaves of free $\mathbb{Z}$ -modules of finite rank on $S_{\text{ét}}$;
- A is an abelian scheme over S
- $c: Y \rightarrow A$ and $c^\vee: X \rightarrow A^\vee$ are group homomorphisms;
- $\tau: 1_{Y \times X} \xrightarrow{\sim} (c \times c^\vee)^* \mathcal{P}_A$ is a trivialization of a $\mathbb{G}_m$ -biextension over $Y \times X$.

Morphisms $f: (X_1, Y_1, A_1, c_1, c_1^\vee, \tau_1) \rightarrow (X_2, Y_2, A_2, c_2, c_2^\vee, \tau_2)$ are group homomorphisms $f^{\text{mult}}: X_2 \rightarrow X_1$, $f^{\text{ab}}: A_1 \rightarrow A_2$, and $f^{\text{ét}}: Y_1 \rightarrow Y_2$ satisfying the following conditions:

- $f^{\text{ab}} c_1 = c_2 f^{\text{ét}}$ and $c_1^\vee f^{\text{mult}} = (f^{\text{ab}})^\vee c_2^\vee$;
- $(\text{id}_{X_1} \times f^{\text{mult}})^* \tau_1$ and $(f^{\text{ét}} \times \text{id}_{X_2})^* \tau_2$ are equal via the isomorphisms

$$\begin{aligned} (\text{id}_{X_1} \times f^{\text{mult}})^* (c_1 \times c_1^\vee)^* \mathcal{P}_{A_1} &\cong (c_1 \times c_2^\vee)^* (\text{id}_{A_1} \times (f^{\text{ab}})^\vee)^* \mathcal{P}_{A_1} \\ &\cong (c_1 \times c_2^\vee)^* (f^{\text{ab}} \times \text{id}_{A_2^\vee})^* \mathcal{P}_{A_2} \\ &\cong (f^{\text{ét}} \times \text{id}_{X_2})^* (c_2 \times c_2^\vee)^* \mathcal{P}_{A_2}. \end{aligned}$$

Then this category is naturally equivalent to the category of 1-motives over S .

Remark 4.3.3. We also refer to an object of the category in Lemma 4.3.2 as a 1-motive over S .

Definition 4.3.4 (Polarization on 1-motives). Let $\mathcal{Q} = (X, Y, A, c, c^\vee, \tau)$ be a 1-motive over S . The tuple $\mathcal{Q}^\vee := (Y, X, A^\vee, c^\vee, c, \tau^\vee)$ is called the *dual 1-motive* of $\mathcal{Q}$. Here, $\tau^\vee$ is the composite of the isomorphisms of $\mathbb{G}_m$ -biextensions over $X \times Y$ defined in the following way:

$$1_{X \times Y} \xrightarrow{s^* \tau} s^* (c \times c^\vee)^* \mathcal{P}_A \cong (c^\vee \times c)^* t^* \mathcal{P}_A \cong (c^\vee \times c)^* \mathcal{P}_{A^\vee},$$

where $s: X \times Y \rightarrow Y \times X$ and $t: A^\vee \times A \rightarrow A \times A^\vee$ are the switching maps.

Let $T^\vee$ denote the torus over S whose character group is Y . The group homomorphism c corresponds to a split semi-abelian scheme $G^\vee$ with an exact sequence $0 \rightarrow T^\vee \rightarrow G^\vee \rightarrow A^\vee \rightarrow 0$. Then $\tau^\vee$ gives a group homomorphism $u^\vee: X \rightarrow G^\vee$, which corresponds to the dual 1-motive $\mathcal{Q}^\vee$ via the equivalence in Lemma 4.3.2.

A *polarization* on $\mathcal{Q}$ is a morphism $\lambda: \mathcal{Q} \rightarrow \mathcal{Q}^\vee$ such that the following conditions are satisfied:

- $\lambda^{\text{mult}} = \lambda^{\text{ét}}: Y \rightarrow X$ and these maps induce an isomorphism $Y \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} X \otimes_{\mathbb{Z}} \mathbb{Q}$;
- $\lambda^{\text{ab}}: A \rightarrow A^\vee$ is a polarization on the abelian scheme A .

4.3.2. Log 1-motives. Next, we consider the log version of 1-motives. Let $(S, \mathcal{M}_S)$ be a locally noetherian fs log scheme.

Let T be a torus over S with character group X . We define a sheaf $T_{\log}$ on $(S, \mathcal{M}_S)_{\text{kfl}}$ by

$$T_{\log} := \mathcal{H}om_{(S, \mathcal{M}_S)_{\text{kfl}}}(X, \mathbb{G}_{m, \log}).$$

The natural injection $\mathbb{G}_m \hookrightarrow \mathbb{G}_{m, \log}$ induces an injection $T \hookrightarrow T_{\log}$. More generally, for a split semi-abelian scheme G over S with torus part T and abelian part A , we define the sheaf $G_{\log}$ on $(S, \mathcal{M}_S)_{\text{kfl}}$ by the following pushout diagram:

$$\begin{array}{ccc} T & \longrightarrow & T_{\log} \\ \downarrow & & \downarrow \\ G & \longrightarrow & G_{\log}. \end{array}$$

Then we have an exact sequence $0 \rightarrow T_{\log} \rightarrow G_{\log} \rightarrow A \rightarrow 0$ of sheaves on $(S, \mathcal{M}_S)_{\text{kfl}}$.

Lemma 4.3.5. The restriction to the small étale site $S_{\text{ét}}$ gives an exact sequence of sheaves on $S_{\text{ét}}$

$$0 \rightarrow (T_{\log})|_{S_{\text{ét}}} \rightarrow (G_{\log})|_{S_{\text{ét}}} \rightarrow A \rightarrow 0.$$

Proof. By working étale locally on S , we may assume that T is a split torus. Consider the morphism of sites $\epsilon: (S, \mathcal{M}_S)_{\text{kfl}} \rightarrow S_{\text{ét}}$ induced from the inclusion functor $S_{\text{ét}} \hookrightarrow (S, \mathcal{M}_S)_{\text{kfl}}$. By [Kat21, Theorem 5.1], we have $R^1\epsilon_* T_{\log} = 0$. Therefore, applying ϵ_* to the exact sequence $0 \rightarrow T_{\log} \rightarrow G_{\log} \rightarrow A \rightarrow 0$ gives the exact sequence in the assertion. $\square$

Definition 4.3.6 (Log 1-motives, [KKN08b, Definition 2.2]). A *log 1-motive* over $(S, \mathcal{M}_S)$ is a morphism $\mathcal{Q}_{\log} := (Y \xrightarrow{u} G_{\log})$ of sheaves of abelian groups on $(S, \mathcal{M}_S)_{\text{kfl}}$, where Y is a strict étale locally constant sheaf of free $\mathbb{Z}$ -modules of finite rank on $(S, \mathcal{M}_S)_{\text{kfl}}$ and G is a split semi-abelian scheme over S .

For an abelian scheme A over S , we let $\mathcal{P}_A^{\log}$ denote the $\mathbb{G}_{m, \log}$ -biextension over $A \times A^{\vee}$ defined as the base change of the Poincaré biextension $\mathcal{P}_A$ along $\mathbb{G}_m \rightarrow \mathbb{G}_{m, \log}$. In the same way as Lemma 4.3.2, we get the following lemma.

Lemma 4.3.7 (The description of log 1-motives of a symmetric form). Consider the category of tuples $(X, Y, A, c, c^{\vee}, \tau)$ consisting of the following objects:

- $(X, Y, A, c, c^{\vee})$ is the same as in Lemma 4.3.2;
- $\tau: 1_{Y \times X} \xrightarrow{\sim} (c \times c^{\vee})^* \mathcal{P}_A^{\log}$ is a trivialization of a $\mathbb{G}_{m, \log}$ -biextension over $Y \times X$.

Morphisms are also defined in the same way as in Lemma 4.3.2. Then this category is naturally equivalent to the category of log 1-motives over $(S, \mathcal{M}_S)$.

Remark 4.3.8. We also refer to an object of the category in Lemma 4.3.7 as a log 1-motive over $(S, \mathcal{M}_S)$.

Definition 4.3.9 (Monodromy pairings associated with log 1-motives, [KKN08b, (2.3)]). Let $\mathcal{Q}_{\log} = (Y \xrightarrow{u} G_{\log}) = (X, Y, A, c, c^{\vee}, \tau)$ be a log 1-motive over $(S, \mathcal{M}_S)$. We have the following commutative diagram whose rows are exact sequences of sheaves on $(S, \mathcal{M}_S)_{\text{kfl}}$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & T & \longrightarrow & G & \longrightarrow & A \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & T_{\log} & \longrightarrow & G_{\log} & \longrightarrow & A \longrightarrow 0. \end{array}$$

By the snake lemma, we get an isomorphism $T_{\log}/T \xrightarrow{\sim} G_{\log}/G$. The homomorphism u induces a homomorphism

$$Y \rightarrow G_{\log} \twoheadrightarrow G_{\log}/G \cong T_{\log}/T \cong \mathcal{H}om_{(S, \mathcal{M}_S)_{\text{kfl}}}(X, \mathbb{G}_{m, \log}/\mathbb{G}_m),$$

which corresponds to a bilinear pairing

$$\langle -, - \rangle: Y \times X \rightarrow \mathbb{G}_{m, \log}/\mathbb{G}_m.$$

This pairing is called the *monodromy pairing* associated with $\mathcal{Q}_{\log}$.

Definition 4.3.10 (Dual on log 1-motives, [KKN08b, Definition 2.7.4]).

Let $\mathcal{Q}_{\log} = (X, Y, A, c, c^{\vee}, \tau)$ be a log 1-motive over $(S, \mathcal{M}_S)$. The tuple $\mathcal{Q}_{\log}^{\vee} := (Y, X, A^{\vee}, c^{\vee}, c, \tau^{\vee})$ is called the *dual log 1-motive* of $\mathcal{Q}_{\log}$. Here, the trivialization $\tau^{\vee}$ of the $\mathbb{G}_m^{\log}$ -biextension $(c^{\vee} \times c)^* \mathcal{P}_{A^{\vee}}^{\log}$ over $X \times Y$ is defined in the same way as in Definition 4.3.4. Then $\tau^{\vee}$ gives a group homomorphism $u^{\vee}: X \rightarrow G_{\log}^{\vee}$, which corresponds to the dual log 1-motive $\mathcal{Q}_{\log}^{\vee}$ via the equivalence in Lemma 4.3.7.

We can associate a log finite group scheme with a log 1-motive by taking n -torsion points in an appropriate sense.

Definition 4.3.11 ([WZ24, Definition 3.4]). Let $\mathcal{Q}_{\log} = (Y \xrightarrow{u} G_{\log})$ be a log 1-motive over $(S, \mathcal{M}_S)$. For an integer $n \geq 1$, consider the sheaf of commutative groups on $(S, \mathcal{M}_S)_{\text{kfl}}$ defined by

$$\mathcal{Q}_{\log}[n] := H^{-1}((Y \xrightarrow{u} G_{\log}) \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}/n),$$

where $(Y \xrightarrow{u} G_{\log})$ is regarded as a complex of sheaves on $(S, \mathcal{M}_S)_{\text{kfl}}$ such that Y lives in the degree -1 part. Concretely, we can write

$$\mathcal{Q}_{\log}[n] = \frac{\text{Ker}(u - (\times n): Y \oplus G_{\log} \rightarrow G_{\log})}{\text{Im}((\times n) + u: Y \rightarrow Y \oplus G_{\log})}.$$

Lemma 4.3.12. Let $\mathcal{Q}_{\log} = (Y \xrightarrow{u} G_{\log}) = (X, Y, A, c, c^{\vee}, \tau)$ be a log 1-motive over $(S, \mathcal{M}_S)$. Let $\pi: G_{\log} \rightarrow A$ and $\pi^{\vee}: G_{\log}^{\vee} \rightarrow A^{\vee}$ be natural surjections. Then there are natural trivializations of $\mathbb{G}_{m, \log}$ -biextensions

$$\rho_1: 1_{G_{\log} \times X} \xrightarrow{\sim} (\pi \times c^{\vee})^* \mathcal{P}_A^{\log}, \quad \rho_2: 1_{Y \times G_{\log}^{\vee}} \xrightarrow{\sim} (c \times \pi^{\vee})^* \mathcal{P}_{A^{\vee}}^{\log},$$

and we have an equality $(u \times \text{id}_X)^*(\rho_1) = (\text{id}_Y \times u^{\vee})^*(\rho_2) = \tau$.

Proof. Take a section $x \in X(S, \mathcal{M}_S)$. We have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_{\log} & \longrightarrow & G_{\log} & \longrightarrow & A \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathbb{G}_{m, \log} & \longrightarrow & \mathcal{P}_A^{\log}|_{A \times \{c^{\vee}(x)\}} & \longrightarrow & A \longrightarrow 0, \end{array}$$

where the left vertical map is induced from $x: T \rightarrow \mathbb{G}_m$. Then the middle vertical map gives a trivialization of the $\mathbb{G}_{m, \log}$ -torsor $((\pi \times c^{\vee})^* \mathcal{P}_A^{\log})|_{G_{\log} \times \{x\}}$ on $G_{\log} \times \{x\} \cong G_{\log}$. By varying x , the trivializations obtained in this way are totalized into a trivialization of $\mathbb{G}_{m, \log}$ -biextensions $\rho_1: 1_{G_{\log} \times X} \xrightarrow{\sim} (\pi \times c^{\vee})^* \mathcal{P}_A^{\log}$. By construction, $(u \times \text{id}_X)^*(\rho_1)$ coincides with τ . The remaining assertions are also proved in the same way. $\square$

Construction 4.3.13 (Weil pairings associated with log 1-motives). Let $\mathcal{Q}_{\log} = (Y \xrightarrow{u} G_{\log}) = (X, Y, A, c, c^{\vee}, \tau)$ be a log 1-motive over $(S, \mathcal{M}_S)$. We shall construct a pairing

$$e_{\mathcal{Q}_{\log}[n]}: \mathcal{Q}_{\log}[n] \times \mathcal{Q}_{\log}^{\vee}[n] \rightarrow \mathbb{G}_{m, \log}$$

as follows: Let $q_1 := (y, g) \in (Y \times G_{\log})(S, \mathcal{M}_S)$ and $q_2 := (x, h) \in (X \times G_{\log}^{\vee})(S, \mathcal{M}_S)$ such that $u(y) = ng$ and $u^{\vee}(x) = nh$. Then there is a unique $e_{\mathcal{Q}_{\log}[n]}(q_1, q_2) \in \mathbb{G}_{m, \log}(S, \mathcal{M}_S)$ fitting into the following commutative diagram of $\mathbb{G}_{m, \log}$ -torsors on $(S, \mathcal{M}_S)$:

$$\begin{array}{ccccc} \mathcal{P}_A^{\log}|_{(ng, h)} & \xrightarrow{\sim} & \mathcal{P}_A^{\log}|_{(u(y), h)} & \xrightarrow[\sim]{\rho_2(y, h)} & \mathbb{G}_{m, \log} \\ \uparrow \sim & & & & \downarrow e_{\mathcal{Q}_{\log}[n]}(q_1, q_2) \\ (\mathcal{P}_A^{\log}|_{(g, h)})^{\otimes n} & & & & \\ \downarrow \sim & & & & \downarrow \\ \mathcal{P}_A^{\log}|_{(g, nh)} & \xrightarrow{\sim} & \mathcal{P}_A^{\log}|_{(g, u^{\vee}(x))} & \xrightarrow[\sim]{\rho_1(g, x)} & \mathbb{G}_{m, \log} \end{array}$$

Here, the left vertical maps are defined by the $\mathbb{G}_{m, \log}$ -biextension structure on $\mathcal{P}_A^{\log}$. The last assertion of Lemma 4.3.12 implies that $(q_1, q_2) \mapsto e_{\mathcal{Q}_{\log}[n]}(q_1, q_2)$ induces a pairing $e_{\mathcal{Q}_{\log}[n]}: \mathcal{Q}_{\log}[n] \times \mathcal{Q}_{\log}^{\vee}[n] \rightarrow \mathbb{G}_m$ (the bilinearity implies the image is contained in $\mathbb{G}_{m, \log}[n] = \mathbb{G}_m[n] \subset \mathbb{G}_m$). The pairing $e_{\mathcal{Q}_{\log}[n]}$ is called the *Weil pairing* associated with the log 1-motive $\mathcal{Q}_{\log}$.

Proposition 4.3.14 (cf. [WZ24, Proposition 3.5]). For a log 1-motive $\mathcal{Q}_{\log} = (Y \xrightarrow{u} G_{\log})$ over $(S, \mathcal{M}_S)$, the following statements hold.

- (1) $\mathcal{Q}_{\log}[n]$ fits into an exact sequence

$$0 \rightarrow G[n] \rightarrow \mathcal{Q}_{\log}[n] \rightarrow Y/nY \rightarrow 0$$

of sheaves of abelian groups on $(S, \mathcal{M}_S)_{\text{kfl}}$.

- (2) $\mathcal{Q}_{\log}[n]$ is a log finite group scheme over $(S, \mathcal{M}_S)$.
(3) The Weil pairing $e_{\mathcal{Q}_{\log}[n]}$ induces an isomorphism $\mathcal{Q}_{\log}^{\vee}[n] \xrightarrow{\sim} (\mathcal{Q}_{\log}[n])^{\vee}$ of log finite group schemes.
(4) For another integer $m \geq 1$, there is a natural exact sequence

$$0 \rightarrow \mathcal{Q}_{\log}[m] \rightarrow \mathcal{Q}_{\log}[mn] \rightarrow \mathcal{Q}_{\log}[n] \rightarrow 0.$$

In particular, for a prime number p , $\mathcal{Q}_{\log}[p^{\infty}] := \bigcup_{n \geq 1} \mathcal{Q}_{\log}[p^n]$ is a log p -divisible group over $(S, \mathcal{M}_S)$.

Proof. (1) is proved by Würthen-Zhao in [WZ24, Proposition 3.5]. (2) follows from [Kat23, Proposition 2.3] and (1). (4) follows from (1) and the snake lemma. (3) is presumably well-known to experts. We shall give a proof here because we could not find proofs in the literature.

We define a filtration $W_{-2, \mathcal{Q}_{\log}} \subset W_{-1, \mathcal{Q}_{\log}} \subset W_{0, \mathcal{Q}_{\log}} = \mathcal{Q}_{\log}[n]$ by

$$W_{-2, \mathcal{Q}_{\log}} := T[n] \cong X^{\vee} \otimes_{\mathbb{Z}} \mu_n, \quad W_{-1, \mathcal{Q}_{\log}} := G[n].$$

Applying (1) to $\mathcal{Q}_{\log}^{\vee}$ allows us to define a filtration

$$W_{-2, \mathcal{Q}_{\log}^{\vee}} \subset W_{-1, \mathcal{Q}_{\log}^{\vee}} \subset W_{0, \mathcal{Q}_{\log}^{\vee}} = \mathcal{Q}_{\log}^{\vee}[n]$$

by

$$W_{-2, \mathcal{Q}_{\log}^{\vee}} := T^{\vee}[n] \cong Y^{\vee} \otimes_{\mathbb{Z}} \mu_n, \quad W_{-1, \mathcal{Q}_{\log}^{\vee}} := G^{\vee}[n].$$

It follows from the definition of the Weil pairing and the last assertion of Lemma 4.3.12 that

$$e_{\mathcal{Q}_{\log}[n]}(W_{-1, \mathcal{Q}_{\log}}, W_{-2, \mathcal{Q}_{\log}^{\vee}}) = e_{\mathcal{Q}_{\log}[n]}(W_{-2, \mathcal{Q}_{\log}}, W_{-1, \mathcal{Q}_{\log}^{\vee}}) = 0,$$

and direct computations imply that natural pairings

$$\begin{aligned} (W_{0, \mathcal{Q}_{\log}}/W_{-1, \mathcal{Q}_{\log}}) \times W_{-2, \mathcal{Q}_{\log}^{\vee}} &\cong Y/nY \times (Y^{\vee} \otimes_{\mathbb{Z}} \mu_n) \rightarrow \mu_n, \\ (W_{-1, \mathcal{Q}_{\log}}/W_{-2, \mathcal{Q}_{\log}}) \times (W_{-1, \mathcal{Q}_{\log}^{\vee}}/W_{-2, \mathcal{Q}_{\log}^{\vee}}) &\cong A[n] \times A^{\vee}[n] \rightarrow \mu_n, \\ W_{-2, \mathcal{Q}_{\log}} \times (W_{0, \mathcal{Q}_{\log}^{\vee}}/W_{-1, \mathcal{Q}_{\log}^{\vee}}) &\cong (X^{\vee} \otimes_{\mathbb{Z}} \mu_n) \times X/nX \rightarrow \mu_n, \end{aligned}$$

coincide with the induced pairings from $e_{\mathcal{Q}_{\log}[n]}$ by construction of the Weil pairing. Since the above three pairings are perfect pairings, $e_{\mathcal{Q}_{\log}[n]}$ is also a perfect pairing. This proves (3). $\square$

Let $(S, \mathcal{M}_S)$ be an fs log regular log scheme. Let U be the interior of $(S, \mathcal{M}_S)$ and $j: U \hookrightarrow S$ be the inclusion map. The pullback along j gives a morphism of sites $(U, \mathcal{M}_U)_{\text{kfl}} \rightarrow (S, \mathcal{M}_S)_{\text{kfl}}$. The associated direct image functor

$$\text{Shv}((U, \mathcal{M}_U)_{\text{kfl}}) \rightarrow \text{Shv}((S, \mathcal{M}_S)_{\text{kfl}})$$

is denoted by $j_{\text{kfl},*}$. In the same way, the direct image functor

$$\text{Shv}(U_{\text{ét}}) \rightarrow \text{Shv}(S_{\text{ét}})$$

induced from j is denoted by $j_{\text{ét},*}$.

Let G be a split semi-abelian scheme over S with torus part T and abelian part A .

Lemma 4.3.15. The natural map $G_{\log} \rightarrow j_{\text{kfl},*}(G|_U)$ of sheaves on $(S, \mathcal{M}_S)_{\text{kfl}}$ induces an isomorphism of sheaves on $S_{\text{ét}}$

$$G_{\log}|_{S_{\text{ét}}} \xrightarrow{\sim} j_{\text{ét},*}(G|_U).$$

Proof. First, we treat the case where $G = T$. By working étale locally on S , we may assume that $T = \mathbb{G}_m$ and that there is an fs chart $P \rightarrow \mathcal{M}_S$. Let $S' \in S_{\text{ét}}$. We have natural maps

$$\mathcal{M}_S(S') \rightarrow \mathcal{O}_{S'}(S') \rightarrow \mathcal{O}_{S'}(U \times_S S')^{\times}$$

By Proposition 3.4.2(2) and (3), both maps are injective. Hence, the map $\mathbb{G}_{m, \log}|_{S_{\text{ét}}} \rightarrow j_{\text{ét},*}(\mathbb{G}_{m, U})$ in the assertion is injective. Further, Lemma 3.4.4 implies that the map $\mathbb{G}_{m, \log}|_{S_{\text{ét}}} \rightarrow j_{\text{ét},*}(\mathbb{G}_{m, U})$ is surjective. This proves the assertion then $G = T$.

Next, consider a general split semi-abelian scheme G . We have the following commutative diagram of sheaves on $S_{\text{ét}}$ whose rows are exact:

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_{\log}|_{S_{\text{ét}}} & \longrightarrow & G_{\log}|_{S_{\text{ét}}} & \longrightarrow & A \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & j_{\text{ét},*}(T|_U) & \longrightarrow & j_{\text{ét},*}(G|_U) & \longrightarrow & j_{\text{ét},*}(A|_U). \end{array}$$

It follows from what we have proved in the previous paragraph that the left vertical map is an isomorphism. By [FC90, Ch.I, Proposition 2.7], the right vertical map is also an isomorphism. Therefore, the snake lemma implies that the middle vertical map is also an isomorphism. $\square$

4.3.3. Degeneration theory. In this subsection, we study a relation between semi-abelian schemes, 1-motives, and log 1-motives over complete regular local rings.

Let $S = \operatorname{Spec}(R)$ be the spectrum of a complete regular local ring R with unique closed point s , and D be a normal crossings divisor on S . Let $(S, \mathcal{M}_S)$ be the fs log scheme defined by D . Set $U := S \setminus D$. Note that every torus over S is split after finite étale base change.

Definition 4.3.16. Let $\mathcal{P}$ be a $\mathbb{G}_m$ -torsor on S . Let $\operatorname{Div}(S, U)$ denote the group of Weil divisors of S whose support is contained in D . Then taking valuations defined by generic points of irreducible components of D gives a map $\nu: \mathcal{P}(U) \rightarrow \operatorname{Div}(S, U)$.

Definition 4.3.17. We define the following categories.

- Let $\operatorname{DEG}(S, U)$ be the category of semi-abelian schemes A over S such that $A \times_S U$ is an abelian scheme over U .
- Let $\operatorname{wDD}(S, U)$ be the category of triples $(Y, G, u: Y|_U \rightarrow G|_U)$ consisting of a locally constant sheaf Y of free $\mathbb{Z}$ -modules of finite rank on $S_{\text{ét}}$, a split semi-abelian scheme G over S , and a group homomorphism $u: Y|_U \rightarrow G|_U$. In the same way as Lemma 4.3.2, $\operatorname{wDD}(S, U)$ is naturally equivalent to the category of 1-motives $\mathcal{Q}_U = (X_U, Y_U, A_U, c_U, c_U^\vee, \tau)$ over U such that the tuple $(X_U, Y_U, A_U, c_U, c_U^\vee)$ (uniquely) extends to $(X, Y, A, c, c^\vee)$ over S . In particular, $\operatorname{wDD}(S, U)$ is a full subcategory of the category of 1-motives over U .
- Let $\operatorname{DD}_{\text{pol}}(S, U)$ be the category of an object $\mathcal{Q}_U = (X_U, Y_U, A_U, c_U, c_U^\vee, \tau) \in \operatorname{wDD}(S, U)$ equipped with a polarization $\lambda_U: \mathcal{Q}_U \rightarrow \mathcal{Q}_U^\vee$ satisfying the following conditions:
 - λ_U^{ab} extends to a polarization on A ;
 - there is a connected finite étale cover $S' \rightarrow S$ such that the pullback of Y to S' is constant and, for each $y \in Y(S')$, we have

$$\nu(\tau(y, \lambda^{\text{ét}}(y))) \in \operatorname{Div}^+(S', U') \setminus \{0\},$$

where $U' := U \times_S S'$ and $\operatorname{Div}^+(S', U')$ is the submonoid of $\operatorname{Div}(S', U')$ consisting of effective divisors. Clearly, this condition is independent of the choice of S' .

Forgetting polarizations gives a functor $\operatorname{DD}_{\text{pol}}(S, U) \rightarrow \operatorname{wDD}(S, U)$. The essential image of this functor is denoted by $\operatorname{DD}(S, U)$.

- Let $\operatorname{wDD}^{\log}(S, U)$ be the category of log 1-motives over $(S, \mathcal{M}_S)$.
- Let $\operatorname{DD}_{\text{pol}}^{\log}(S, U)$ be the category of a log 1-motive $\mathcal{Q}_{\log} = (X, Y, A, c, c^\vee, \tau^{\log})$ equipped with a morphism $\lambda: \mathcal{Q}_{\log} \rightarrow \mathcal{Q}_{\log}^\vee$ satisfying the following conditions:
 - λ^{ab} is a polarization on A ;
 - for $y \in Y_{\bar{s}} \setminus \{0\}$, we have $\langle y, \lambda^{\text{ét}}(y) \rangle \in (\mathcal{M}_{S, \bar{s}} / \mathcal{O}_{S, \bar{s}}^\times) \setminus \{1\}$, where $\bar{s}$ is a geometric point on S above s , and $\langle -, - \rangle$ is the monodromy pairing $Y \times X \rightarrow \mathbb{G}_{m, \log} / \mathbb{G}_m$ (see Definition 4.3.9).

Forgetting λ gives a functor $\operatorname{DD}_{\text{pol}}^{\log}(S, U) \rightarrow \operatorname{wDD}^{\log}(S, U)$. The essential image of this functor is denoted by $\operatorname{DD}^{\log}(S, U)$.

Remark 4.3.18. The notion of polarizations on log 1-motives is also defined in [KKN08b, Definition 2.8]. However, we do not use it in this chapter. Note that, for an object $(\mathcal{Q}_{\log}, \lambda)$ of the category $\operatorname{DD}_{\text{pol}}^{\log}(S, U)$, the morphism λ is not a polarization in the sense of *loc. cit.* unless $S = U$.

Theorem 4.3.19 (Mumford's degeneration theory, cf. [FC90, Lan13, Mad19]). There is a natural equivalence of categories

$$\mathrm{DEG}(S, U) \simeq \mathrm{DD}(S, U).$$

Proof. See [Mad19, (1.2.2)]. $\square$

By reinterpreting the right hand side in the equivalence in Theorem 4.3.19 in terms of log 1-motives, we obtain the following result.

Theorem 4.3.20. There are natural equivalences of categories

$$\mathrm{DEG}(S, U) \simeq \mathrm{DD}(S, U) \simeq \mathrm{DD}^{\log}(S, U).$$

Proof. It is enough to prove the second equivalence. For a locally constant sheaf Y of free $\mathbb{Z}$ -modules of finite rank on $S_{\text{ét}}$ and a split semi-abelian scheme G on S , giving a group homomorphism $Y|_U \rightarrow G|_U$ is equivalent to giving a homomorphism $Y \rightarrow G_{\log}$ by Lemma 4.3.15. Hence, there is a natural equivalence of categories

$$\mathrm{wDD}(S, U) \simeq \mathrm{wDD}^{\log}(S, U).$$

If we take a connected finite étale cover $S' \rightarrow S$ such that every irreducible component of $S' - U'$ is regular, we have an isomorphism of monoids $\mathrm{Div}^+(S', U') \cong \mathcal{M}_{S, \bar{s}} / \mathcal{O}_{S, \bar{s}}^\times$, where we put $U' := U \times_S S'$. Therefore, the above equivalence induces an equivalence

$$\mathrm{DD}_{\mathrm{pol}}(S, U) \simeq \mathrm{DD}_{\mathrm{pol}}^{\log}(S, U),$$

and so we obtain an equivalence

$$\mathrm{DD}(S, U) \simeq \mathrm{DD}^{\log}(S, U).$$

$\square$

Proposition 4.3.21. Let $A \in \mathrm{DEG}(S, U)$. Let $\mathcal{Q}_U$ (resp. $\mathcal{Q}_{\log}$) be the object of $\mathrm{DD}(S, U)$ (resp. $\mathrm{DD}^{\log}(S, U)$) corresponding to A via the equivalences in Theorem 4.3.20. Then there are natural isomorphisms of finite flat group schemes over U

$$A|_U[n] \cong \mathcal{Q}_U[n] \cong \mathcal{Q}_{\log}[n]|_U$$

for any integer $n \geq 1$.

Proof. We have a natural isomorphism $A|_U[n] \cong \mathcal{Q}_U[n]$ (see [FC90, Ch.III, Corollary 7.3] or [Mad19, (1.2.2.1)]). Since the restriction of $\mathcal{Q}_{\log}$ to U coincides with $\mathcal{Q}_U$ by construction, we have natural isomorphisms

$$\mathcal{Q}_U[n] \cong (\mathcal{Q}_{\log})|_U[n] \cong \mathcal{Q}_{\log}[n]|_U.$$

$\square$

4.4. The proof of the main theorems

Lemma 4.4.1. Let R be a discrete valuation ring with fraction field K , and $\mathcal{M}_R$ be the log structure on $\mathrm{Spec}(R)$ defined by the unique closed point.

- (1) Let $n \geq 1$ be an integer invertible in R . Let G_1 and G_2 be weak log finite group schemes over $(\mathrm{Spec}(R), \mathcal{M}_R)$ killed by n . Then the restriction map

$$\mathrm{Hom}(G_1, G_2) \rightarrow \mathrm{Hom}(G_1|_K, G_2|_K)$$

is an isomorphism.

- (2) Let p be a prime number. Let G_1 and G_2 be log p -divisible groups over $(\mathrm{Spec}(R), \mathcal{M}_R)$. Then the restriction map

$$\mathrm{Hom}(G_1, G_2) \rightarrow \mathrm{Hom}(G_1|_K, G_2|_K)$$

is an isomorphism.

Proof. (1) follows from Lemma 4.2.3 and the surjectivity of

$$\mathrm{Gal}(\overline{K}/K) \rightarrow \pi_{1,\mathrm{k\acute{e}t}}(\mathrm{Spec}(R), \mathcal{M}_R).$$

(2) is nothing but the log version of the theorem of Tate and de Jong ([BWZ24, Theorem 5.19]). Note that, although [BWZ24, Theorem 5.19] assumes that K is of mixed characteristic $(0, p)$, and that the residue field of R is perfect, the fully faithfulness part is essentially proved in [BWZ24, Lemma 4.8], in which the argument works without such an additional assumption. $\square$

Proposition 4.4.2. Let notations be as in Lemma 4.4.1. Let A be a semi-abelian scheme over R with $A_K := A \otimes_R K$ being an abelian variety over K .

- (1) Let $n \geq 1$ be an integer invertible in R . Then the finite group scheme $A_K[n]$ over K uniquely extends to a log finite group scheme over $(\mathrm{Spec}(R), \mathcal{M}_R)$.
- (2) Let p be a prime number. Then the p -divisible group $A_K[p^\infty]$ over K uniquely extends to a log p -divisible group over $(\mathrm{Spec}(R), \mathcal{M}_R)$.

Proof. For both assertions, the uniqueness follows from Lemma 4.4.1. By Proposition 4.2.2, we may assume that R is complete. Let $\mathcal{Q}_{\log}$ be the log 1-motive on $(\mathrm{Spec}(R), \mathcal{M}_R)$ corresponding to $A \in \mathrm{DEG}(R, K)$ via the equivalence in Theorem 4.3.20. Then the log finite group scheme $\mathcal{Q}_{\log}[n]$ and the log p -divisible group $\mathcal{Q}_{\log}[p^\infty]$ are the desired extensions. $\square$

Theorem 4.4.3. Let $n \geq 1$ be an integer. Let $(X, \mathcal{M}_X)$ be an fs log scheme defined by a locally noetherian regular scheme X with a normal crossings divisor D , and A be a semi-abelian scheme over X . Let $U := X \setminus D$. Suppose that $A_U := A \times_X U$ is an abelian scheme over U and that $D \otimes_{\mathbb{Z}} \mathbb{Z}[1/n]$ is dense in D . Then the finite flat group scheme $A_U[n]$ over U uniquely extends to a log finite group scheme $A[n]^{\log}$ over $(X, \mathcal{M}_X)$.

Proof. First, we prove the following claim: for weak log finite group schemes G_1 and G_2 over $(X, \mathcal{M}_X)$ killed by n , the restriction map

$$\mathrm{Hom}(G_1, G_2) \rightarrow \mathrm{Hom}(G_1|_U, G_2|_U)$$

is an isomorphism. Take a homomorphism $f_U: G_1|_U \rightarrow G_2|_U$. By the assumption, n is invertible in $\mathcal{O}_{X,\eta}$ for each generic point η of D . Hence, there exist an open subset V of X containing U with $\mathrm{codim}_X(X \setminus V) \geq 2$ and an extension $f_V: G_1|_V \rightarrow G_2|_V$ of f_U by Lemma 4.4.1 (1) and the limit argument. Then f_V uniquely extends to a homomorphism $f: G_1 \rightarrow G_2$ by Proposition 3.4.10. This argument also shows that f is a unique extension of f_U .

We turn to proving Theorem 4.4.3. Since the uniqueness of the extension follows from the claim in the previous paragraph, we may work Zariski locally on X . Hence, the limit argument allows us to assume that $X = \mathrm{Spec}(R)$ for a local ring R . By the assumption, n is invertible in $\mathcal{O}_{X,\eta}$ for each generic point η of D . Hence, there exist an open subset V of X containing U with $\mathrm{codim}_X(X \setminus V) \geq 2$ and a log finite group scheme $A_V[n]^{\log}$ over $(V, \mathcal{M}_V)$ extending $A_U[n]$ by Proposition 4.4.2 (1) and the limit argument. It is enough to extend $A_V[n]^{\log}$ to a log finite group scheme over $(X, \mathcal{M}_X)$. Let $X' := \mathrm{Spec}(\widehat{R})$ and

$X'' := X' \times_X X'$. Let $p_i: (X'', \mathcal{M}_{X''}) \rightarrow (X', \mathcal{M}_{X'})$ be the projection maps for $i = 1, 2$. Let U' (resp. V') denotes the preimage of U (resp. V) via $X' \rightarrow X$. Let $\mathcal{Q}_{\log}$ be the object of $\mathrm{DD}^{\log}(X', U')$ corresponding to $A \times_X X' \in \mathrm{DEG}(X', U')$. By Proposition 4.3.21, the log finite group scheme $\mathcal{Q}_{\log}[n]$ is an extension of $(A \times_X U')[n]$. By the claim in the previous paragraph, we have an isomorphism $\mathcal{Q}_{\log}[n]|_{V'} \cong A_V[n]^{\log}|_{V'}$. Applying Proposition 3.4.10 to the strict flat map $(X'', \mathcal{M}_{X''}) \rightarrow (X, \mathcal{M}_X)$ and the open set $V \subset X$, we obtain a unique isomorphism $p_1^* \mathcal{Q}_{\log}[n] \cong p_2^* \mathcal{Q}_{\log}[n]$ extending the isomorphism

$$(p_1^* \mathcal{Q}_{\log}[n])|_{V''} \cong A_V[n]^{\log}|_{V''} \cong p_2^* \mathcal{Q}_{\log}[n]|_{V''},$$

where V'' is the preimage of V via $X'' \rightarrow X$. This defines descent datum of $\mathcal{Q}_{\log}[n]$, and we obtain a log finite group scheme $A[n]^{\log}$ extending $A_V[n]^{\log}$ by strict fpqc descent for finite Kummer log flat schemes ([Kat21, Theorem 7.1 and Theorem 8.1]). This finishes the proof. $\square$

Theorem 4.4.4. Let $(X, \mathcal{M}_X)$ be an fs log scheme defined by a locally noetherian regular scheme X with a normal crossings divisor D , and A be a semi-abelian scheme over X . Let $U := X \setminus D$. Suppose that A_U is an abelian scheme over U . Then the p -divisible group $A_U[p^\infty]$ over U uniquely extends to a log p -divisible group $A^{\log}[p^\infty]$ over $(X, \mathcal{M}_X)$.

Proof. Although almost all arguments of Theorem 4.4.3 work in this setting, we have to notice that we need to pass to finite levels when we use the limit argument.

For each generic point η of D , there exists a unique log p -divisible group $A_{\mathcal{O}_{X,\eta}}[p^\infty]^{\log}$ over $(\mathrm{Spec}(\mathcal{O}_{X,\eta}), \mathcal{M}_{\mathcal{O}_{X,\eta}})$ extending $A_U[p^\infty]|_{\mathrm{Spec}(K(\eta))}$ by Lemma 4.4.2, where $K(\eta)$ denotes the fraction field of $\mathcal{O}_{X,\eta}$. For every $n \geq 1$, by the same argument as Theorem 4.4.3, there exists a unique log finite group $A[p^n]^{\log}$ over $(X, \mathcal{M}_X)$ with compatible isomorphisms $A[p^n]^{\log}|_U \cong A_U[p^n]$ and $A[p^n]^{\log}|_{(\mathrm{Spec}(\mathcal{O}_{X,\eta}), \mathcal{M}_{\mathcal{O}_{X,\eta}})} \cong A_{\mathcal{O}_{X,\eta}}[p^n]$ for each generic point η of D .

Natural inclusion maps $A_U[p^n] \hookrightarrow A_U[p^{n+1}]$ and $A_{\mathcal{O}_{X,\eta}}[p^n] \hookrightarrow A_{\mathcal{O}_{X,\eta}}[p^{n+1}]$ uniquely extend to a homomorphism $A[p^n]^{\log} \rightarrow A[p^{n+1}]^{\log}$ for every $n \geq 1$ by the limit argument and Proposition 3.4.10. It is enough to check that $A[p^\infty]^{\log} := \varinjlim_{n \geq 1} A[p^n]^{\log}$ is a log p -divisible group with

$$A[p^n]^{\log} \cong \mathrm{Ker}(\times p^n: A[p^\infty]^{\log} \rightarrow A[p^\infty]^{\log}).$$

To check this, it suffices show that it is so after taking the base change to $(\mathrm{Spec}(\hat{\mathcal{O}}_{X,x}), \mathcal{M}_{\hat{\mathcal{O}}_{X,x}})$ for every $x \in X$. Hence, we may assume that $X = \mathrm{Spec}(R)$ for a complete regular local ring R . Let $\mathcal{Q}_{\log}$ be the object of $\mathrm{DD}^{\log}(X, U)$ corresponding to A under the equivalence in Theorem 4.3.20. By the uniqueness of extension, $A[p^n]^{\log}$ is isomorphic to $\mathcal{Q}_{\log}[p^n]$. Therefore, the claim follows from Proposition 4.3.14 (3). $\square$

CHAPTER 5

Log prismatic F -crystals and realization functors

5.1. Introduction

The first goal of this chapter is to generalize the construction of realization functors and the relationship between them to log prismatic F -crystals on semi-stable log formal schemes over $\mathcal{O}_K$.

Theorem 5.1.1 (see [Theorem 5.6.18](#)). For a semi-stable log formal scheme $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ over $\mathcal{O}_K$, there exist an étale realization functor

$$T_{\text{ét}}: \text{Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \text{Loc}_{\mathbb{Z}_p}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta})$$

and a crystalline realization functor

$$T_{\text{fisoc}}: \text{Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \text{FillIsoc}^{\varphi}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}),$$

where $\text{Loc}_{\mathbb{Z}_p}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta})$ denotes the category of Kummer étale $\mathbb{Z}_p$ -local systems on the generic fiber $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta}$ and $\text{FillIsoc}^{\varphi}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ denotes the category of filtered F -isocrystals on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Moreover, for a log prismatic F -crystal $\mathcal{E}$, coefficient objects $T_{\text{ét}}(\mathcal{E})$ and $T_{\text{fisoc}}(\mathcal{E})$ are associated in the sense of [Definition 5.4.18](#) (2) (cf. [\[Fal89, AI12, Tsu\]](#)).

Remark 5.1.2. In fact, we work with the category of analytic log prismatic F -crystals, denoted by $\text{Vect}^{\text{an}, \varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$, which is larger than $\text{Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$.

Remark 5.1.3. When $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ has framings with $m = n = 0$ (resp. $m = 0$ and $n \geq 1$) étale locally, this theorem is proved in [\[DLMS23, GR24\]](#) (resp. [\[DLMS24\]](#)). Moreover, the work [\[IKY25\]](#) gives the description of T_{fisoc} using Nygaard filtrations when $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ has a framing with $m = n = 0$ étale locally. We also generalize this description for arbitrary m, n .

The next goal is to prove the fully faithfulness and bi-exactness of étale realization functors.

Theorem 5.1.4 (see [Theorem 5.6.27](#) and [Theorem 5.6.32](#)). Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a semi-stable log formal scheme over $\mathcal{O}_K$. Consider the étale realization functor

$$T_{\text{ét}}: \text{Vect}^{\text{an}, \varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \text{Loc}_{\mathbb{Z}_p}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta}).$$

- (1) The functor $T_{\text{ét}}$ is fully faithful.
- (2) When $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ admits framings with $n \leq 1$ étale locally, $T_{\text{ét}}$ gives a bi-exact equivalence to the essential image. In other words, a sequence of analytic log prismatic F -crystals

$$0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_2 \rightarrow \mathcal{E}_3 \rightarrow 0$$

is exact if and only if the corresponding sequence of Kummer étale local systems

$$0 \rightarrow T_{\text{ét}}(\mathcal{E}_1) \rightarrow T_{\text{ét}}(\mathcal{E}_2) \rightarrow T_{\text{ét}}(\mathcal{E}_3) \rightarrow 0$$

is exact.

Remark 5.1.5. When $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ has framings with $m = n = 0$ (resp. $m = 0$ and $n \geq 1$) étale locally, the fully faithfulness of the étale realization functor is proved in [BS23, DLMS23, GR24] (resp. [DLMS24]). While the detailed calculation concerning Breuil-Kisin rings is the core of the proof in [DLMS23, DLMS24], the works [BS23, GR24] use the fully faithfulness of Fargues's functor. In this chapter, we adopt the method using the fully faithfulness of Fargues's functor.

Remark 5.1.6. When $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ has framings with $m = n = 0$, the bi-exactness is proved in [IKY24].

Remark 5.1.7. It is an important problem to specify the essential image of the étale realization functor

$$T_{\text{ét}}: \text{Vect}^{\text{an}, \varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \text{Loc}_{\mathbb{Z}_p}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta}).$$

This problem is indeed solved in some cases: When $\mathfrak{X} = \text{Spf}(\mathcal{O}_K)$ and $\mathcal{M}_{\mathfrak{X}}$ is trivial, it is proved that the functor $T_{\text{ét}}$ gives the equivalence to the category of crystalline $\mathbb{Z}_p$ -representation of $\text{Gal}(\overline{K}/K)$ in [BS23]. More generally, when $\mathcal{M}_{\mathfrak{X}}$ is trivial and $\mathfrak{X}$ is higher dimensional, the essential image of $T_{\text{ét}}$ is the full subcategory of crystalline $\mathbb{Z}_p$ -local systems. The work [GR24] proves this by the method generalizing [BS23], and the work [DLMS23] proves this by using the detailed computation concerning Breuil-Kisin rings. When $\mathfrak{X} = \text{Spf}(\mathcal{O}_K)$ and $\mathcal{M}_{\mathfrak{X}}$ is the standard log structure, the work [DL24] shows that the essential image of $T_{\text{ét}}$ is the full subcategory of semi-stable $\mathbb{Z}_p$ -representations of $\text{Gal}(\overline{K}/K)$. When $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ has framings with $m = 0$ and $n \geq 1$ étale locally, it is shown that $T_{\text{ét}}$ gives the equivalences to the category of semi-stable $\mathbb{Z}_p$ -local systems in [DLMS24] by a similar approach to [DLMS23].

In this thesis, we do not treat this problem. Naive expectation is that the essential image of $T_{\text{ét}}$ is the full subcategory of semi-stable $\mathbb{Z}_p$ -local systems in the sense of Definition 5.4.18.

We briefly explain the structure of this chapter. In Section 5.2, we define semi-stable log formal schemes and arrange notation and terminology concerning them. When we refer to semi-stable log formal schemes, we use notation in this section without reference. In Section 5.3, we recall basics on absolute log crystalline sites and F -isocrystals. One of targets of this section is Proposition 5.3.36, which generalizes [GY24, Theorem 2.21]. This allows us to describe F -isocrystals on the special fibers of semi-stable log formal schemes admitting framings with $n = 1$ étale locally as F -isocrystals equipped with monodromy operators on the special fibers of horizontally semi-stable log formal schemes. In Section 5.4, we introduce de Rham (resp. crystalline)(resp. semi-stable) period sheaves and the notion of semi-stable $\mathbb{Z}_p$ -local systems on the generic fibers of semi-stable log formal schemes. A key idea to do this is to consider the notion of *big log affinoid perfectoids* (Definition 5.4.4). Although de Rham period sheaves on log adic spaces are already defined in [DLLZ23b], we give another construction which is similar to one in [Sch16] in spirit. In Section 5.5, we review fundamental results on log prismatic sites and prismatic F -crystals. This descent property enables us to make an analogous argument to [BS22, GR24, IKY24] in order to prove Theorem 5.1.4. In Section 5.6, we construct several realization functors and prove Theorem 5.1.1 and Theorem 5.1.4.

Notation and conventions.

- Limits of categories mean 2-limits.
- The i -th standard basis of $\mathbb{N}^r$ is denote by e_i for $1 \leq i \leq r$.

- For I -adically complete ring A and a monoid M , a ring $A\langle M \rangle$ denotes the I -adic completion of the monoid algebra $A[M]$.

5.2. Setups for semi-stable log formal schemes

In this section, we summarize notations around small affine log formal schemes and framings on them. Throughout this thesis, we use the notation in this subsection without mentioning it.

Fix a uniformizer π of $\mathcal{O}_K$ and $E(t) \in W[t]$ be a corresponding Eisenstein polynomial. Let $l, m, n \in \mathbb{Z}_{\geq 0}$, $r := m + n$. Consider the prelog ring $(R^0, \mathbb{N}^r)$ defined as follows:

$$R^0 := \begin{cases} \mathcal{O}_K \langle x_1^{\pm 1}, \dots, x_l^{\pm 1}, y_1, \dots, y_m \rangle & (n = 0) \\ \mathcal{O}_K \langle x_1^{\pm 1}, \dots, x_l^{\pm 1}, y_1, \dots, y_m, z_1, \dots, z_n \rangle / (\prod_{k=1}^n z_k - \pi) & (n \geq 1) \end{cases}$$

equipped with a prelog structure $\mathbb{N}^r \rightarrow R^0$ given by

$$e_i \mapsto \begin{cases} y_i & (1 \leq i \leq m) \\ z_{i-m} & (m+1 \leq i \leq r). \end{cases}$$

We define an integral perfectoid $\mathcal{O}_C$ -algebra $\tilde{R}_\infty^0$ as follows:

$$\tilde{R}_\infty^0 := \begin{cases} \mathcal{O}_C \langle \{x_i^{\pm 1/p^\infty}\}, \{y_j^{\mathbb{Q}_{\geq 0}}\} \rangle & (n = 0) \\ \mathcal{O}_C \langle \{x_i^{\pm 1/p^\infty}\}, \{y_j^{\mathbb{Q}_{\geq 0}}\}, \{z_k^{\mathbb{Q}_{\geq 0}}\} \rangle / (\prod_{k=1}^n z_k^{1/s} - \pi^{1/s})_{s \geq 1} & (n \geq 1). \end{cases}$$

We define a prelog structure $\mathbb{Q}_{\geq 0}^r \rightarrow \tilde{R}_\infty^0$ by

$$se_i \mapsto \begin{cases} y_i^s & (1 \leq i \leq m) \\ z_{i-m}^s & (m+1 \leq i \leq r). \end{cases}$$

for $s \in \mathbb{Q}_{\geq 0}$. If we put $R_\infty^0 := R^0 \hat{\otimes}_{\mathbb{Z}_p \langle \mathbb{N}^r \rangle} \mathbb{Z}_p \langle \mathbb{Q}_{\geq 0}^r \rangle$, there is a sequence of p -complete prelog rings

$$(5.2.1) \quad (R^0, \mathbb{N}^r) \rightarrow (R_\infty^0, \mathbb{Q}_{\geq 0}^r) \rightarrow (\tilde{R}_\infty^0, \mathbb{Q}_{\geq 0}^r).$$

The sequence of p -adic fs log formal schemes associated with the sequence (5.2.1) is denoted by

$$(5.2.2) \quad (\mathrm{Spf}(\tilde{R}_\infty^0), \mathcal{M}_{\tilde{R}_\infty^0}) \rightarrow (\mathrm{Spf}(R_\infty^0), \mathcal{M}_{R_\infty^0}) \rightarrow (\mathrm{Spf}(R^0), \mathcal{M}_R).$$

Let $\tilde{U}_\infty^0$ be the object of $(\mathrm{Spf}(R^0), \mathbb{N}^r)_{\eta, \mathrm{prok\acute{e}t}}^a$ written as the limit of finite Kummer étale coverings

$$(\mathrm{Spf}(\mathcal{O}_L \langle \{x_i^{\pm 1/p^m}\}, \{y_j^{1/m}\}, \{z_k^{1/m}\} \rangle / (\prod_{k=1}^n z_k^{1/m} - \pi^{1/m})), \frac{1}{m} \mathbb{N}^r)_\eta^a \rightarrow (\mathrm{Spf}(R^0), \mathbb{N}^r)_\eta^a,$$

where $m \geq 1$ and L is a finite extension of $K[\pi^{1/m}]$ contained in C . Then $\tilde{U}_\infty^0$ is a log affinoid perfectoid, and its associated affinoid perfectoid space is isomorphic to $\mathrm{Spf}(\tilde{R}_\infty^0)_\eta$.

Let

$$(x_i^s)^\flat := (x_i^s, x_i^{s/p}, x_i^{s/p^2}, \dots) \in (\tilde{R}_\infty^0)^\flat$$

for $s \in \mathbb{Z}[1/p]$, and let

$$(y_j^s)^\flat := (y_j^s, y_j^{s/p}, y_j^{s/p^2}, \dots) \in (\tilde{R}_\infty^0)^\flat$$

$$(z_k^s)^\flat := (z_k^s, z_k^{s/p}, z_k^{s/p^2}, \dots) \in (\tilde{R}_\infty^0)^\flat$$

for $s \in \mathbb{Q}_{\geq 0}$. When $n \geq 1$, we set

$$(\pi^s)^b := \prod_{k=1}^n (z_k^s)^b$$

for $s \in \mathbb{Q}_{\geq 0}$.

Notation 5.2.1. When we would like to ease the notation, we write

$$t_i := \begin{cases} x_i & (1 \leq i \leq l) \\ y_{i-l} & (l+1 \leq i \leq l+m) \\ z_{i-l-m} & (l+m+1 \leq i \leq l+r) \end{cases}$$

$$s := \begin{cases} l+m & (n=0) \\ l+m+n-1 = l+r-1 & (n \geq 1). \end{cases}$$

Definition 5.2.2 (Small affine log formal schemes). A p -adic affine log formal scheme $(\mathrm{Spf}(R), \mathcal{M}_R)$ over $\mathcal{O}_K$ is called *small affine* if $(\mathrm{Spf}(R), \mathcal{M}_R)$ admits a strict étale map to $(\mathrm{Spf}(R^0), \mathbb{N}^r)^a$ for some l, m, n , called a *framing*.

Let $(\mathrm{Spf}(R), \mathcal{M}_R)$ be a small affine log formal scheme over $\mathcal{O}_K$ and fix a framing $(\mathrm{Spf}(R), \mathcal{M}_R) \rightarrow (\mathrm{Spf}(R^0), \mathbb{N}^r)^a$. Let $\alpha: \mathbb{N}^r \rightarrow \mathcal{M}_R$ denote the induced chart. We set

$$R_{\infty, \alpha} := R_{\infty}^0 \widehat{\otimes}_{R^0} R, \quad \tilde{R}_{\infty, \alpha} := \tilde{R}_{\infty}^0 \widehat{\otimes}_{R^0} R.$$

Taking p -completed base change of the sequence (5.2.1) along $R^0 \rightarrow R$, we obtain a sequence of p -complete prelog rings

$$(5.2.3) \quad (R, \mathbb{N}^r) \rightarrow (R_{\infty, \alpha}, \mathbb{Q}_{\geq 0}^r) \rightarrow (\tilde{R}_{\infty, \alpha}, \mathbb{Q}_{\geq 0}^r).$$

The sequence of p -adic fs log formal schemes associated with the sequence (5.2.3) is denoted by

$$(5.2.4) \quad (\mathrm{Spf}(\tilde{R}_{\infty, \alpha}), \mathcal{M}_{\tilde{R}_{\infty, \alpha}}) \rightarrow (\mathrm{Spf}(R_{\infty, \alpha}), \mathcal{M}_{R_{\infty, \alpha}}) \rightarrow (\mathrm{Spf}(R), \mathcal{M}_R),$$

which is also obtained as the base change of the sequence (5.2.2) along $(\mathrm{Spf}(R), \mathcal{M}_R) \rightarrow (\mathrm{Spf}(R^0), \mathbb{N}^r)^a$.

Let $\tilde{U}_{\infty, \alpha} := \tilde{U}_{\infty}^0 \times_{(\mathrm{Spf}(R^0), \mathbb{N}^r)_{\eta}^a} (\mathrm{Spf}(R), \mathcal{M}_R)_{\eta}$. Then $\tilde{U}_{\infty, \alpha}$ is a log affinoid perfectoid in $(\mathrm{Spf}(R), \mathcal{M}_R)_{\eta, \mathrm{prok\acute{e}t}}$ with the associated perfectoid space isomorphic to $\mathrm{Spf}(\tilde{R}_{\infty, \alpha})_{\eta}$.

The following lemma is used afterward.

Lemma 5.2.3. The natural map $(\mathrm{Spf}(\tilde{R}_{\infty, \alpha}), \mathcal{M}_{\tilde{R}_{\infty, \alpha}}) \rightarrow (\mathrm{Spf}(R_{\infty, \alpha}), \mathcal{M}_{R_{\infty, \alpha}})$ is a strict quasi-syntomic cover.

Proof. It suffices to prove that $R_{\infty}^0 \rightarrow \tilde{R}_{\infty}^0$ is a quasi-syntomic cover, where $R_{\infty}^0 := R^0 \widehat{\otimes}_{\mathbb{Z}_p \langle \mathbb{N}^r \rangle} \mathbb{Z}_p \langle \mathbb{Q}_{\geq 0}^r \rangle$. When $n = 0$, the assertion is easy. Hence, we treat the case that $n \geq 1$. The ring map

$$R_{\infty}^0 = \mathcal{O}_K \langle \{x_i^{\pm 1}\}, \{y_j^{\mathbb{Q}_{\geq 0}}\}, \{z_k^{\mathbb{Q}_{\geq 0}}\} \rangle / \left(\prod_{k=1}^n z_k - \pi \right)$$

$$\rightarrow R_{\infty}^{\prime 0} := \mathcal{O}_K \langle \{x_i^{\pm 1/p^{\infty}}\}, \{y_j^{\mathbb{Q}_{\geq 0}}\}, \{z_k^{\mathbb{Q}_{\geq 0}}\} \rangle / \left(\prod_{k=1}^n z_k - \pi \right)$$

is a quasi-syntomic cover. Let $\mathcal{O}_{K_\infty}$ denote the p -adic completion of

$$\bigcup_{s \geq 1} \mathcal{O}_K[\pi^{1/s}] (\subset \mathcal{O}_C),$$

and consider a map $\mathcal{O}_{K_\infty} \rightarrow R_\infty^0$ mapping $\pi^{1/s}$ to $\prod_{k=1}^n z_k^{1/s}$. Then $R_\infty^0 \widehat{\otimes}_{\mathcal{O}_{K_\infty}} \mathcal{O}_C$ is naturally isomorphic to $\widetilde{R}_\infty^0$, and so $R_\infty^0 \rightarrow \widetilde{R}_\infty^0$ is a quasi-syntomic cover. This proves the claim. $\square$

Definition 5.2.4 (Semi-stable log formal schemes). A p -adic log formal scheme $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ over $\mathcal{O}_K$ is called *horizontally semi-stable* (resp. *vertically semi-stable*) if $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ admits a framing with $n = 0$ (resp. $n \geq 1$) étale locally, and $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is called *semi-stable* if it admits a framing étale locally. When $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is horizontally semi-stable (resp. vertically semi-stable), we let $\mathcal{M}_{\mathcal{O}_K}$ denote the trivial (resp. standard) log structure on $\mathrm{Spf}(\mathcal{O}_K)$. Then $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow (\mathrm{Spf}(\mathcal{O}_K), \mathcal{M}_{\mathcal{O}_K})$ is log smooth.

Definition 5.2.5 (Unramified models). Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a horizontally semi-stable log formal scheme over $\mathcal{O}_K$. A horizontally semi-stable log formal scheme $(\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})$ over W with $(\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0}) \otimes_W \mathcal{O}_K \cong (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is called an *unramified model* of $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$.

Let $(\mathrm{Spf}(R), \mathcal{M}_R)$ be a small affine log formal scheme over $\mathcal{O}_K$ with a fixed framing such that $n = 0$. Consider a prelog ring $(R_0^0, \mathbb{N}^r)$ defined as follows:

$$R_0^0 := W \langle x_1^{\pm 1}, \dots, x_l^{\pm 1}, y_1, \dots, y_m \rangle$$

equipped with a prelog structure $\mathbb{N}^r \rightarrow R_0^0$ given by $e_i \mapsto y_i$. Note that now we have $m = r$ by $n = 0$. The framing $(\mathrm{Spf}(R), \mathcal{M}_R) \rightarrow (\mathrm{Spf}(R_0^0), \mathbb{N}^r)^a$ uniquely descends to a strict étale map $(\mathrm{Spf}(R_0), \mathcal{M}_{R_0}) \rightarrow (\mathrm{Spf}(R_0^0), \mathbb{N}^r)^a$. Then $(\mathrm{Spf}(R_0), \mathcal{M}_{R_0})$ is an unramified model of $(\mathrm{Spf}(R), \mathcal{M}_R)$. We call it the *unramified model defined by the framing*. The ring map $R_0^0 \rightarrow R_0^0$ given by $x_i \mapsto x_i^p, y_j \mapsto y_j^p$ and the monoid map $\times p: \mathbb{N}^r \rightarrow \mathbb{N}^r$ gives a Frobenius lift on the log formal scheme $(\mathrm{Spf}(R_0^0), \mathbb{N}^r)^a$, which uniquely lifts to a Frobenius lift on $(\mathrm{Spf}(R_0), \mathcal{M}_{R_0})$.

5.3. F -isocrystals on log crystalline sites

5.3.1. Log crystalline sites. In this subsection, we review fundamental properties of crystals on absolute log crystalline sites.

Definition 5.3.1 (Big absolute log crystalline sites). Let $(X, \mathcal{M}_X)$ be a log scheme over $\mathbb{F}_p$. Let $(X, \mathcal{M}_X)_{\mathrm{crys}}$ be the site defined as follows: an object consists of

- a log scheme $(T, \mathcal{M}_T)$ on which p is nilpotent and a PD-thickening $U \hookrightarrow T$ such that the PD-structure on $\mathrm{Ker}(\mathcal{O}_T \twoheadrightarrow \mathcal{O}_U)$ is compatible with the natural PD-structure on $(p) \subset \mathbb{Z}_p$,
- a map of log schemes $f: (U, \mathcal{M}_U) \rightarrow (X, \mathcal{M}_X)$, where $\mathcal{M}_U$ is the inverse image log structure of $\mathcal{M}_T$.

Morphisms are obvious ones. We simply write $(U, T, \mathcal{M}_T)$ or $(T, \mathcal{M}_T)$ for an object of $(X, \mathcal{M}_X)_{\mathrm{crys}}$ when there exists no confusion. When $T = \mathrm{Spec}(R)$ and $U = \mathrm{Spec}(R/I)$, we write $(R \twoheadrightarrow R/I, \mathcal{M}_R)$ for a log PD-thickening $(U, T, \mathcal{M}_T)$. We say that a map $(U_1, T_1, \mathcal{M}_{T_1}) \rightarrow (U_2, T_2, \mathcal{M}_{T_2})$ in $(X, \mathcal{M}_X)_{\mathrm{crys}}$ is *Cartesian* if the natural map $U_1 \rightarrow U_2 \times_{T_2} T_1$ is an isomorphism. A family of Cartesian maps $\{(U_i, T'_i, \mathcal{M}_{T'_i}) \rightarrow (U, T, \mathcal{M}_T)\}_i$ is covering if $\{(T'_i, \mathcal{M}_{T'_i}) \rightarrow (T, \mathcal{M}_T)\}_i$ is a strict fpqc covering. For a morphism $f: (Y, \mathcal{M}_Y) \rightarrow (X, \mathcal{M}_X)$, there exists a natural morphism of sites

$$(X, \mathcal{M}_X)_{\mathrm{crys}} \rightarrow (Y, \mathcal{M}_Y)_{\mathrm{crys}}.$$

Let $\mathcal{O}_{\text{crys}}$ denote the ring sheaf given by $(T, \mathcal{M}_T) \mapsto \Gamma(T, \mathcal{O}_T)$.

Let $(X, \mathcal{M}_X)_{\text{crys}}^{\text{str}}$ denote the full subcategory of $(X, \mathcal{M}_X)_{\text{crys}}$ consisting of an object $(U, T, \mathcal{M}_T)$ such that the structure map $(U, \mathcal{M}_U) \rightarrow (X, \mathcal{M}_X)$ is strict.

Remark 5.3.2. In many references, the site $(X, \mathcal{M}_X)_{\text{crys}}^{\text{str}}$ is called an (absolute) log crystalline site.

Definition 5.3.3 (Crystalline crystals on log schemes).

Let $\text{Crys}((X, \mathcal{M}_X)_{\text{crys}})$ be the category of crystals of quasi-coherent sheaves of finite type on the log crystalline site $(X, \mathcal{M}_X)_{\text{crys}}$. An object of $\text{Crys}((X, \mathcal{M}_X)_{\text{crys}})$ is called a *crystalline crystal* on $(X, \mathcal{M}_X)$. The Frobenius morphism $F: (X, \mathcal{M}_X) \rightarrow (X, \mathcal{M}_X)$ induces a functor

$$F^*: \text{Crys}((X, \mathcal{M}_X)_{\text{crys}}) \rightarrow \text{Crys}((X, \mathcal{M}_X)_{\text{crys}}).$$

Let $\text{Vect}((X, \mathcal{M}_X)_{\text{crys}})$ denote the full subcategory of the category $\text{Crys}((X, \mathcal{M}_X)_{\text{crys}})$ consisting of crystals of vector bundles on $(X, \mathcal{M}_X)_{\text{crys}}$.

Let $\text{Isoc}((X, \mathcal{M}_X)_{\text{crys}})$ denote the isogeny category $\text{Crys}((X, \mathcal{M}_X)_{\text{crys}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. An object $\mathcal{E}$ of $\text{Isoc}((X, \mathcal{M}_X)_{\text{crys}})$ is called an *isocrystal*. For $\mathcal{E}^+ \in \text{Crys}((X, \mathcal{M}_X)_{\text{crys}})$, the image of $\mathcal{E}^+$ by the natural functor

$$\text{Crys}((X, \mathcal{M}_X)_{\text{crys}}) \rightarrow \text{Isoc}((X, \mathcal{M}_X)_{\text{crys}})$$

is denoted by $\mathcal{E}^+[1/p]$. For a p -adic log PD-thickening $(\text{Spec}(A/I), \text{Spf}(A), \mathcal{M}_A) \in (X, \mathcal{M}_X)$, there is an evaluation functor

$$\text{Isoc}((X, \mathcal{M}_X)_{\text{crys}}) \rightarrow \text{Mod}(A[1/p]).$$

The image of an isocrystal $\mathcal{E}$ via this functor is simply denoted by $\mathcal{E}_A$. An isocrystal $\mathcal{E}$ is called *locally free* if $\mathcal{E}_A$ is a finite projective $A[1/p]$ -module for every $(\text{Spec}(A/I), \text{Spf}(A), \mathcal{M}_A)$. The full subcategory of $\text{Isoc}((X, \mathcal{M}_X)_{\text{crys}})$ consisting of locally free objects is denoted by $\text{Isoc}_{\text{lf}}((X, \mathcal{M}_X)_{\text{crys}})$.

Let $\text{Vect}^\varphi((X, \mathcal{M}_X)_{\text{crys}})$ be the category of pairs $(\mathcal{E}, \varphi_{\mathcal{E}})$ consisting of a locally free crystalline crystal $\mathcal{E}$ on $(X, \mathcal{M}_X)$ and an isomorphism $\varphi_{\mathcal{E}}: F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ in the isogeny category $\text{Crys}((X, \mathcal{M}_X)_{\text{crys}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Morphisms $(\mathcal{E}_1, \varphi_{\mathcal{E}_1}) \rightarrow (\mathcal{E}_2, \varphi_{\mathcal{E}_2})$ are morphisms $\mathcal{E}_1 \rightarrow \mathcal{E}_2$ that are compatible with $\varphi_{\mathcal{E}_1}$ and $\varphi_{\mathcal{E}_2}$. When no confusion occurs, we simply write $\mathcal{E}$ for $(\mathcal{E}, \varphi_{\mathcal{E}})$.

Let $\text{Isoc}^\varphi((X, \mathcal{M}_X)_{\text{crys}})$ be the category of pairs $(\mathcal{E}, \varphi_{\mathcal{E}})$ consisting of an isocrystal $\mathcal{E}$ on $(X, \mathcal{M}_X)$ and an isomorphism $\varphi_{\mathcal{E}}: F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ of isocrystals. Morphisms are ones in the category of isocrystals that are compatible with Frobenius isomorphisms. When no confusion occurs, we simply write $\mathcal{E}$ for $(\mathcal{E}, \varphi_{\mathcal{E}})$. An object of $\text{Isoc}^\varphi((X, \mathcal{M}_X)_{\text{crys}})$ is called an *F-isocrystal* on $(X, \mathcal{M}_X)$. An *F-isocrystal* is called *locally free* if its underlying isocrystal is locally free, and the full subcategory of $\text{Isoc}^\varphi((X, \mathcal{M}_X)_{\text{crys}})$ consisting of locally free objects is denoted by $\text{Isoc}_{\text{lf}}^\varphi((X, \mathcal{M}_X)_{\text{crys}})$.

A *crystalline Dieudonné crystal* on $(X, \mathcal{M}_X)$ is a triple consisting of $\mathcal{E} \in \text{Vect}((X, \mathcal{M}_X)_{\text{crys}})$ and maps $\varphi_{\mathcal{E}}: F^*\mathcal{E} \rightarrow \mathcal{E}$, $\psi_{\mathcal{E}}: \mathcal{E} \rightarrow F^*\mathcal{E}$ such that the composite $\mathcal{E} \xrightarrow{\psi_{\mathcal{E}}} F^*\mathcal{E} \xrightarrow{\varphi_{\mathcal{E}}} \mathcal{E}$ coincides with the multiplication by p . The category of crystalline Dieudonné crystals on $(X, \mathcal{M}_X)$ is denoted by $\text{DM}((X, \mathcal{M}_X)_{\text{crys}})$.

Lemma 5.3.4. For an integral log scheme $(X, \mathcal{M}_X)$ over $\mathbb{F}_p$, the inclusion functor

$$(X, \mathcal{M}_X)_{\text{crys}}^{\text{str}} \hookrightarrow (X, \mathcal{M}_X)_{\text{crys}}$$

has a right adjoint functor

$$(X, \mathcal{M}_X)_{\text{crys}} \rightarrow (X, \mathcal{M}_X)_{\text{crys}}^{\text{str}} \quad ((U, T, \mathcal{M}_T) \mapsto (U, T, \mathcal{N}_T)).$$

Proof. Let $(U, T, \mathcal{M}_T) \in (X, \mathcal{M}_X)_{\text{crys}}^{\text{str}}$ and $f: (U, \mathcal{M}_U) \rightarrow (X, \mathcal{M}_X)$ denote the structure morphism. We define a log structure $\mathcal{N}_T$ on T by

$$\mathcal{N}_T := \mathcal{M}_T \times_{\mathcal{M}_U} f^* \mathcal{M}_X,$$

under the identification $T_{\text{ét}} \simeq U_{\text{ét}}$. Since $(X, \mathcal{M}_X)$ is integral, the surjection $\mathcal{M}_T \twoheadrightarrow \mathcal{M}_U$ is $1 + \mathcal{I}$ -torsor, where $\mathcal{I} := \text{Ker}(\mathcal{O}_T \twoheadrightarrow \mathcal{O}_U)$. Therefore, $(U, f^* \mathcal{M}_X) \hookrightarrow (T, \mathcal{N}_T)$ is strict, and $(U, T, \mathcal{N}_T)$ belongs to $(X, \mathcal{M}_X)_{\text{crys}}^{\text{str}}$. Then the functor sending $(U, T, \mathcal{M}_T)$ to $(U, T, \mathcal{N}_T)$ is the desired one. $\square$

For a morphism of integral log schemes $(Y, \mathcal{M}_Y) \rightarrow (X, \mathcal{M}_X)$, we have a functor

$$(Y, \mathcal{M}_Y)_{\text{crys}}^{\text{str}} \hookrightarrow (Y, \mathcal{M}_Y)_{\text{crys}} \rightarrow (X, \mathcal{M}_X)_{\text{crys}} \rightarrow (X, \mathcal{M}_X)_{\text{crys}}^{\text{str}},$$

where the last functor is one given in Lemma 5.3.4. This gives a morphism of site

$$(X, \mathcal{M}_X)_{\text{crys}}^{\text{str}} \rightarrow (Y, \mathcal{M}_Y)_{\text{crys}}^{\text{str}},$$

which induces pullback functors

$$\begin{aligned} f^* &: \text{Crys}((X, \mathcal{M}_X)_{\text{crys}}^{\text{str}}) \rightarrow \text{Crys}((Y, \mathcal{M}_Y)_{\text{crys}}^{\text{str}}), \\ f^* &: \text{Vect}((X, \mathcal{M}_X)_{\text{crys}}^{\text{str}}) \rightarrow \text{Vect}((Y, \mathcal{M}_Y)_{\text{crys}}^{\text{str}}), \\ f^* &: \text{Isoc}((X, \mathcal{M}_X)_{\text{crys}}^{\text{str}}) \rightarrow \text{Isoc}((Y, \mathcal{M}_Y)_{\text{crys}}^{\text{str}}). \end{aligned}$$

This functoriality allows us to define variants

$$\text{Vect}^\varphi((X, \mathcal{M}_X)_{\text{crys}}^{\text{str}}), \text{Isoc}^\varphi((X, \mathcal{M}_X)_{\text{crys}}^{\text{str}}), \text{DM}((X, \mathcal{M}_X)_{\text{crys}}^{\text{str}})$$

by replacing $(X, \mathcal{M}_X)_{\text{crys}}$ with a strict site $(X, \mathcal{M}_X)_{\text{crys}}^{\text{str}}$ in Definition 5.3.3.

Remark 5.3.5. For $\star \in \{\text{str}, \emptyset\}$, we have canonical bi-exact equivalences

$$\begin{aligned} \text{Crys}((X, \mathcal{M}_X)_{\text{crys}}^\star) &\simeq \varprojlim_{(T, \mathcal{M}_T) \in (X, \mathcal{M}_X)_{\text{crys}}^\star} \text{FQCoh}(T), \\ \text{Vect}((X, \mathcal{M}_X)_{\text{crys}}^\star) &\simeq \varprojlim_{(T, \mathcal{M}_T) \in (X, \mathcal{M}_X)_{\text{crys}}^\star} \text{Vect}(T). \end{aligned}$$

Remark 5.3.6. Let $(X, \mathcal{M}_X)$ be an integral log scheme over $\mathbb{F}_p$. For a crystalline crystal $\mathcal{E}$ on $(X, \mathcal{M}_X)$ and a log PD-thickening $(U, T, \mathcal{M}_T) \in (X, \mathcal{M}_X)_{\text{crys}}$, the crystal property gives an isomorphism

$$\mathcal{E}_{(U, T, \mathcal{M}_T)} \cong \mathcal{E}_{(U, T, \mathcal{N}_T)},$$

where $(U, T, \mathcal{N}_T)$ is the image of $(U, T, \mathcal{M}_T)$ via the functor in Lemma 5.3.4. Therefore, the restriction functors

$$\begin{aligned} \text{Crys}((X, \mathcal{M}_X)_{\text{crys}}) &\rightarrow \text{Crys}((X, \mathcal{M}_X)_{\text{crys}}^{\text{str}}), \\ \text{Vect}^*(X, \mathcal{M}_X)_{\text{crys}} &\rightarrow \text{Vect}^*(X, \mathcal{M}_X)_{\text{crys}}^{\text{str}}, \\ \text{Isoc}^*(X, \mathcal{M}_X)_{\text{crys}} &\rightarrow \text{Isoc}^*(X, \mathcal{M}_X)_{\text{crys}}^{\text{str}}, \\ \text{DM}((X, \mathcal{M}_X)_{\text{crys}}) &\rightarrow \text{DM}((X, \mathcal{M}_X)_{\text{crys}}^{\text{str}}), \end{aligned}$$

give equivalences for $\star \in \{\emptyset, \varphi\}$.

Lemma 5.3.7 (Dwork's trick). Let $\iota: (X, \mathcal{M}_X) \hookrightarrow (X', \mathcal{M}_{X'})$ be a strict nilpotent immersion of log schemes over $\mathbb{F}_p$.

(1) The restriction functor

$$\iota^*: \text{Isoc}^\varphi((X', \mathcal{M}_{X'})_{\text{crys}}) \rightarrow \text{Isoc}^\varphi((X, \mathcal{M}_X)_{\text{crys}})$$

is an equivalence.

(2) Let $\mathcal{E}$ be an F -isocrystal on $(X', \mathcal{M}_{X'})$ and $(A \twoheadrightarrow A/I, \mathcal{M}_A)$ be a p -adic log PD-thickening in $(X', \mathcal{M}_{X'})_{\text{crys}}$. Suppose that ι has a retraction $r: (X', \mathcal{M}_{X'}) \rightarrow (X, \mathcal{M}_X)$. Let $(A \twoheadrightarrow A/I, \mathcal{M}_A)^r := (ir)^{-1}(A \twoheadrightarrow A/I, \mathcal{M}_A)$. Then there exists a natural isomorphism

$$\mathcal{E}_{(A \twoheadrightarrow A/I, \mathcal{M}_A)^r} \cong \mathcal{E}_{(A \twoheadrightarrow A/I, \mathcal{M}_A)}.$$

Proof. (1): See [DLMS24, Remark B.23].

(2): Take an integer $n \geq 1$ such that the iterated Frobenius map $F^n: (X', \mathcal{M}_{X'}) \rightarrow (X', \mathcal{M}_{X'})$ factors through $(X, \mathcal{M}_X) \hookrightarrow (X', \mathcal{M}_{X'})$. Then, by $irF^n = F^n$, we have isomorphisms

$$\mathcal{E}_{(A \twoheadrightarrow A/I, \mathcal{M}_A)} \cong \mathcal{E}_{(F^n)^{-1}(A \twoheadrightarrow A/I, \mathcal{M}_A)} = \mathcal{E}_{(F^n)^{-1}(A \twoheadrightarrow A/I, \mathcal{M}_A)^r} \cong \mathcal{E}_{(A \twoheadrightarrow A/I, \mathcal{M}_A)^r},$$

where the first and third isomorphisms are induced by the Frobenius structure on $\mathcal{E}$. We can check directly that the resulting isomorphism $\mathcal{E}_{(A \twoheadrightarrow A/I, \mathcal{M}_A)} \cong \mathcal{E}_{(A \twoheadrightarrow A/I, \mathcal{M}_A)^r}$ is independent of the choice of n . $\square$

Remark 5.3.8. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a horizontally semi-stable log formal scheme over W . We set $(X, \mathcal{M}_X) := (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \otimes_W k$, $(X', \mathcal{M}_{X'}) := (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \otimes_W \mathcal{O}_K/p$. Then the natural map $k \rightarrow \mathcal{O}_K/p$ induces a retraction $r: (X', \mathcal{M}_{X'}) \rightarrow (X, \mathcal{M}_X)$ of the natural inclusion $\iota: (X, \mathcal{M}_X) \rightarrow (X', \mathcal{M}_{X'})$, and so the above lemma can be applied.

The following notion plays an important role when we clarify the relationship between isocrystals and modules with connection.

Definition 5.3.9 (cf. [ES18, p.2117]). Let $\mathcal{E}$ be an object of $\text{Crys}((X, \mathcal{M}_X)_{\text{crys}})$. We say that $\mathcal{E}$ is p -torsion free if, étale locally on X , there exist a p -adic log PD-thickening $(\mathcal{D}, \mathcal{M}_{\mathcal{D}}) \in (X, \mathcal{M}_X)_{\text{crys}}^{\text{str}}$ satisfying the following conditions:

- $\mathcal{D}$ is $\mathbb{Z}_p$ -flat ;
- $(\mathcal{D}, \mathcal{M}_{\mathcal{D}})$ covers the final object of the topos associated with $(X, \mathcal{M}_X)_{\text{crys}}$;
- there exists an $n+1$ -fold self product $(\mathcal{D}^{(n)}, \mathcal{M}_{\mathcal{D}^{(n)}})$ of $(\mathcal{D}, \mathcal{M}_{\mathcal{D}})$ in $(X, \mathcal{M}_X)_{\text{crys}}$ for each $n \geq 0$;
- $\mathcal{E}_{\mathcal{D}^{(n)}}$ is p -torsion free for each $n \geq 0$.

Let $\text{Crys}_{\text{tf}}((X, \mathcal{M}_X)_{\text{crys}})$ be the full subcategory of $\text{Crys}((X, \mathcal{M}_X)_{\text{crys}})$ consisting of p -torsion free objects. Let $\text{Isoc}_{\text{tf}}((X, \mathcal{M}_X)_{\text{crys}})$ denote the essential image of the natural functor

$$\text{Crys}_{\text{tf}}((X, \mathcal{M}_X)_{\text{crys}}) \hookrightarrow \text{Crys}((X, \mathcal{M}_X)_{\text{crys}}) \rightarrow \text{Isoc}((X, \mathcal{M}_X)_{\text{crys}}).$$

We say that an isocrystal $\mathcal{E}$ is p -torsion free if $\mathcal{E}$ belongs to $\text{Isoc}_{\text{tf}}((X, \mathcal{M}_X)_{\text{crys}})$ as well. Let $\text{Isoc}_{\text{lftf}}((X, \mathcal{M}_X)_{\text{crys}})$ denote the full subcategory of $\text{Isoc}((X, \mathcal{M}_X)_{\text{crys}})$ consisting of locally free and p -torsion free objects and $\text{Isoc}_{\text{lftf}}^\varphi((X, \mathcal{M}_X)_{\text{crys}})$ denote the category of F -isocrystals whose underlying isocrystals belong to $\text{Isoc}_{\text{lftf}}((X, \mathcal{M}_X)_{\text{crys}})$.

Lemma 5.3.10. Suppose that we can take $(\mathcal{D}, \mathcal{M}_{\mathcal{D}})$ in Definition 5.3.9 such that $\mathcal{D}$ is noetherian. Then every isocrystal on $(X, \mathcal{M}_X)$ is p -torsion free.

Proof. Let $\mathcal{E}$ be an isocrystal on $(X, \mathcal{M}_X)$, and take $\mathcal{E}^+ \in \text{Crys}((X, \mathcal{M}_X)_{\text{crys}})$ with $\mathcal{E} \cong \mathcal{E}^+[1/p]$. Then $\mathcal{E}^+$ corresponds to a quasi-coherent $\mathcal{O}_{\mathcal{D}}$ -module $\mathcal{M} := \mathcal{E}_A^+$ equipped with

HPD-stratification. Let $\overline{\mathcal{M}}$ be the maximal p -torsion free quotient of $\mathcal{M}$. Since $\mathcal{D}^{(1)} \rightarrow \mathcal{D}$ is flat, $\overline{\mathcal{M}}$ has the induced HPD-stratification (see Lemma 5.6.12 below). The $\mathcal{O}_{\mathcal{D}}$ -module $\overline{\mathcal{M}}$ equipped with HPD-stratification defines an object $\overline{\mathcal{E}}^+ \in \text{Crys}((X, \mathcal{M}_X)_{\text{crys}})$. Since A is noetherian, M and $\overline{M}$ is isomorphic in the isogeny category $\text{Mod}(\mathcal{O}_{\mathcal{D}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Therefore, we get $\mathcal{E} \cong \mathcal{E}^+[1/p] \cong \overline{\mathcal{E}}^+[1/p]$. This proves that $\mathcal{E}$ is p -torsion free. $\square$

Remark 5.3.11. By the above proposition, when $(X, \mathcal{M}_X)$ is log smooth over k , every isocrystal on $(X, \mathcal{M}_X)$ is p -torsion free. Indeed, a log smooth formal lift $(\mathcal{D}, \mathcal{M}_{\mathcal{D}})$ of $(X, \mathcal{M}_X)$ is enough.

5.3.2. Isocrystals and connections. In this subsection, we recall the relation between crystals and modules with integrable connections studied in [Kat89, Section 6].

Let $(\mathfrak{S}, \mathcal{M}_{\mathfrak{S}})$ be a p -adic fine noetherian $\mathbb{Z}_p$ -flat log formal scheme with a PD-ideal $\mathcal{I}$ containing p that is compatible with the natural PD-structure on $(p) \subset \mathbb{Z}_p$. Let $(S, \mathcal{M}_S)$ be the strict closed subscheme of $(\mathfrak{S}, \mathcal{M}_{\mathfrak{S}})$ defined by $\mathcal{I}$. Consider a fine log scheme $(X, \mathcal{M}_X)$ over $(S, \mathcal{M}_S)$. Suppose that there is a strict closed immersion $(X, \mathcal{M}_X) \hookrightarrow (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$, where $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ is a p -adic fine log formal scheme that is log smooth and integral over $(\mathfrak{S}, \mathcal{M}_{\mathfrak{S}})$. Let $(\mathcal{D}, \mathcal{M}_{\mathcal{D}})$ be the log PD-envelope of $(X, \mathcal{M}_X) \hookrightarrow (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ and $(\mathcal{D}^{(n)}, \mathcal{M}_{\mathcal{D}^{(n)}})$ be the log PD-envelope of the diagonal map

$$(X, \mathcal{M}_X) \hookrightarrow \underbrace{(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \times_{(\mathfrak{S}, \mathcal{M}_{\mathfrak{S}})} \cdots \times_{(\mathfrak{S}, \mathcal{M}_{\mathfrak{S}})} (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})}_{n+1 \text{ copies of } (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})}$$

for $n \geq 1$.

Construction 5.3.12. Let $\mathcal{J}$ denote the kernel of the surjection $\mathcal{O}_{\mathcal{D}^{(1)}} \twoheadrightarrow \mathcal{O}_{\mathcal{D}}$ induced from the diagonal map $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \hookrightarrow (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \times_{(\mathfrak{S}, \mathcal{M}_{\mathfrak{S}})} (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$. Suppose that we are given $t_1, \dots, t_s \in \Gamma(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ such that $\text{dlog}(t_1), \dots, \text{dlog}(t_s)$ forms a basis of $\Omega_{(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})/(\mathfrak{S}, \mathcal{M}_{\mathfrak{S}})}^1$ (this is the case étale locally on $\mathfrak{Y}$). By [Kat89, Proposition 6.5], there is an isomorphism of $\mathcal{O}_{\mathcal{D}}$ -algebras

$$\mathcal{O}_{\mathcal{D}}\{\xi_1, \dots, \xi_s\} \xrightarrow{\sim} \mathcal{O}_{\mathcal{D}^{(1)}}$$

given by $\xi_i \mapsto p_1^*(t_i)p_2^*(t_i)^{-1} - 1$, where $\mathcal{O}_{\mathcal{D}^{(1)}}$ is viewed as an $\mathcal{O}_{\mathcal{D}}$ -algebra via p_2 and $\mathcal{O}_{\mathcal{D}}\{\xi_1, \dots, \xi_s\}$ is the p -completed PD-polynomial ring with variables $\xi_1, \dots, \xi_s$ over $\mathcal{O}_{\mathcal{D}}$.

Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_{\mathcal{D}}$ -module with a HPD-stratification $p_1^*\mathcal{F} \xrightarrow{\sim} p_2^*\mathcal{F}$. Then, by cutting off the HPD-stratification on $\mathcal{F}$ modulo $\mathcal{J}^{[\geq 2]}$, we obtain an integrable connection

$$\nabla: \mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \Omega_{(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})/(\mathfrak{S}, \mathcal{M}_{\mathfrak{S}})}^1.$$

We define a $\mathcal{O}_{\mathfrak{S}}$ -linear map $\nabla_{t_i \frac{\partial}{\partial t_i}}: \mathcal{F} \rightarrow \mathcal{F}$ as follows:

$$\nabla(x) = \sum_{i=1}^s \nabla_{t_i \frac{\partial}{\partial t_i}}(x) \otimes \text{dlog}(t_i) \quad (x \in \mathcal{F}).$$

More generally, let $\nabla_{t_i^n \frac{\partial^n}{\partial t_i^n}}: \mathcal{F} \rightarrow \mathcal{F}$ be the $\mathcal{O}_{\mathfrak{S}}$ -linear map defined by

$$\nabla_{t_i^n \frac{\partial^n}{\partial t_i^n}} := \prod_{j=0}^{n-1} (\nabla_{t_i \frac{\partial}{\partial t_i}} - j)$$

for $1 \leq i \leq s$ and $n \geq 1$. Then, by the proof of [Kat89, Theorem 6.2], the $\mathcal{O}_{\mathfrak{S}}$ -linear map

$$\eta: \mathcal{F} \rightarrow p_1^*\mathcal{F} \xrightarrow{\sim} p_2^*\mathcal{F}$$

defined from the HPD-stratification is written as follows:

$$\eta(x) = \sum_{\underline{n}=(n_1, \dots, n_s) \in \mathbb{N}^s} \nabla_{\underline{n}}(x) \otimes \xi^{\underline{n}} \quad (x \in \mathcal{F}),$$

where $\nabla_{\underline{n}} := \prod_{1 \leq i \leq s} \nabla_{t_i^{n_i} \frac{\partial^{n_i}}{\partial t_i^{n_i}}}$, and $\xi^{\underline{n}} := \prod_{1 \leq i \leq s} \xi_i^{[n_i]}$.

This procedure gives the following equivalence.

Proposition 5.3.13 (cf. [Ber74, Théorème 4.2.11], [Kat89, Theorem 6.2]). There is a natural equivalence between the following categories:

- the category of quasi-coherent $\mathcal{O}_{\mathcal{D}}$ -modules with HPD-stratification;
- the category of quasi-coherent $\mathcal{O}_{\mathcal{D}}$ -modules with topologically quasi-nilpotent integrable connection.

Here, an integrable connection $\mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \Omega_{(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})/(\mathfrak{S}, \mathcal{M}_{\mathfrak{S}})}^1$ is called *topologically quasi-nilpotent* if, for some (or equivalently any) local sections $t_1, \dots, t_s \in \Gamma(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ as in Construction 5.3.12, $x \in \mathcal{F}$, and $1 \leq i \leq s$, $\nabla_{t_i^{n_i} \frac{\partial^{n_i}}{\partial t_i^{n_i}}}(x)$ converges to 0 when $n \rightarrow \infty$.

Definition 5.3.14. We define the following categories.

- Let $\text{StrFQCoh}(\mathcal{D})$ be the category of quasi-coherent $\mathcal{O}_{\mathcal{D}}$ -modules of finite type equipped with HPD-stratification.
- Let $\text{FQCoh}^{\nabla}(\mathcal{D})$ be the category of quasi-coherent $\mathcal{O}_{\mathcal{D}}$ -modules $\mathcal{M}$ of finite type equipped with topologically quasi-nilpotent integrable connection.

Let $\mathcal{E} \in \text{Crys}((X, \mathcal{M}_X)_{\text{crys}})$. By the crystal property of $\mathcal{E}$, there are isomorphisms

$$p_1^* \mathcal{E}_{\mathcal{D}} \cong \mathcal{E}_{\mathcal{D}^{(1)}} \cong p_2^* \mathcal{E}_{\mathcal{D}},$$

where $p_i: (\mathcal{D}^{(1)}, \mathcal{M}_{\mathcal{D}^{(1)}}) \rightarrow (\mathcal{D}, \mathcal{M}_{\mathcal{D}})$ is the natural projection map for $i = 1, 2$. This isomorphism satisfies the cocycle condition on $\mathcal{D}^{(2)}$ and gives a HPD-stratification on $\mathcal{E}_{\mathcal{D}}$. This construction and the equivalence in Proposition 5.3.13 gives the following proposition.

Proposition 5.3.15. There is a natural functor

$$\text{Crys}((X, \mathcal{M}_X)_{\text{crys}}) \rightarrow \text{StrFQCoh}(\mathcal{D}) (\simeq \text{FQCoh}^{\nabla}(\mathcal{D})).$$

Corollary 5.3.16. Suppose that $(\mathfrak{S}, \mathcal{M}_{\mathfrak{S}}) = (\text{Spf}(W), \mathcal{M}_W)$ is the formal spectrum of W equipped with the trivial log structure and $\mathcal{I} = (p)$. Then there are natural equivalences

$$\text{Crys}((X, \mathcal{M}_X)_{\text{crys}}) \simeq \text{StrFQCoh}(\mathcal{D}) \simeq \text{FQCoh}(\mathcal{D}).$$

Proof. Since $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ is log smooth over W , the p -adic log PD-thickening $(\mathcal{D}, \mathcal{M}_{\mathcal{D}})$ covers the final object of the topos associated with $(X, \mathcal{M}_X)_{\text{crys}}$. Therefore, the functor in Proposition 5.3.15 gives an equivalence. $\square$

Next, we turn to studying isocrystals. We start with some lemmas on completions of modules.

Lemma 5.3.17. (cf. [DLMS23, Lemma 3.24(4)]) Let $A \rightarrow B$ be a ring map from a p -complete and p -torsion free ring A to a p -complete ring B . Let M be a finitely generated A -module. Suppose that $M[1/p]$ is finite projective over $A[1/p]$. Then there is a natural isomorphism

$$M[1/p] \otimes_A B \xrightarrow{\sim} (M \otimes_A B)^{\wedge}[1/p].$$

Proof. Since $M \otimes_A B$ is a finitely generated B -module, the natural map

$$M \otimes_A B \rightarrow (M \otimes_A B)^\wedge$$

is surjective. It suffices to show that the induced surjection $M[1/p] \otimes_A B \rightarrow (M \otimes_A B)^\wedge[1/p]$ is injective.

Since $M[1/p]$ is finite projective over $A[1/p]$, we can take a finite free A -module N and an injection $M \rightarrow N$ such that $M[1/p] \rightarrow N[1/p]$ admits a retract. Then the map $M \otimes_A B \rightarrow N \otimes_A B$ factors as

$$M \otimes_A B \rightarrow (M \otimes_A B)^\wedge \rightarrow N \otimes_A B.$$

Since $M[1/p] \rightarrow N[1/p]$ admits a retract, $M[1/p] \otimes_A B \rightarrow N[1/p] \otimes_A B$ is also injective. Therefore, $M[1/p] \otimes_A B \rightarrow (M \otimes_A B)^\wedge[1/p]$ is also injective. $\square$

Lemma 5.3.18. Let A be a p -complete and p -torsion free ring. Let M be a finitely generated p -torsion free A -module. Suppose that $M[1/p]$ is finite projective over $A[1/p]$. Then M is p -complete.

Proof. Since M is finitely generated, the natural map $M \rightarrow M^\wedge$ is surjective. To prove that this map is injective, it suffices to show that $M[1/p] \rightarrow M^\wedge[1/p]$ is an isomorphism due to the p -torsion freeness of M . The natural map

$$M^\wedge[1/p] = (\varprojlim_{n \geq 1} M/p^n M)[1/p] \rightarrow \varprojlim_{n \geq 1} M[1/p]/p^n M = (M[1/p])^\wedge$$

is an isomorphism. Since the topology on $M[1/p]$ is independent of the choice of a lattice, the assumption that $M[1/p]$ is finite projective over $A[1/p]$ implies that we have $(M[1/p])^\wedge \cong M[1/p]$. Therefore, we conclude that $M[1/p] \cong M^\wedge[1/p]$. $\square$

Definition 5.3.19. We define the following categories:

- Let $\text{StrVect}(\mathcal{D})$ be the category of p -torsion free quasi-coherent $\mathcal{O}_{\mathcal{D}}$ -module $\mathcal{M}^+$ such that $\mathcal{M}^+[1/p]$ is locally projective $\mathcal{O}_{\mathcal{D}}[1/p]$ -module equipped with a $\mathcal{O}_{\mathcal{D}(1)}$ -linear isomorphism

$$\epsilon: p_1^* \mathcal{M}^+ \xrightarrow{\sim} p_2^* \mathcal{M}^+;$$

- Let $\text{StrVect}(\mathcal{D}[1/p])$ be the category of locally finite projective $\mathcal{O}_{\mathcal{D}}[1/p]$ -modules $\mathcal{M}$ with a $\mathcal{O}_{\mathcal{D}(1)}[1/p]$ -linear isomorphism

$$\epsilon: p_1^* \mathcal{M} \xrightarrow{\sim} p_2^* \mathcal{M}$$

satisfying the cocycle condition over $\mathcal{D}^{(2)}$;

- Let $\text{Vect}^\nabla(\mathcal{D}[1/p])$ be the category of locally finite projective $\mathcal{O}_{\mathcal{D}}[1/p]$ -modules $\mathcal{M}$ equipped with a topologically quasi-nilpotent integrable connection

$$\nabla_{\mathcal{M}}: \mathcal{M} \rightarrow \mathcal{M} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \Omega_{(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})/(\mathfrak{S}, \mathcal{M}_{\mathfrak{S}})}^1.$$

Lemma 5.3.20. There is a natural equivalence

$$\text{StrVect}(\mathcal{D}[1/p]) \xrightarrow{\sim} \text{Vect}^\nabla(\mathcal{D}[1/p]).$$

Proof. Both of the source and the target satisfies étale descent by [Dri22, Proposition 2.6]. We may assume that every formal scheme is affine and there are sections $t_1, \dots, t_s \in \Gamma(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ such that $\text{dlog}(t_1), \dots, \text{dlog}(t_s)$ forms a basis of $\Omega_{(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})/(\mathfrak{S}, \mathcal{M}_{\mathfrak{S}})}^1$. Let $\mathfrak{Y} = \text{Spf}(B)$, $\mathcal{D} = \text{Spf}(D)$, and $\mathcal{D}^{(n)} = \text{Spf}(D^{(n)})$.

An object of the source category is described as a pair (M, ϵ) consisting of a finite projective $D[1/p]$ -module M and an $D^{(1)}[1/p]$ -linear isomorphism

$$\epsilon: M \otimes_{D, p_1} D^{(1)} \xrightarrow{\sim} M \otimes_{D, p_2} D^{(1)}$$

satisfying the cocycle condition over $D^{(2)}$. We have an exact sequence of D -modules

$$0 \rightarrow D \otimes_B \Omega_{(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})/(\mathfrak{S}, \mathcal{M}_{\mathfrak{S}})}^1 \rightarrow D^{(1)}/J^{[\geq 2]} \rightarrow D \rightarrow 0.$$

Hence, the map $M \rightarrow M \otimes_D D^{(1)}/J^{[\geq 2]}$ given by $x \mapsto \epsilon(x \otimes 1) - x \otimes 1$ defines a connection

$$\nabla_M: M \rightarrow M \otimes_B \Omega_{(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})/(\mathfrak{S}, \mathcal{M}_{\mathfrak{S}})}^1.$$

We define a map $\nabla_{t_i \frac{\partial}{\partial t_i}}: M \rightarrow M$ by $\nabla_M(x) = \sum_{i=1}^s \nabla_{t_i \frac{\partial}{\partial t_i}}(x) \otimes \text{dlog}(t_i)$ for $1 \leq i \leq s$.

Since M is finite projective over $D[1/p]$, there is an isomorphism

$$M \otimes_{D, p_2} D^{(1)} \cong \widehat{\bigoplus_{\underline{n}=(n_1, \dots, n_s) \in \mathbb{N}^s} M \cdot \xi^{[\underline{n}]}}$$

(see Construction 5.3.12). Then, in the same way as [Ber74, Théorème 4.2.11] or [Kat89, Theorem 6.2], we conclude that the map

$$M \rightarrow M \otimes_{D, p_1} D^{(1)} \xrightarrow{\epsilon} M \otimes_{D, p_2} D^{(1)} \cong \widehat{\bigoplus_{\underline{n}=(n_1, \dots, n_s) \in \mathbb{N}^s} M \cdot \xi^{[\underline{n}]}}$$

is described as

$$x \mapsto \sum_{\underline{n}=(n_1, \dots, n_s) \in \mathbb{N}^s} \nabla_{\underline{n}}(x) \otimes \xi^{[\underline{n}]}$$

under the same notation as Construction 5.3.12. This description allows us to check that (M, ∇_M) belongs to $\text{Vect}^{\nabla}(\mathcal{D}[1/p])$ and the functor $\text{StrVect}(\mathcal{D}[1/p]) \rightarrow \text{Vect}^{\nabla}(\mathcal{D}[1/p])$ given by $(M, \epsilon) \mapsto (M, \nabla_M)$ is an equivalence $\square$

Lemma 5.3.21. There is a natural equivalence

$$\text{StrVect}(\mathcal{D}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} \text{StrVect}(\mathcal{D}[1/p]),$$

where the source category is the isogeny category.

Proof. Both of the source and the target satisfies étale descent by [Dri22, Proposition 2.6 and Corollary 2.5]. We may assume that every formal scheme is affine and there are sections $t_1, \dots, t_s \in \Gamma(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ such that $\text{dlog}(t_1), \dots, \text{dlog}(t_s)$ forms a basis of $\Omega_{(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})/(\mathfrak{S}, \mathcal{M}_{\mathfrak{S}})}^1$. Let $\mathfrak{Y} = \text{Spf}(B)$, $\mathcal{D} = \text{Spf}(D)$, and $\mathcal{D}^{(n)} = \text{Spf}(D^{(n)})$.

By Lemma 5.3.17, inverting p gives the functor in the statement by Lemma 5.3.17. Note that, for a ring A and finitely generated p -torsion free A -modules M, N , the natural map

$$\text{Hom}_A(M, N)[1/p] \rightarrow \text{Hom}_{A[1/p]}(M[1/p], N[1/p])$$

is an isomorphism. Since $p_i: D \rightarrow D^{(1)}$ is p -adically flat, $M^+ \widehat{\otimes}_{D, p_i} D^{(1)}$ is p -torsion free for each object $(M^+, \epsilon) \in \text{StrVect}(D)$. Hence, the functor in the statement is fully faithful.

We use the notation in Construction 5.3.12 and the proof of Proposition 5.3.20. Let $(M, \epsilon) \in \text{StrVect}(D[1/p])$. To show the essential surjectivity, it is enough to prove that there exists a finitely generated D -submodule M^+ of M that is preserved by $\nabla_{t_i \frac{\partial}{\partial t_i}}$ for each $1 \leq i \leq s$. Fix a finitely generated D -submodule M_0^+ of M . By the convergence

condition, there exists an integer $N \geq 1$ such that $\nabla_{\underline{n}}(M_0^+)$ is contained in M_0^+ for every $\underline{n} \in \mathbb{N}^s \setminus [0, N]^s$. We put

$$M^+ := \sum_{\underline{n} \in [0, N]^s} \nabla_{\underline{n}}(M_0^+).$$

The formula $D \cdot \nabla_{\underline{n}}(M_0^+) \subset M_0^+ + \nabla_{\underline{n}}(M_0^+)$ for each $\underline{n} \in \mathbb{N}^s$ coming from the Leibniz rule implies that M^+ is a finitely generated D -submodule. The formula

$$\nabla_{t_i \frac{\partial}{\partial t_i}} \circ \nabla_{\underline{n}} = \nabla_{(n_1, \dots, n_i+1, \dots, n_s)} + n_i \nabla_{\underline{n}} \quad (\underline{n} \in \mathbb{N}^s, 1 \leq i \leq s)$$

implies that M^+ is preserved by $\nabla_{t_i \frac{\partial}{\partial t_i}}$ for any $1 \leq i \leq s$. Then the essential surjectivity follows. $\square$

Proposition 5.3.22. There is a natural functor

$$\text{Isoc}_{\text{lf}}((X, \mathcal{M}_X)_{\text{crys}}) \rightarrow \text{StrVect}(\mathcal{D}[1/p])(\simeq \text{Vect}^{\nabla}(\mathcal{D}[1/p])).$$

Proof. This can be defined in the same way as Proposition 5.3.15. $\square$

Proposition 5.3.23. Suppose that $(\mathfrak{S}, \mathcal{M}_{\mathfrak{S}})$ is $\text{Spf}(W)$ equipped with the trivial log structure and $\mathcal{I} = (p)$. Then there are natural equivalences

$$\text{Isoc}_{\text{lf}}((X, \mathcal{M}_X)_{\text{crys}}) \simeq \text{StrVect}(\mathcal{D}[1/p]) \simeq \text{Vect}^{\nabla}(\mathcal{D}[1/p]).$$

In particular, when $(X, \mathcal{M}_X)$ is log smooth over k , there are natural equivalences

$$\text{Isoc}_{\text{lf}}((X, \mathcal{M}_X)_{\text{crys}}) \simeq \text{StrVect}(\mathcal{D}[1/p]) \simeq \text{Vect}^{\nabla}(\mathcal{D}[1/p]).$$

Proof. Since the latter assertion follows from the former and Remark 5.3.11, it is enough to prove the former one. The restriction of the functor in Proposition 5.3.22 gives a functor

$$(5.3.1) \quad \text{ev}_{\mathcal{D}}: \text{Isoc}_{\text{lf}}((X, \mathcal{M}_X)_{\text{crys}}) \rightarrow \text{StrVect}(\mathcal{D}[1/p]).$$

We consider the inverse direction. Corollary 5.3.16 gives a fully faithful functor

$$\text{StrVect}(\mathcal{D}) \rightarrow \text{Crys}_{\text{tf}}((X, \mathcal{M}_X)_{\text{crys}}).$$

By passing to isogeny categories, we obtain a fully faithful functor

$$\text{StrVect}(\mathcal{D}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \rightarrow \text{Isoc}_{\text{lf}}((X, \mathcal{M}_X)_{\text{crys}}).$$

Taking the composition with the equivalence in Lemma 5.3.21 gives a fully faithful functor

$$(5.3.2) \quad \text{StrVect}(\mathcal{D}[1/p]) \rightarrow \text{Isoc}_{\text{lf}}((X, \mathcal{M}_X)_{\text{crys}}).$$

The composite

$$\text{StrVect}(\mathcal{D}[1/p]) \rightarrow \text{Isoc}_{\text{lf}}((X, \mathcal{M}_X)_{\text{crys}}) \rightarrow \text{StrVect}(\mathcal{D}[1/p])$$

is naturally isomorphic to the identity functor by definition. Hence, in order to show that functors (5.3.1) and (5.3.2) are inverse to each other, it suffices to verify that the functor (5.3.1) is fully faithful. Let $\mathcal{E}_1^+, \mathcal{E}_2^+ \in \text{Crys}_{\text{tf}}((X, \mathcal{M}_X)_{\text{crys}})$ with $\mathcal{E}_1^+[1/p]$ and $\mathcal{E}_2^+[1/p]$ being locally free. Take a p -adic log PD-thickening $(\mathcal{D}_2, \mathcal{M}_{\mathcal{D}_2}) \in (X, \mathcal{M}_X)_{\text{crys}}$ satisfying conditions in Definition 5.3.9 for a torsion free crystalline crystal $\mathcal{E}_2^+$. We can define a category $\text{StrVect}(\mathcal{D}_2[1/p])$ and a functor $\text{ev}_{\mathcal{D}_2}$ in a similar way to $\text{StrVect}(\mathcal{D}[1/p])$ and $\text{ev}_{\mathcal{D}}$, and there is the following commutative diagram:

$$\begin{array}{ccc} \text{Isoc}_{\text{lf}}((X, \mathcal{M}_X)_{\text{crys}}) & \xrightarrow{\text{ev}_{\mathcal{D}}} & \text{StrVect}(\mathcal{D}[1/p]) \\ & \searrow \text{ev}_{\mathcal{D}_2} & \downarrow \sim \\ & & \text{StrVect}(\mathcal{D}_2[1/p]). \end{array}$$

Here, the right vertical equivalence is defined from the fact that both of $(\mathcal{D}, \mathcal{M}_{\mathcal{D}})$ and $(\mathcal{D}_2, \mathcal{M}_{\mathcal{D}_2})$ cover the final object of the topos associated with $(X, \mathcal{M}_X)_{\text{crys}}$. The natural map

$$\text{Hom}_{\text{Isoc}((X, \mathcal{M}_X)_{\text{crys}})}(\mathcal{E}_1^+[1/p], \mathcal{E}_2^+[1/p]) \rightarrow \text{Hom}_{\text{StrVect}(\mathcal{D}_2[1/p])}(\text{ev}_{\mathcal{D}_2}(\mathcal{E}_1^+[1/p]), \text{ev}_{\mathcal{D}_2}(\mathcal{E}_2^+[1/p]))$$

is an isomorphism because of the following general claim: for a ring A , a finitely generated A -module M , and a p -torsion free A -module N , the natural map

$$\text{Hom}_A(M, N)[1/p] \rightarrow \text{Hom}_{A[1/p]}(M[1/p], N[1/p])$$

is an isomorphism. Therefore the natural map

$$\text{Hom}_{\text{Isoc}((X, \mathcal{M}_X)_{\text{crys}})}(\mathcal{E}_1^+[1/p], \mathcal{E}_2^+[1/p]) \rightarrow \text{Hom}_{\text{StrVect}(\mathcal{D}[1/p])}(\text{ev}_{\mathcal{D}}(\mathcal{E}_1^+[1/p]), \text{ev}_{\mathcal{D}}(\mathcal{E}_2^+[1/p]))$$

is also an isomorphism. This proves the fully faithfulness of the functor (5.3.1). $\square$

Remark 5.3.24. Under the assumption of Proposition 5.3.23, we have functors

$$\text{Isoc}_{\text{lf}}((X, \mathcal{M}_X)_{\text{crys}}) \rightarrow \text{StrVect}(\mathcal{D}[1/p])$$

given in Proposition 5.3.22 and

$$\text{StrVect}(\mathcal{D}[1/p]) \rightarrow \text{Isoc}_{\text{lf}}((X, \mathcal{M}_X)_{\text{crys}})$$

given in Proposition 5.3.23 (or its proof). Although the composition

$$\text{StrVect}(\mathcal{D}[1/p]) \rightarrow \text{Isoc}_{\text{lf}}((X, \mathcal{M}_X)_{\text{crys}}) \rightarrow \text{StrVect}(\mathcal{D}[1/p])$$

is isomorphic to the identity functor, we do not know whether the composition

$$\text{Isoc}_{\text{lf}}((X, \mathcal{M}_X)_{\text{crys}}) \rightarrow \text{StrVect}(\mathcal{D}[1/p]) \rightarrow \text{Isoc}_{\text{lf}}((X, \mathcal{M}_X)_{\text{crys}})$$

is isomorphic to the identity functor. In particular, we do not know whether the category $\text{Isoc}_{\text{lf}}((X, \mathcal{M}_X)_{\text{crys}})$ is indeed larger than $\text{Isoc}_{\text{lf}}((X, \mathcal{M}_X)_{\text{crys}})$.

Definition 5.3.25. Let $(X, \mathcal{M}_X)$ be an fs log smooth log adic space over K . We have the sheaf of differentials $\Omega_{(X, \mathcal{M}_X)/K}^1$ defined in [DLLZ23b, Definition 3.3.6]. Let $\text{MIC}(X, \mathcal{M}_X)$ denote the category of vector bundles with integrable connection on $(X, \mathcal{M}_X)$.

Proposition 5.3.26. Let $\mathcal{M}_{\mathcal{O}_K}$ be a log structure on $\text{Spf}(\mathcal{O}_K)$ which is either of the trivial log structure or the log structure associated with the prelog structure $\mathcal{O}_K \setminus \{0\} \hookrightarrow \mathcal{O}_K$. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a log smooth log formal scheme over $(\text{Spf}(\mathcal{O}_K), \mathcal{M}_{\mathcal{O}_K})$ and $(X_{p=0}, \mathcal{M}_{X_{p=0}})$ be the mod- p fiber of $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Then there exists a natural functor

$$\text{Isoc}_{\text{lf}}((X_{p=0}, \mathcal{M}_{X_{p=0}})_{\text{crys}}) \rightarrow \text{MIC}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta}) \quad (\mathcal{E} \mapsto (E := \mathcal{E}_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})}, \nabla_E)),$$

where $\mathcal{E}_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})}$ is the vector bundle on $\mathfrak{X}_{\eta}$ obtained as the evaluation of $\mathcal{E}$ at the p -adic log PD-thickening $(X_0, \mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Moreover, when $\mathcal{O}_K = W$ and $\mathcal{M}_W$ is trivial, this functor is fully faithful.

Proof. When $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) = (\text{Spf}(R), \mathcal{M}_R)$ is affine, $\text{MIC}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta})$ is equivalent to the category of finite projective $R[1/p]$ -modules M equipped with an integrable connection

$$M \rightarrow M \otimes_R \Omega_{(R, \mathcal{M}_R)/(\mathcal{O}_K, \mathcal{M}_{\mathcal{O}_K})}^1,$$

which include $\text{Vect}^{\nabla}(R[1/p])$ as a full subcategory. Hence, the assertion follows from Proposition 5.3.22 and Proposition 5.3.23. $\square$

Definition 5.3.27 (Filtered isocrystals). Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a semi-stable log formal scheme over $\mathcal{O}_K$. Consider a pair $(\mathcal{E}, \text{Fil}^\bullet(E))$ consisting of $\mathcal{E} \in \text{Isoc}_{\text{Iftf}}^\varphi((X_{p=0}, \mathcal{M}_{X_{p=0}})_{\text{crys}})$ and a descending filtration $\text{Fil}^\bullet(E)$ on E , where (E, ∇_E) is the object of $\text{MIC}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_\eta)$ obtained via the functor in the Proposition 5.3.26. We say that $(\mathcal{E}, \text{Fil}^\bullet(E))$ is a *filtered F -isocrystal* on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ if the following conditions are satisfied:

- $\text{Fil}^\bullet(E)$ is a separated and exhausted filtration with locally free grading pieces;
- the triple $(E, \nabla_E, \text{Fil}^\bullet(E))$ satisfies the Griffiths transversality.

5.3.3. F -isocrystals and monodromy operators. Let k be a perfect field of characteristic $p > 0$. Let $W := W(k)$ be the Witt ring of k . Let $(X, \mathcal{M}_X) = (\text{Spec}(R), \mathcal{M}_R)$ be a fine log scheme over k admitting a strict étale map $(X, \mathcal{M}_X) \rightarrow (\text{Spec}(k[P]), P)^a$, where P is an fs monoid such that the order of the torsion part of P^{gp} is prime to p .

The goal of this subsection is to describe F -isocrystals on $(X, \mathcal{M}_X \oplus 0^\mathbb{N})^a$ as F -isocrystals on $(X, \mathcal{M}_X)$ equipped with monodromy operators (Proposition 5.3.36). We follow the argument in [GY24, §2].

Take a strict closed immersion

$$(X, \mathcal{M}_X) \hookrightarrow (\text{Spf}(B), \mathcal{M}_B),$$

where $(\text{Spf}(B), \mathcal{M}_B)$ is a fine log smooth log formal scheme over W equipped with a Frobenius lift φ_B . Let $(\text{Spf}(D), \mathcal{M}_D)$ be its p -completed log PD-envelope. Let $D\{u\}$ be the p -completed PD-polynomial ring over D with a single variable u . Let $\mathcal{M}_{B\langle u \rangle} := (\mathcal{M}_B \oplus u^\mathbb{N})^a$ be the log structure on $\text{Spf}(B\langle u \rangle)$ and $\mathcal{M}_{D\{u\}}$ be the pullback log structure of $\mathcal{M}_{B\langle u \rangle}$ to $\text{Spf}(D\{u\})$. Then $(\text{Spf}(D\{u\}), \mathcal{M}_{D\{u\}})$ is the p -completed log PD-envelope of the strict closed immersion

$$(X, \mathcal{M}_X \oplus 0^\mathbb{N})^a \hookrightarrow (\text{Spf}(B\langle u \rangle), \mathcal{M}_{B\langle u \rangle})$$

given by $u \mapsto 0$. Let $\varphi_{B\langle u \rangle}$ denote the Frobenius lift on $(\text{Spf}(B\langle u \rangle), \mathcal{M}_{B\langle u \rangle})$ induced from φ_B and $u \mapsto u^p$. Let φ_D and $\varphi_{D\{u\}}$ be induced Frobenius lifts on respective log formal schemes.

Definition 5.3.28. We define the following categories.

- Let $\text{Vect}^{\varphi, \nabla}(D[1/p])$ be the category of tuples (M, φ_M, ∇_M) consisting of an object (M, ∇_M) of $\text{Vect}^\nabla(\text{Spf}(D)[1/p])$ and a $D[1/p]$ -linear isomorphism $\varphi_M: \varphi_D^* M \xrightarrow{\sim} M$ such that φ_M is horizontal with respect to ∇_M .
- Let $\text{Vect}^{\varphi, \nabla}(D\{u\}[1/p])$ be the category of tuples (M, φ_M, ∇_M) consisting of an object (M, ∇_M) of $\text{Vect}^\nabla(\text{Spf}(D\{u\})[1/p])$ and a $D\{u\}[1/p]$ -linear isomorphism $\varphi_M: \varphi_{D\{u\}}^* M \xrightarrow{\sim} M$ such that φ_M is horizontal with respect to ∇_M .
- Let $\text{Vect}^{\varphi, \nabla, N}(D[1/p])$ be the category of tuples $(M, \varphi_M, \nabla_M, N_M)$ consisting of an object (M, φ_M, ∇_M) of $\text{Vect}^{\varphi, \nabla}(D[1/p])$ and a $D[1/p]$ -linear endomorphism $N_M: M \rightarrow M$ such that the following conditions are satisfied:
 - N_M is horizontal with respect to ∇_M ;
 - $N_M \varphi_M = p \varphi_M (N_{\varphi_D^* M})$, where $N_{\varphi_D^* M}: \varphi_D^* M \rightarrow \varphi_D^* M$ is the base change of N_M along φ_D .

Construction 5.3.29. We define a functor

$$\text{Vect}^{\varphi, \nabla}(D\{u\}[1/p]) \rightarrow \text{Vect}^{\varphi, \nabla, N}(D[1/p]) \quad ((M, \varphi_M, \nabla_M) \mapsto (M_0, \varphi_{M_0}, \nabla_{M_0}, N_{M_0}))$$

as follows. Let $J: \text{Ker}(D\{u\}[1/p] \xrightarrow{u \rightarrow 0} D[1/p])$. Let $M_0 := M/JM$ and φ_{M_0} is the induced map from φ_M . Since there is a decomposition

$$\begin{aligned} \Omega_{(B\langle u \rangle, \mathcal{M}_{B\langle u \rangle})/W}^1 &\cong (\Omega_{(B, \mathcal{M}_B)/W}^1 \otimes_B B\langle u \rangle) \oplus (\Omega_{(W\langle u \rangle, u^{\mathbb{N}})^a/W}^1 \otimes_{W\langle u \rangle} B\langle u \rangle) \\ &\cong (\Omega_{(B, \mathcal{M}_B)/W}^1 \otimes_B B\langle u \rangle) \oplus B\langle u \rangle \cdot \text{dlog}(u), \end{aligned}$$

the connection ∇_M induces maps

$$\begin{aligned} M &\rightarrow M \otimes_B \Omega_{(B, \mathcal{M}_B)/W}^1 \\ M &\rightarrow M \otimes_{W\langle u \rangle} \Omega_{(W\langle u \rangle, u^{\mathbb{N}})^a/W}^1 \end{aligned}$$

by taking the composition with natural projection maps. These are denoted by ∇_M^{hol} and N_M , respectively. By Leibniz rule, we have $\nabla_M(JM) \subset J \cdot M \otimes_{B\langle u \rangle} \Omega_{(B\langle u \rangle, \mathcal{M}_{B\langle u \rangle})/W}^1$. We define ∇_{M_0} and N_{M_0} as the mod- J reduction of ∇_M^{hol} and N_M , respectively. To check that $(M_0, \varphi_{M_0}, \nabla_{M_0}, N_{M_0})$ is indeed an object of $\text{Vect}^{\varphi, \nabla, N}(D[1/p])$, it is enough to show that $N_{M_0}\varphi_{M_0} = p\varphi_{M_0}(N_{\varphi_D^* M_0})$. This is because the fact that φ_M is horizontal and $\varphi_{D\{u\}}(\text{dlog}(u)) = p \cdot \text{dlog}(u)$ implies that $N_M\varphi_M = p\varphi_M(N_{\varphi_D^* M})$.

Definition 5.3.30. Let $\text{Isoc}_{\text{lf}}^{\varphi, N}((X, \mathcal{M}_X)_{\text{crys}})$ denote the category of triples $(\mathcal{E}, \varphi_{\mathcal{E}}, N_{\mathcal{E}})$ such that $(\mathcal{E}, \varphi_{\mathcal{E}})$ is a locally free F -isocrystal on $(X, \mathcal{M}_X)$ and $N_{\mathcal{E}}: (\mathcal{E}, \varphi_{\mathcal{E}}) \rightarrow (\mathcal{E}, p\varphi_{\mathcal{E}})$ is a map of F -isocrystals.

Lemma 5.3.31. For every $(\mathcal{E}, \varphi_{\mathcal{E}}, N_{\mathcal{E}}) \in \text{Isoc}_{\text{lf}}^{\varphi, N}((X, \mathcal{M}_X)_{\text{crys}})$, the endomorphism $N_{\mathcal{E}}$ is nilpotent.

Proof. Let U be the biggest open subset of X where the log structure $\mathcal{M}_X$ is trivial. Concretely, we can write $U = X \times_{\text{Spec}(k[P])} \text{Spec}(k[P^{\text{gp}}])$. By [GY24, Proposition 2.14], $N_{\mathcal{E}}|_U$ is nilpotent. Take a formal log smooth lift $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ of $(X, \mathcal{M}_X)$ over W and $\mathfrak{U}$ be the open subset of $\mathfrak{X}$ corresponding to $U \subset X$. There is the following commutative diagram:

$$\begin{array}{ccc} \text{Isoc}((X, \mathcal{M}_X)_{\text{crys}}) & \longrightarrow & \text{MIC}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta}) \\ \downarrow & & \downarrow \\ \text{Isoc}(U_{\text{crys}}) & \longrightarrow & \text{MIC}(\mathfrak{U}_{\eta}). \end{array}$$

Since U is dense in X and $\mathfrak{X}$ is reduced, the restriction functor

$$\text{Vect}(\mathfrak{X}_{\eta}) \rightarrow \text{Vect}(\mathfrak{U}_{\eta})$$

is faithful. In particular, the right vertical functor in the diagram is also faithful. By Proposition 5.3.26, both horizontal functors are fully faithful. Therefore, the left vertical functor is faithful, and so $N_{\mathcal{E}}$ is also nilpotent. $\square$

Construction 5.3.32. By Proposition 5.3.23, there are natural equivalences

$$\begin{aligned} \text{Isoc}_{\text{lf}}^{\varphi, N}((X, \mathcal{M}_X)_{\text{crys}}) &\simeq \text{Vect}^{\varphi, \nabla, N}(D[1/p]) \\ \text{Isoc}_{\text{ltf}}^{\varphi}((X, \mathcal{M}_X \oplus 0^{\mathbb{N}})_{\text{crys}}^a) &\simeq \text{Vect}^{\varphi, \nabla}(D\{u\}[1/p]) \end{aligned}$$

Combining with the functor constructed in Construction 5.3.29, we obtain a functor

$$\text{Isoc}_{\text{ltf}}^{\varphi}((X, \mathcal{M}_X \oplus 0^{\mathbb{N}})_{\text{crys}}^a) \rightarrow \text{Isoc}_{\text{lf}}^{\varphi, N}((X, \mathcal{M}_X)_{\text{crys}}),$$

denoted by Res . The functor Res is independent of the choice of an embedding $(X, \mathcal{M}_X) \hookrightarrow (\text{Spf}(B), \mathcal{M}_B)$ because, if we are given two such embeddings, they are dominated by a common embedding (for example, the exactification of their product).

Remark 5.3.33. We define a forgetting functor

$$U: \text{Isoc}_{\text{lft}}((X, \mathcal{M}_X \oplus 0^{\mathbb{N}})_{\text{crys}}^a) \rightarrow \text{Isoc}_{\text{lf}}((X, \mathcal{M}_X)_{\text{crys}})$$

by $U(\mathcal{E})(T, \mathcal{M}_T) := \mathcal{E}((T, \mathcal{M}_T \oplus 0^{\mathbb{N}})^a)$ for each $\mathcal{E} \in \text{Isoc}((X, \mathcal{M}_X \oplus 0^{\mathbb{N}})_{\text{crys}}^a)$ and $(T, \mathcal{M}_T) \in (X, \mathcal{M}_X)_{\text{crys}}$. Also, we can consider a similar functor for the category of F -isocrystals.

Let $\mathcal{E}$ be an object of $\text{Isoc}_{\text{lft}}^{\varphi}((X, \mathcal{M}_X \oplus 0^{\mathbb{N}})_{\text{crys}}^a)$. By construction of the functor Res , the $D[1/p]$ -module with an Frobenius isomorphism $\text{Res}(\mathcal{E})_D$ is isomorphic to $\mathcal{E}_{D\{u\}} \otimes_{D\{u\}} D$, which is isomorphic to $\mathcal{E}_{(D, \mathcal{M}_D \oplus 0^{\mathbb{N}})^a}$ by the crystal property of $\mathcal{E}$ for the map of p -adic log PD-thickenings $(D\{u\}, \mathcal{M}_{D\{u\}}) \rightarrow (D, \mathcal{M}_D \oplus 0^{\mathbb{N}})^a$ given by $u \mapsto 0$. Therefore, the following triangle is commutative:

$$\begin{array}{ccc} \text{Isoc}_{\text{lf}}^{\varphi}((X, \mathcal{M}_X \oplus 0^{\mathbb{N}})_{\text{crys}}^a) & \xrightarrow{\text{Res}} & \text{Isoc}_{\text{lf}}^{\varphi, N}((X, \mathcal{M}_X)_{\text{crys}}) \\ & \searrow U & \downarrow \\ & & \text{Isoc}_{\text{lf}}^{\varphi}((X, \mathcal{M}_X)_{\text{crys}}). \end{array}$$

Here, the right vertical arrow is the functor forgetting N .

Lemma 5.3.34. Let A, A_0 be p -complete and p -torsion free rings equipped with Frobenius lifts φ_A and φ_{A_0} , respectively. Suppose that we are given a surjection $A \twoheadrightarrow A_0$ and its section $A_0 \rightarrow A$ that are compatible with Frobenius lifts. We assume that the following condition holds:

- if we set $I := \text{Ker}(A \twoheadrightarrow A_0)$, for an arbitrary integer $n \geq 1$, there exists an integer $m \geq 1$ such that $\varphi_A^m(J) \subset p^n A$.

Let $\text{Vect}^{\varphi}(A[1/p])$ (resp. $\text{Vect}^{\varphi}(A_0[1/p])$) be the category of finite projective $A[1/p]$ -modules (resp. A_0 -modules) M equipped with $A[1/p]$ -linear isomorphism $\varphi_M: \varphi_A^* M \xrightarrow{\sim} M$ (resp. $A_0[1/p]$ -linear isomorphism A_0 -linear isomorphism $\varphi_{A_0}^* M \rightarrow M$). Then the base change functor

$$\text{Vect}^{\varphi}(A[1/p]) \rightarrow \text{Vect}^{\varphi}(A_0[1/p])$$

along $A \twoheadrightarrow A_0$ is an equivalence, and the inverse of this functor is given by

$$(M, \varphi_M) \mapsto (M \otimes_{A_0} A, \varphi_M \otimes \varphi_A).$$

Proof. This follows from the same argument as [GY24, Theorem 2.21]. $\square$

Lemma 5.3.35. The functor $\text{Vect}^{\varphi, \nabla}(D\{u\}[1/p]) \rightarrow \text{Vect}^{\varphi, \nabla, N}(D[1/p])$ constructed in Construction 5.3.29 is an equivalence, and the inverse sends $(M, \varphi_M, \nabla_M, N_M)$ to a $D\{u\}[1/p]$ -module $M' := M \widehat{\otimes}_W W\{u\}$ equipped with additional structures

$$(\varphi_{M'}, \nabla_{M'}) := (\varphi_M \otimes \varphi_{W\{u\}}, \nabla_M \otimes \text{id}_{W\{u\}} \oplus (N_M \otimes \text{id}_{W\{u\}} + \text{id}_M \otimes N_{W\{u\}}) \cdot \text{dlog}(u)),$$

where we used the identification

$$M' \otimes_{B\langle u \rangle} \Omega_{(B\langle u \rangle, \mathcal{M}_{B\langle u \rangle})/W}^1 \cong (M' \otimes_B \Omega_{(B, \mathcal{M}_B)/W}^1) \oplus M' \cdot \text{dlog}(u).$$

Proof. First, we check that $(M', \varphi_{M'}, \nabla_{M'})$ defined in the assertion indeed belongs to $\text{Vect}^{\varphi, \nabla}(D\{u\}[1/p])$. The fact that φ_M is horizontal and the equations $N_M \varphi_M = p \varphi_M N_{\varphi_D^* M}$ and $N_{W\{u\}} \varphi_{W\{u\}} = p \varphi_{W\{u\}} N_{W\{u\}}$ imply that $\varphi_{M'}$ is horizontal. To check the topologically

quasi-nilpotence of $\nabla_{M'}$, it suffices to prove that, for $x \otimes u^{[n]} \in M'$, the element

$$\begin{aligned} & \left(\prod_{j=0}^m (N_M \otimes \text{id}_{W\{u\}} + \text{id}_M \otimes N_{W\{u\}} - j) \right) (x \otimes u^{[n]}) \\ &= \left(\prod_{j=0}^m (N_M \otimes \text{id}_{W\{u\}} + (n - j)) \right) (x \otimes u^{[n]}) \end{aligned}$$

converges to 0 when $m \rightarrow \infty$. By Lemma 5.3.31, N_M is nilpotent. Take $r \in \mathbb{N}$ such that $N_M^{r+1} = 0$. Then the above element is written as a $\mathbb{Z}$ -linear combination of $x \otimes u^{[n]}$, $N_M(x) \otimes u^{[n]}$, $\dots$, $N_M^r(x) \otimes u^{[n]}$, and their coefficients converges to 0 in the p -adic topology. This proves that $(M', \varphi_{M'}, \nabla_{M'})$ belongs $\text{Vect}^{\varphi, \nabla}(D\{u\}[1/p])$. It follows from Lemma 5.3.34 and the same argument as [GY24, Corollary 2.23] that the functor $\text{Vect}^{\varphi, \nabla, N}(D[1/p]) \rightarrow \text{Vect}^{\varphi, \nabla}(D\{u\}[1/p])$ defined in the assertion is indeed the inverse. $\square$

Proposition 5.3.36. The functor

$$\text{Res}: \text{Isoc}_{\text{lftf}}^{\varphi}((X, \mathcal{M}_X \oplus 0^{\mathbb{N}})_{\text{crys}}^a) \rightarrow \text{Isoc}_{\text{lf}}^{\varphi, N}((X, \mathcal{M}_X)_{\text{crys}})$$

is an equivalence.

Proof. This follows from the construction of the functor Res and Lemma 5.3.35. $\square$

Corollary 5.3.37. The functor $\text{Isoc}_{\text{lf}}^{\varphi}((X, \mathcal{M}_X)_{\text{crys}}) \rightarrow \text{Isoc}_{\text{lftf}}^{\varphi}((X, \mathcal{M}_X \oplus 0^{\mathbb{N}})_{\text{crys}}^a)$ defined as the pullback along the natural map $(X, \mathcal{M}_X \oplus 0^{\mathbb{N}})^a \rightarrow (X, \mathcal{M}_X)$ is fully faithful, and the essential image of the functor

$$\text{Isoc}_{\text{lf}}^{\varphi}((X, \mathcal{M}_X)_{\text{crys}}) \rightarrow \text{Isoc}_{\text{lftf}}^{\varphi}((X, \mathcal{M}_X \oplus 0^{\mathbb{N}})_{\text{crys}}^a) \xrightarrow{\text{Res}} \text{Isoc}_{\text{lf}}^{\varphi, N}((X, \mathcal{M}_X)_{\text{crys}})$$

consists of objects with $N_{\mathcal{E}} = 0$.

Proof. We can directly check that there is a commutative diagram

$$\begin{array}{ccc} \text{Isoc}_{\text{lf}}^{\varphi}((X, \mathcal{M}_X)_{\text{crys}}) & \longrightarrow & \text{Isoc}_{\text{lftf}}^{\varphi}((X, \mathcal{M}_X \oplus 0^{\mathbb{N}})_{\text{crys}}^a) \\ \downarrow \sim & & \downarrow \sim \\ \text{Vect}^{\varphi, \nabla}(D[1/p]) & \longrightarrow & \text{Vect}^{\varphi, \nabla}(D\{u\}[1/p]), \end{array}$$

where the bottom horizontal functor sends (M, φ_M, ∇_M) to

$$(M \widehat{\otimes}_W W\{u\}, \varphi_M \otimes \varphi_{W\{u\}}, \nabla_M \otimes \text{id}_{W\{u\}} \oplus (\text{id}_M \otimes N_{W\{u\}}) \cdot \text{dlog}(u)).$$

Then the assertion follows from Proposition 5.3.36. $\square$

5.4. Local systems on log adic spaces

Let k be a perfect field, $W := W(k)$ the Witt ring of k , K a totally ramified finite extension of $K_0 := W[1/p]$, and $\mathcal{O}_K$ a valuation ring of K . Let C be the completion of an algebraic closure of K and $\mathcal{O}_C$ be the valuation ring of C . Fix a compatible system of p^n -th power roots of unity for $n \geq 1$ denoted by $(\zeta_p, \zeta_{p^2}, \dots)$ and let $\epsilon := (1, \zeta_p, \zeta_{p^2}, \dots) \in \mathcal{O}_C^{\flat}$.

In this section, let $(X, \mathcal{M}_X)$ be a locally noetherian fs log adic space over $\text{Spa}(K, \mathcal{O}_K)$ ([DLLZ23b]).

5.4.1. Horizontal period sheaves on log adic spaces. We start with the definition of various period sheaves on log adic spaces.

Definition 5.4.1. Let S be a p -torsion free integral perfectoid $\mathcal{O}_C$ -algebra and $S^\flat := \varprojlim_{x \mapsto x^p} S/p$ be the tilt of S .

- Set $A_{\text{inf}}(S) := W(S^\flat)$. The natural surjection $A_{\text{inf}}(S) \twoheadrightarrow S^\flat \twoheadrightarrow S/p$ uniquely lifts to a surjection $\theta: A_{\text{inf}}(S) \twoheadrightarrow S$, and the ideal $\text{Ker}(\theta)$ is generated by a regular element. Let $\varphi_{A_{\text{inf}}(S)}$ be the Witt vector Frobenius map. Let $q := [\epsilon] \in A_{\text{inf}}(\mathcal{O}_C)$ and $\mu := q - 1 \in A_{\text{inf}}(\mathcal{O}_C)$.
- Set $B_{\text{inf}}(S) := A_{\text{inf}}(S)[1/p]$. Let $\varphi_{B_{\text{inf}}(S)}$ be the induced ring map from $\varphi_{A_{\text{inf}}(S)}$.
- Let $B_{\text{dR}}^+(S)$ be the $\text{Ker}(\theta)$ -adic completion of $B_{\text{inf}}(S)$ equipped with the $\text{Ker}(\theta)$ -adic filtration $\text{Fil}^\bullet(B_{\text{dR}}^+(S))$. We put

$$t := \log(q) := \sum_{n \geq 1} (-1)^{n+1} \frac{\mu^n}{n} \in B_{\text{dR}}^+(\mathcal{O}_C).$$

- Set

$$B_{\text{dR}}(S) := B_{\text{dR}}^+(S)[1/\xi] = B_{\text{dR}}^+(S)[1/\mu] = B_{\text{dR}}^+(S)[1/t]$$

and $\text{Fil}^\bullet(B_{\text{dR}}(S))$ be the filtration defined by $\text{Fil}^n(B_{\text{dR}}(S)) := \xi^n B_{\text{dR}}^+(S)$ for $n \in \mathbb{Z}$.

- Let $A_{\text{crys}}^0(S) := A_{\text{inf}}(S)[\{\xi^{[n]}\}_{n \geq 1}]$ be the PD-envelope of the surjection θ equipped with the filtration $\text{Fil}^\bullet(A_{\text{crys}}^0(S))$ whose n -th filtration is the ideal generated by $\xi^{[i]}$ for $i \geq n$, where ξ is a generator of $\text{Ker}(\theta) \subset A_{\text{inf}}(S)$. The ring map $A_{\text{crys}}^0(S) \rightarrow A_{\text{crys}}^0(S)$ induced from $\varphi_{A_{\text{inf}}(S)}$ is denoted by $\varphi_{A_{\text{crys}}^0(S)}$.
- Let $A_{\text{crys}}(S)$ denote the p -completion of $A_{\text{crys}}^0(S)$ with the filtration $\text{Fil}^\bullet(A_{\text{crys}}(S))$ whose n -th filtration is the closure of $\text{Fil}^n(A_{\text{crys}}^0(S))$. In the same way as [TT19, Proposition 2.23], a natural map $A_{\text{crys}}(S) \rightarrow B_{\text{dR}}^+(S)$ is defined, and this map is injective by [CK19, Proposition 5.36] (see also Lemma 5.4.13). We put

$$t := \log(q) := \sum_{n \geq 1} (-1)^{n+1} (n-1)! \mu^{[n]} \in A_{\text{crys}}(\mathcal{O}_C).$$

This definition is compatible with the one defined in $\mathbb{B}_{\text{dR}}^+(S)$ via the above embedding. Let $\varphi_{A_{\text{crys}}(S)}$ denote the Frobenius lift on $A_{\text{crys}}(S)$ induced from $\varphi_{A_{\text{crys}}^0(S)}$.

- Set $B_{\text{crys}}^+(S) := A_{\text{crys}}(S)[1/p]$ and let $\text{Fil}^\bullet(B_{\text{crys}}^+(S))$ be the induced filtration. Let $\varphi_{B_{\text{crys}}^+(S)}$ the ring map $B_{\text{crys}}^+(S) \rightarrow B_{\text{crys}}^+(S)$ induced from $\varphi_{A_{\text{crys}}(S)}$.
- Set

$$B_{\text{crys}}(S) := B_{\text{crys}}^+(S)[1/t] = B_{\text{crys}}^+(S)[1/\mu] = A_{\text{crys}}(S)[1/\mu]$$

let $\text{Fil}^\bullet(B_{\text{crys}}(S))$ be the filtration defined by

$$\text{Fil}^n(B_{\text{crys}}(S)) := \sum_{i \in \mathbb{Z}} t^{-i} \text{Fil}^{i+n}(B_{\text{crys}}^+(S))$$

for $n \in \mathbb{Z}$. Let $\varphi_{B_{\text{crys}}(S)}$ the ring map $B_{\text{crys}}(S) \rightarrow B_{\text{crys}}(S)$ induced from $\varphi_{B_{\text{crys}}^+(S)}$.

- Let $\mathcal{M}_{A_{\text{inf}}(S)}$ be the log structure on $\text{Spf}(A_{\text{inf}}(S))$ associated with a prelog structure $\mathcal{O}_C^\flat \setminus \{0\} \rightarrow A_{\text{inf}}(S)$ given by $x \mapsto [x]$ and $\mathcal{M}_S$ be the pullback structure on $\text{Spf}(S)$ of it along $\theta: A_{\text{inf}}(S) \twoheadrightarrow S$. We define $(\text{Spf}(\widehat{A}_{\text{st}}(S)), \mathcal{M}_{\widehat{A}_{\text{st}}(S)})$ as the p -completed log PD-envelope of the closed immersion

$$(\text{Spf}(S), \mathcal{M}_S) \hookrightarrow (\text{Spf}(A_{\text{inf}}(S) \widehat{\otimes}_W W\langle u \rangle), \mathcal{M}_{A_{\text{inf}}(S)} \oplus u^{\mathbb{N}})^a$$

defined by θ and $u \mapsto p$. Explicitly, we can define an isomorphism

$$\widehat{A}_{\text{st}}(S) \cong A_{\text{crys}}(S)\{v\} \subset B_{\text{dR}}^+(S)[[v]]$$

sending v to $\frac{u}{[p^b]} - 1$ by choosing $p^b = (p, p^{1/p}, \dots) \in \mathcal{O}_C^b$, where $A_{\text{crys}}(S)\{v\}$ is the p -completed PD-polynomial ring with a single variable v over $A_{\text{crys}}(S)$.

We define a monodromy operator $N_{\widehat{A}_{\text{st}}(S)}: \widehat{A}_{\text{st}}(S) \rightarrow \widehat{A}_{\text{st}}(S)$ as follows. The $A_{\text{crys}}(S)$ -linear map $A_{\text{crys}}(S)\{v\} \rightarrow A_{\text{crys}}(S)\{v\}$ defined by $f(v) \mapsto (v+1)f'(v)$ induces the $A_{\text{crys}}(S)$ -linear map $\widehat{A}_{\text{st}}(S) \rightarrow \widehat{A}_{\text{st}}(S)$, denoted by $N_{\widehat{A}_{\text{st}}(S)}$, via the above isomorphism. We can easily check that $N_{\widehat{A}_{\text{st}}(S)}$ is independent of the choice of p^b . The $A_{\text{crys}}(S)$ -subalgebra $A_{\text{crys}}(S)[\log(\frac{u}{[p^b]})]$ of $\widehat{A}_{\text{st}}(S)(S)$ is the subset of nilpotent elements for $N_{\widehat{A}_{\text{st}}(S)}$ as checked inductively.

The Frobenius lift $\varphi_{A_{\text{inf}}(S)}$ induces a Frobenius lift on $(\text{Spf}(A_{\text{inf}}(S)), \mathcal{M}_{A_{\text{inf}}(S)})$, and we define a Frobenius lift $\varphi_{W\{u\}}$ on $(\text{Spf}(W\{u\}), u^{\mathbb{N}})^a$ by $u \mapsto u^p$. Then $\varphi_{A_{\text{inf}}(S)}$ and $\varphi_{W\{u\}}$ induces a Frobenius lift on $(\text{Spf}(\widehat{A}_{\text{st}}(S)), \mathcal{M}_{\widehat{A}_{\text{st}}(S)})$, denoted by $\varphi_{\widehat{A}_{\text{st}}(S)}$. The underlying ring map $\widehat{A}_{\text{st}}(S) \rightarrow \widehat{A}_{\text{st}}(S)$ is also denoted by $\varphi_{\widehat{A}_{\text{st}}(S)}$.

- Set $\widehat{B}_{\text{st}}^+(S) := \widehat{A}_{\text{st}}(S)[1/p]$. Let $N_{\widehat{B}_{\text{st}}^+(S)}: \widehat{B}_{\text{st}}^+(S) \rightarrow \widehat{B}_{\text{st}}^+(S)$ be the monodromy operator induced from $N_{\widehat{A}_{\text{st}}(S)}$. Let $\varphi_{\widehat{B}_{\text{st}}^+(S)}: \widehat{B}_{\text{st}}^+(S) \rightarrow \widehat{B}_{\text{st}}^+(S)$ be the Frobenius structure induced from $\varphi_{\widehat{A}_{\text{st}}(S)}$.
- Set $\widehat{B}_{\text{st}}(S) := \widehat{B}_{\text{st}}^+(S)[1/t]$. Let $N_{\widehat{B}_{\text{st}}(S)}: \widehat{B}_{\text{st}}(S) \rightarrow \widehat{B}_{\text{st}}(S)$ be the monodromy operator induced from $N_{\widehat{B}_{\text{st}}^+(S)}$. Let $\varphi_{\widehat{B}_{\text{st}}(S)}: \widehat{B}_{\text{st}}(S) \rightarrow \widehat{B}_{\text{st}}(S)$ be the Frobenius structure induced from $\varphi_{\widehat{B}_{\text{st}}^+(S)}$.
- Let $A_{\text{st}}(S) := A_{\text{crys}}(S)[\log(\frac{p}{[p^b]})]$ be the $A_{\text{crys}}(S)$ -subalgebra of $B_{\text{dR}}^+(S)$ generated by $\log(\frac{p}{[p^b]})$. This is independent of the choice of p^b because $\log(\frac{p}{[ep^b]}) = \log(\frac{p}{[p^b]}) - \log([e])$ and $\log([e]) \in A_{\text{crys}}(S)$. By [Shi22, Corollary 2.28], $\log(\frac{p}{[p^b]})$ is transcendental over $A_{\text{crys}}(S)$. Let $N_{A_{\text{st}}(S)}: A_{\text{st}}(S) \rightarrow A_{\text{st}}(S)$ be the monodromy operator defined as the $A_{\text{crys}}(S)$ -linear map $A_{\text{st}}(S) \rightarrow A_{\text{st}}(S)$ given by $(\log(\frac{p}{[p^b]}))^n \mapsto n(\log(\frac{p}{[p^b]}))^{n-1}$ for each $n \geq 1$. Let $\varphi_{A_{\text{st}}(S)}: A_{\text{st}}(S) \rightarrow A_{\text{st}}(S)$ be the ring map induced from $\varphi_{A_{\text{crys}}(S)}$ and $\log(\frac{p}{[p^b]}) \mapsto p\log(\frac{p}{[p^b]})$. The maps $N_{A_{\text{st}}(S)}$ and $\varphi_{A_{\text{crys}}(S)}$ are independent of a choice of p^b . The map of $A_{\text{crys}}(S)$ -algebras $A_{\text{st}}(S) \rightarrow \widehat{A}_{\text{st}}(S)$ mapping $\log(\frac{p}{[p^b]})$ to $\log(\frac{u}{[p^b]})$ is an injective map which is independent of a choice of p^b . This injection is compatible with Frobenius structures and monodromy operators and identifies $A_{\text{st}}(S)$ with the subset of $\widehat{A}_{\text{st}}(S)$ consisting of all nilpotent elements for $N_{\widehat{A}_{\text{st}}(S)}$.
- Set $B_{\text{st}}^+(S) := A_{\text{st}}(S)[1/p]$. Let $N_{B_{\text{st}}^+(S)}: B_{\text{st}}^+(S) \rightarrow B_{\text{st}}^+(S)$ be the monodromy operator induced from $N_{A_{\text{st}}(S)}$, and let $\varphi_{B_{\text{st}}^+(S)}: B_{\text{st}}^+(S) \rightarrow B_{\text{st}}^+(S)$ be the Frobenius structure induced from $\varphi_{A_{\text{st}}(S)}$.
- Set $B_{\text{st}}(S) := B_{\text{st}}^+(S)[1/t]$. Let $N_{B_{\text{st}}(S)}: B_{\text{st}}(S) \rightarrow B_{\text{st}}(S)$ be the monodromy operator induced from $N_{B_{\text{st}}^+(S)}$, and let $\varphi_{B_{\text{st}}(S)}: B_{\text{st}}(S) \rightarrow B_{\text{st}}(S)$ be the Frobenius structure induced from $\varphi_{B_{\text{st}}^+(S)}$.

Let $(X, \mathcal{M}_X)_{\text{prokét}}$ be the pro-Kummer-étale site of $(X, \mathcal{M}_X)$ ([DLLZ23b, Definition 5.1.2]) and $(\widehat{\mathcal{O}}, \widehat{\mathcal{O}}^+)$ be completed structure sheaves ([DLLZ23b, Definition 5.4.1]). For

a log affinoid perfectoid U with the associated affinoid perfectoid space $\widehat{U} = \mathrm{Spa}(S, S^+)$ ([DLLZ23b, Definition 5.3.1 and Remark 5.3.5]), there exists a natural morphism of sites

$$(\mathrm{Spa}(S, S^+))_{\mathrm{pro\acute{e}t}} \rightarrow (X, \mathcal{M}_X)_{\mathrm{prok\acute{e}t}/U}$$

inducing an equivalence of the associated topoi ([DLLZ23b, Lemma 5.3.8]). The log affinoid perfectoid objects form a basis of $(X, \mathcal{M}_X)_{\mathrm{prok\acute{e}t}}$ by [DLLZ23b, Proposition 5.3.12].

Lemma 5.4.2. Let $(X, \mathcal{M}_X)$ be a locally noetherian fs log adic space over $\mathrm{Spa}(K, \mathcal{O}_K)$ and $B \in \{A_{\mathrm{inf}}, B_{\mathrm{inf}}, B_{\mathrm{st}}, \widehat{B}_{\mathrm{st}}\}$ (resp. $B \in \{B_{\mathrm{dR}}^+, B_{\mathrm{dR}}, B_{\mathrm{crys}}\}$). Then

$$U \mapsto (B(S^+)) \quad (\text{resp. } (B(S^+), \mathrm{Fil}^\bullet(B(S^+))))$$

defines a sheaf (resp. filtered sheaf) on $(X, \mathcal{M}_X)_{\mathrm{prok\acute{e}t}}$, where U is a log affinoid perfectoid in $(X, \mathcal{M}_X)_{C, \mathrm{prok\acute{e}t}}$ and $\widehat{U} = \mathrm{Spa}(S, S^+)$ is the associated affinoid perfectoid space (in the sense of [DLLZ23b, Definition 5.3.1 and Remark 5.3.5]).

Proof. For the case that $B = A_{\mathrm{inf}}, B_{\mathrm{inf}}, B_{\mathrm{dR}}^+, B_{\mathrm{dR}}$, see [DLLZ23a, Proposition 2.2.4 (1)].

For the case that $B = B_{\mathrm{crys}}, \widehat{B}_{\mathrm{st}}$, the same argument as the proof of [TT19, Corollary 2.8 (2)] works. In their proof, [Sch13, Lemma 4.10] is used as a key input, which we replace with [DLLZ23b, Theorem 5.4.3]. In the case that $B = B_{\mathrm{st}}(S)$, the assertion follows from the result for $B_{\mathrm{crys}}(S)$ $\square$

Definition 5.4.3. The (filtered) sheaves defined in the above lemma are denoted by $\mathbb{A}_{\mathrm{inf}}, \mathbb{B}_{\mathrm{inf}}, \mathbb{B}_{\mathrm{dR}}^+, \mathbb{B}_{\mathrm{dR}}, \mathbb{B}_{\mathrm{crys}}, \mathbb{B}_{\mathrm{st}}, \widehat{\mathbb{B}}_{\mathrm{st}}$ respectively.

5.4.2. Period sheaves with connections for semi-stable log formal schemes.

Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a semi-stable log formal scheme over $\mathcal{O}_K$.

Definition 5.4.4. Let U be a log affinoid perfectoid in $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \mathrm{prok\acute{e}t}}$ with the associated affinoid perfectoid space $\widehat{U} := \mathrm{Spa}(S, S^+)$. We say that U is *big* (with respect to the formal model $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$) if $U \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta}$ factors as

$$U \rightarrow (\mathrm{Spf}(\widetilde{R}_{\infty, \alpha}), \mathcal{M}_{\widetilde{R}_{\infty, \alpha}})_{\eta} \rightarrow (\mathrm{Spf}(R), \mathcal{M}_R)_{\eta} \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta},$$

where $(\mathrm{Spf}(R), \mathcal{M}_R)$ is a small affine log formal scheme with a fixed framing and the morphism $(\mathrm{Spf}(R), \mathcal{M}_R) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is strict étale. In this situation, we define a log structure on $\mathcal{M}_{S^+}$ as the pullback log structure of $\mathcal{M}_{\widetilde{R}_{\infty, \alpha}}$ by $\mathrm{Spf}(S^+) \rightarrow \mathrm{Spf}(\widetilde{R}_{\infty, \alpha})$. This is independent of the choice of the above factorization. To check this, take another factorization

$$U \rightarrow (\mathrm{Spf}(\widetilde{R}'_{\infty, \alpha'}), \mathcal{M}_{\widetilde{R}'_{\infty, \alpha'}})_{\eta} \rightarrow (\mathrm{Spf}(R'), \mathcal{M}_{R'})_{\eta} \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta},$$

and set

$$(\mathfrak{V}, \mathcal{M}_{\mathfrak{V}}) := (\mathrm{Spf}(\widetilde{R}_{\infty, \alpha}), \mathcal{M}_{\widetilde{R}_{\infty, \alpha}}) \times_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})}^{\mathrm{sat}} (\mathrm{Spf}(\widetilde{R}'_{\infty, \alpha'}), \mathcal{M}_{\widetilde{R}'_{\infty, \alpha'}}).$$

We have the induced map

$$U \rightarrow (\mathrm{Spf}(\widetilde{R}_{\infty, \alpha}), \mathcal{M}_{\widetilde{R}_{\infty, \alpha}})_{\eta} \times_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta}}^{\mathrm{sat}} (\mathrm{Spf}(\widetilde{R}'_{\infty, \alpha'}), \mathcal{M}_{\widetilde{R}'_{\infty, \alpha'}})_{\eta} \cong (\mathfrak{V}, \mathcal{M}_{\mathfrak{V}})_{\eta},$$

and so the natural map $\mathrm{Spf}(S^+) \rightarrow \mathfrak{X}$ factors through $\mathfrak{V}$. Since both of two projection maps from $(\mathfrak{V}, \mathcal{M}_{\mathfrak{V}})$ are strict by Lemma 6.A.2, the log structure $\mathcal{M}_{S^+}$ is independent of the choice of a factorization.

By Lemma 5.A.5, $\mathcal{M}_{S^+}$ lifts uniquely to a log structure $\mathcal{M}_{A_{\text{inf}}(S^+)}$ on $\text{Spf}(A_{\text{inf}}(S^+))$, and $\mathcal{M}_{A_{\text{inf}}(S^+)}$ is isomorphic to the associated log structure with the prelog ring $\mathbb{Q}_{\geq 0} \rightarrow A_{\text{inf}}(S^+)$ defined by

$$se_i \mapsto \begin{cases} [(y_i^s)^b] & (1 \leq i \leq m) \\ [(z_{i-m}^s)^b] & (m+1 \leq i \leq r) \end{cases}$$

for $s \in \mathbb{Q}_{\geq 0}$. Let $\mathcal{M}_{A_{\text{crys}}(S^+)}$ denote the log structure on $\text{Spf}(A_{\text{crys}}(S^+))$ obtained as the pullback of $\mathcal{M}_{A_{\text{inf}}(S^+)}$. We define a log PD-thickening $(A_{\text{crys},K}(S^+) \rightarrow S^+/p, \mathcal{M}_{A_{\text{crys},K}(S^+)})$ in $(X_0, \mathcal{M}_{X_0})_{\text{crys}}$ as the p -adic log PD-envelope of the closed immersion

$$(\text{Spec}(S^+/p), \mathcal{M}_{S^+/p}) \hookrightarrow (\text{Spf}(A_{\text{crys}}(S^+) \otimes_W \mathcal{O}_K), (\mathcal{M}_{A_{\text{crys}}(S^+)} \oplus \mathcal{M}_{\mathcal{O}_K})^a),$$

where $\mathcal{M}_{S^+/p}$ is the restriction of $\mathcal{M}_{S^+}$. When $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is horizontally semi-stable (resp. vertically semi-stable), we have the explicit description

$$\begin{aligned} A_{\text{crys},K}(S^+) &= (A_{\text{crys}}(S^+) \otimes_W \mathcal{O}_K)[\{\frac{([\pi^b] - \pi)^n}{n!}\}_{n \geq 1}]_p^\wedge \\ (\text{resp. } A_{\text{crys},K}(S^+) &= (A_{\text{crys}}(S^+) \otimes_W \mathcal{O}_K)[\{\frac{(([\pi^b]/\pi) - 1)^n}{n!}\}_{n \geq 1}]_p^\wedge). \end{aligned}$$

Hence, we can define a natural map

$$A_{\text{crys},K}(S^+) \rightarrow B_{\text{dR}}^+(S^+)$$

in the same way as the map $A_{\text{crys}}(S^+) \rightarrow B_{\text{dR}}^+(S^+)$.

De Rham period sheaves with connections on log adic spaces are introduced in [DLLZ23a] and denoted by $\mathcal{O}\mathbb{B}_{\text{dR},\log}^+$ there. These objects are regarded as the log version of de Rham period sheaves defined in [Sch13].

We give another construction of $\mathcal{O}\mathbb{B}_{\text{dR},\log}^+$ which is similar to the definition in [Sch16] under the setting that a formal model is fixed. Our construction is used in Proposition 5.4.19 afterward.

Construction 5.4.5. Let U be a big log affinoid perfectoid in $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \text{prokét}}$ with the associated affinoid perfectoid space $\widehat{U} = \text{Spa}(S, S^+)$. Choose a factorization $U \rightarrow (\text{Spf}(R), \mathcal{M}_R)_\eta \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_\eta$, where $(\text{Spf}(R), \mathcal{M}_R) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is strict étale. The natural surjection $\theta: A_{\text{inf}}(S^+) \twoheadrightarrow S^+$ induces a closed immersion

$$(\text{Spf}(S^+), \mathcal{M}_{S^+}) \hookrightarrow (\text{Spf}(R \widehat{\otimes}_W A_{\text{inf}}(S^+)), \mathcal{M}_R \oplus \mathcal{M}_{A_{\text{inf}}(S^+)})^a.$$

Let $(\text{Spf}(\mathcal{O}\mathbb{A}_{\text{inf}}(U)), \mathcal{M}_{\mathcal{O}\mathbb{A}_{\text{inf}}(U)})$ be its exactification. The surjection $\mathcal{O}\mathbb{A}_{\text{inf}}(U)[1/p] \twoheadrightarrow S$ is also denoted by θ . Let $\mathcal{O}\mathbb{B}_{\text{dR}}^+(U)$ be the $\text{Ker}(\theta)$ -adic completion of $\mathcal{O}\mathbb{A}_{\text{inf}}(U)[1/p]$. Then $\mathcal{O}\mathbb{B}_{\text{dR}}^+(U)$ is independent of the choice of $(\text{Spf}(R), \mathcal{M}_R)$. We have an induced map $\theta: \mathcal{O}\mathbb{B}_{\text{dR}}^+(U) \twoheadrightarrow S$ and define a filtration $\text{Fil}^\bullet(\mathcal{O}\mathbb{B}_{\text{dR}}^+(U))$ as the $\text{Ker}(\theta)$ -adic filtration. Let $(\mathcal{O}\mathbb{B}_{\text{dR}}^+, \text{Fil}^\bullet(\mathcal{O}\mathbb{B}_{\text{dR}}^+))$ be the filtered sheaf associated with the presheaf given by $U \mapsto (\mathcal{O}\mathbb{B}_{\text{dR}}^+(U), \text{Fil}^\bullet(\mathcal{O}\mathbb{B}_{\text{dR}}^+(U)))$.

We define a connection

$$\nabla_{\mathcal{O}\mathbb{B}_{\text{dR}}^+} : \mathcal{O}\mathbb{B}_{\text{dR}}^+ \rightarrow \mathcal{O}\mathbb{B}_{\text{dR}}^+ \otimes_{\mathcal{O}_{\mathfrak{X}_\eta}} \Omega_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_\eta/K}^1$$

as a unique $\mathbb{B}_{\text{dR}}^+$ -linear continuous connection extending the derivation $\mathcal{O}_{\mathfrak{X}_\eta} \rightarrow \Omega_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_\eta/K}^1$ such that

$$\nabla_{\mathcal{O}\mathbb{B}_{\text{dR}}^+}(m/m') = (m/m') \otimes \text{dlog}(m)$$

for sections $m \in \Gamma(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ and $m' \in \Gamma(\mathrm{Spf}(A_{\mathrm{inf}}(S^+)), \mathcal{M}_{A_{\mathrm{inf}}(S^+)})$ with the common image in $\Gamma(\mathrm{Spf}(S^+), \mathcal{M}_{S^+})$. By the Leibniz rule, we have

$$\nabla_{\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+}(\mathrm{Fil}^n(\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+)) \subset \mathrm{Fil}^{n-1}(\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+) \otimes_{\mathcal{O}_{\mathfrak{X}_\eta}} \Omega_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_\eta/K}^1$$

for $n \geq 1$.

Suppose that $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) = (\mathrm{Spf}(R), \mathcal{M}_R)$ is small affine log formal scheme over $\mathcal{O}_K$ with a fixed framing and that there is a factorization $U \rightarrow \tilde{U}_{\infty, \alpha} \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_\eta$. Then we have an isomorphism $\mathcal{O}\mathbb{A}_{\mathrm{inf}}(U) \cong (R \widehat{\otimes}_W A_{\mathrm{inf}}(S^+))[\{([y_j^b]/y_j)^{\pm 1}\}, \{([z_k^b]/z_k)^{\pm 1}\}]_{(p, \xi)}^\wedge$. The connection

$$\nabla_{\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+(U)} : \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+(U) \rightarrow \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+(U) \otimes_{R[1/p]} \Omega_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_\eta/K}^1$$

is characterized as a unique $B_{\mathrm{dR}}^+(S^+)$ -linear continuous connection extending the derivation $R[1/p] \rightarrow \Omega_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_\eta/K}^1$ such that

$$\nabla_{\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+(U)}(y_j/[y_j^b]) = (y_j/[y_j^b]) \otimes \mathrm{dlog}(y_j), \quad \nabla_{\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+(U)}(z_k/[z_k^b]) = (z_k/[z_k^b]) \otimes \mathrm{dlog}(z_k)$$

for each j, k .

Consider the sheaf $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+[1/t]$ equipped with a filtration defined by

$$\mathrm{Fil}^n(\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+[1/t]) := \sum_{m \geq 0} t^{n-m} \mathrm{Fil}^m(\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+)$$

for $n \in \mathbb{Z}$. Let $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ denote the completion of $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+[1/t]$ with respect to the filtration. We define a filtration on $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ by

$$\mathrm{Fil}^n(\mathcal{O}\mathbb{B}_{\mathrm{dR}}) := \varprojlim_{m \geq 0} \mathrm{Fil}^n(\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+[1/t]) / \mathrm{Fil}^{n+m}(\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+[1/t])$$

for $n \in \mathbb{Z}$. The connection $\nabla_{\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+}$ induces a connection

$$\nabla_{\mathcal{O}\mathbb{B}_{\mathrm{dR}}} : \mathcal{O}\mathbb{B}_{\mathrm{dR}} \rightarrow \mathcal{O}\mathbb{B}_{\mathrm{dR}} \otimes_{\mathcal{O}_{\mathfrak{X}_\eta}} \Omega_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_\eta/K}^1$$

satisfying $\nabla_{\mathcal{O}\mathbb{B}_{\mathrm{dR}}}(\mathrm{Fil}^n(\mathcal{O}\mathbb{B}_{\mathrm{dR}})) \subset \mathrm{Fil}^{n-1}(\mathcal{O}\mathbb{B}_{\mathrm{dR}})$ for $n \in \mathbb{Z}$.

We shall give a local description of $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$. Suppose that $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) = (\mathrm{Spf}(R), \mathcal{M}_R)$ is small affine log formal scheme over $\mathcal{O}_K$ with a fixed framing. Consider a surjection $\theta : \mathbb{B}_{\mathrm{dR}}^+[X_1, \dots, X_s] \twoheadrightarrow \mathcal{O}^\flat$ induced from the natural surjection $\theta : \mathbb{B}_{\mathrm{dR}}^+ \twoheadrightarrow \mathcal{O}^\flat$ and $X_i \mapsto 1$. Let $\mathbb{B}_{\mathrm{dR}}^+[[X_1 - 1, \dots, X_s - 1]]$ be the $\mathrm{Ker}(\theta)$ -adic completion of $\mathbb{B}_{\mathrm{dR}}^+[X_1, \dots, X_s]$. We have the induced surjection $\mathbb{B}_{\mathrm{dR}}^+[[X_1 - 1, \dots, X_s - 1]] \twoheadrightarrow \mathcal{O}^\flat$, which is also denoted by θ . Let $\mathrm{Fil}^\bullet(\mathbb{B}_{\mathrm{dR}}^+[[X_1 - 1, \dots, X_s - 1]])$ be the $\mathrm{Ker}(\theta)$ -adic filtration of $\mathbb{B}_{\mathrm{dR}}^+[X_1, \dots, X_s]$.

Lemma 5.4.6. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) = (\mathrm{Spf}(R), \mathcal{M}_R)$ be a small affine log formal scheme over $\mathcal{O}_K$ with a fixed framing. Then there is an isomorphism of $\mathbb{B}_{\mathrm{dR}}^+$ -algebras

$$\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+|_{\tilde{U}_{\infty, \alpha}} \cong \mathbb{B}_{\mathrm{dR}}^+[[X_1 - 1, \dots, X_s - 1]]|_{\tilde{U}_{\infty, \alpha}}$$

that is compatible with filtrations.

Proof. We follow the argument of [Sch16]. Define a map of $\mathbb{B}_{\mathrm{dR}}^+|_{\tilde{U}_{\infty, \alpha}}$ -algebras

$$\mathbb{B}_{\mathrm{dR}}^+[[X_1 - 1, \dots, X_s - 1]]|_{\tilde{U}_{\infty, \alpha}} \rightarrow \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+|_{\tilde{U}_{\infty, \alpha}}$$

sending X_i to $t_i/[t_i^b]$ for each i . We shall construct the inverse map. Let U be a log affinoid perfectoid over $\tilde{U}_{\infty, \alpha}$ with the associated affinoid perfectoid space $\widehat{U} = \mathrm{Spa}(S, S^+)$. Consider a map of $\mathcal{O}_K$ -algebras $R^0 \rightarrow B_{\mathrm{dR}}^+(S^+)[[X_1 - 1, \dots, X_s - 1]]$ given by

$$x_i \mapsto [x_i^b]X_i \quad (1 \leq i \leq l), \quad y_j \mapsto [y_j^b]X_{l+j} \quad (1 \leq j \leq m)$$

$$z_k \mapsto \begin{cases} [z_k^b]X_{l+m+k} & (1 \leq k \leq n-1) \\ \pi \cdot (\prod_{k=1}^{n-1} [z_k^b]X_{l+m+k})^{-1} & (k = n). \end{cases}$$

Since $R^0 \rightarrow R$ is étale, there exists a unique map $R \rightarrow B_{\mathrm{dR}}^+(S^+)[[X_1 - 1, \dots, X_s - 1]]$ commuting the following diagram:

$$\begin{array}{ccc} R^0 & \longrightarrow & B_{\mathrm{dR}}^+(S^+)[[X_1 - 1, \dots, X_s - 1]] \\ \downarrow & \nearrow & \downarrow \theta \\ R & \longrightarrow & S. \end{array}$$

This map and the natural map $A_{\mathrm{inf}}(S^+) \rightarrow B_{\mathrm{dR}}^+(S^+)[[X_1 - 1, \dots, X_s - 1]]$ induce a desired map $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+|_{\tilde{U}_{\infty,\alpha}} \rightarrow B_{\mathrm{dR}}^+[[X_1 - 1, \dots, X_s - 1]]|_{\tilde{U}_{\infty,\alpha}}$.

It is enough to show that the map constructed above is indeed an inverse map. The composition

$$B_{\mathrm{dR}}^+[[X_1 - 1, \dots, X_s - 1]]|_{\tilde{U}_{\infty,\alpha}} \rightarrow \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+|_{\tilde{U}_{\infty,\alpha}} \rightarrow B_{\mathrm{dR}}^+[[X_1 - 1, \dots, X_s - 1]]|_{\tilde{U}_{\infty,\alpha}}$$

is the identity map by definition. Hence, it suffices to show that

$$(B_{\mathrm{dR}}^+(S^+)/\xi^n)[X_1, \dots, X_s]/I^n \rightarrow \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+(U)/\mathrm{Ker}(\theta)^n$$

is surjective for $n \geq 1$, where I is the ideal generated by $X_1 - 1, \dots, X_s - 1$. Let $(\mathrm{Spf}(E_n), \mathcal{M}_{E_n})$ denote the n -th infinitesimal neighborhood of $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ over $\mathrm{Spf}(W)$. Since $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow \mathrm{Spf}(W)$ is log smooth after passing to generic fibers, the ring map

$$f: R[X_1, \dots, X_s]/I^n \rightarrow E_n$$

defined by $f(X_i) = (t_i \otimes 1)(1 \otimes t_i)^{-1}$ is an isomorphism after inverting p . Since R is topologically of finite type over $\mathcal{O}_K$ and the image of the ring map

$$R \rightarrow B_{\mathrm{dR}}^+(S^+)[[X_1 - 1, \dots, X_s - 1]] \twoheadrightarrow S$$

is contained in S^+ , the image of the ring map

$$R \rightarrow B_{\mathrm{dR}}^+(S^+)[[X_1 - 1, \dots, X_s - 1]] \twoheadrightarrow B_{\mathrm{dR}}^+(S^+)/\xi^n$$

is contained in a p -complete ring $A_k := (A_{\mathrm{inf}}(S^+)/\xi^n)[\xi/p^k] \subset B_{\mathrm{dR}}^+(S^+)/\xi^n$ for some $k \geq 1$. Taking the p -complete base change of f along $R \rightarrow A_k$, we get a map

$$A_k[X_1, \dots, X_s]/I^n \rightarrow E_n \hat{\otimes}_R A_k$$

which is an isomorphism up to bounded p -power torsions. By composing a natural surjection $E_n \hat{\otimes}_R A_k \rightarrow \mathcal{O}\mathbb{A}_{\mathrm{inf}}(U)/\mathrm{Ker}(\theta)^n$ and inverting p , we conclude that the map

$$(B_{\mathrm{dR}}^+(S^+)/\xi^n)[X_1, \dots, X_s]/I^n \rightarrow \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+(U)/\mathrm{Ker}(\theta)^n$$

is surjective. $\square$

In [DLLZ23a, Definition 2.2.10], they introduced de Rham period sheaves $\mathcal{O}\mathbb{B}_{\mathrm{dR},\log}^+$ on locally noetherian fs log adic space. The following corollary shows that the sheaf $\mathcal{O}\mathbb{B}_{\mathrm{dR},\log}^+$ on the generic fiber of a semi-stable log formal scheme coincides with a de Rham period sheaf defined above by using a fixed formal model.

Corollary 5.4.7. For a semi-stable log formal scheme $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ over $\mathcal{O}_K$, there exists a natural isomorphism of sheaves on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \mathrm{prokét}}$

$$\mathcal{O}\mathbb{B}_{\mathrm{dR},\log}^+ \cong \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$$

that is compatible with filtrations and connections.

Proof. By working étale locally on $\mathfrak{X}$, we may assume that $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is a small affine log formal scheme with a fixed framing. Lemma 5.4.6 and [DLLZ23a, Proposition 2.3.15] give an isomorphism that is compatible with filtrations and connections

$$\mathcal{O}\mathbb{B}_{\mathrm{dR}, \log}^+|_{\tilde{U}_{\infty, \alpha}} \cong \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+|_{\tilde{U}_{\infty, \alpha}},$$

which descends to an isomorphism

$$\mathcal{O}\mathbb{B}_{\mathrm{dR}, \log}^+ \cong \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$$

that is compatible with filtrations and connections. $\square$

Let $\nu: (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \mathrm{prokét}} \rightarrow \mathfrak{X}_{\eta, \mathrm{an}}$ be the natural map of sites.

Proposition 5.4.8. There is a natural isomorphism

$$\mathcal{O}_{\mathfrak{X}_{\eta}} \cong \nu_* \mathcal{O}\mathbb{B}_{\mathrm{dR}}$$

that is compatible with connections.

Proof. Let $D_{\mathrm{dR}, \log}$ be the log p -adic Riemann-Hilbert functor defined in [DLLZ23a, (3.2.6) and Theorem 3.2.7 (1)]. Then we have isomorphisms

$$\mathcal{O}_{\mathfrak{X}_{\eta}} \cong D_{\mathrm{dR}, \log}(\widehat{\mathbb{Q}}_p) \cong \nu_* \mathcal{O}\mathbb{B}_{\mathrm{dR}}$$

by Corollary 5.4.7. $\square$

Next, we turn to defining crystalline and semi-stable period sheaves with connections. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a horizontally semi-stable log formal scheme over W .

Construction 5.4.9 (Crystalline period sheaves with connections). Let U be a big log affinoid perfectoid in $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \mathrm{prokét}}$ with the associated affinoid perfectoid space $\widehat{U} = \mathrm{Spa}(S, S^+)$. Choose a factorization $U \rightarrow (\mathrm{Spf}(R), \mathcal{M}_R)_{\eta} \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta}$, where the map $(\mathrm{Spf}(R), \mathcal{M}_R) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is strict étale (this indeed exists because U is big). The natural surjection $\theta: A_{\mathrm{inf}}(S^+) \rightarrow S^+$ induces a closed immersion

$$(\mathrm{Spf}(S^+), \mathcal{M}_{S^+}) \hookrightarrow (\mathrm{Spf}(R \widehat{\otimes}_W A_{\mathrm{inf}}(S^+)), \mathcal{M}_R \oplus \mathcal{M}_{A_{\mathrm{inf}}(S^+)})^a.$$

Let $(\mathrm{Spf}(\mathcal{O}\mathbb{A}_{\mathrm{crys}}(U)), \mathcal{M}_{\mathcal{O}\mathbb{A}_{\mathrm{crys}}(U)})$ be its p -adic log PD-envelope. This is independent of the choice of a factorization. We set

$$\begin{aligned} \mathcal{O}\mathbb{B}_{\mathrm{crys}}^+(U) &:= \mathcal{O}\mathbb{A}_{\mathrm{crys}}(U)[1/p] \\ \mathcal{O}\mathbb{B}_{\mathrm{crys}}(U) &:= \mathcal{O}\mathbb{B}_{\mathrm{crys}}^+(U)[1/\mu]. \end{aligned}$$

Suppose that $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is equipped with a Frobenius lift $\varphi_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})}$. Let φ_R denote a unique Frobenius lift on $(\mathrm{Spf}(R), \mathcal{M}_R)$ such that the strict étale map $(\mathrm{Spf}(R), \mathcal{M}_R) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is compatible with Frobenius lifts on both sides. We let $\varphi_{\mathcal{O}\mathbb{A}_{\mathrm{crys}}(U)}$ denote a Frobenius lift on $(\mathrm{Spf}(\mathcal{O}\mathbb{A}_{\mathrm{crys}}(U)), \mathcal{M}_{\mathcal{O}\mathbb{A}_{\mathrm{crys}}(U)})$ induced from the product Frobenius $\varphi_R \times \varphi_{A_{\mathrm{crys}}(S^+)}$ on $(\mathrm{Spf}(R \widehat{\otimes}_W A_{\mathrm{crys}}(S^+)), \mathcal{M}_R \oplus \mathcal{M}_{A_{\mathrm{crys}}(S^+)})^a$. Let $\varphi_{\mathcal{O}\mathbb{B}_{\mathrm{crys}}^+(U)}$ and $\varphi_{\mathcal{O}\mathbb{B}_{\mathrm{crys}}(U)}$ be the induced ring maps on respective rings.

We define a connection

$$\nabla_{\mathcal{O}\mathbb{A}_{\mathrm{crys}}(U)}: \mathcal{O}\mathbb{A}_{\mathrm{crys}}(U) \rightarrow \mathcal{O}\mathbb{A}_{\mathrm{crys}}(U) \otimes_R \Omega_{(\mathrm{Spf}(R), \mathcal{M}_R)/W}^1$$

(note that the right hand side is independent of the choice of $(\mathrm{Spf}(R), \mathcal{M}_R)$) as a unique $A_{\mathrm{crys}}(S^+)$ -linear continuous connection extending the derivation

$$R \rightarrow \Omega_{(\mathrm{Spf}(R), \mathcal{M}_R)/W}^1$$

satisfying the following condition:

- Suppose that we have a factorization $U \rightarrow (\mathrm{Spf}(R_1), \mathcal{M}_{R_1})_\eta \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_\eta$, where $(\mathrm{Spf}(R_1), \mathcal{M}_{R_1}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is strict étale. Then the equation

$$\nabla_{\mathcal{O}_{\mathrm{Acrys}}(U)}(m/m') = (m/m') \otimes \mathrm{dlog}(m)$$

holds for each $m \in \Gamma(\mathrm{Spf}(R_1), \mathcal{M}_{R_1})$ and $m' \in \Gamma(\mathrm{Spf}(A_{\mathrm{crys}}(S^+)), \mathcal{M}_{A_{\mathrm{crys}}(S^+)})$ with the common image in $\Gamma(\mathrm{Spf}(S^+), \mathcal{M}_{S^+})$. Here, we used a natural identification

$$\begin{aligned} & \mathcal{O}_{\mathrm{Acrys}}(U) \otimes_R \Omega^1_{(\mathrm{Spf}(R), \mathcal{M}_R)/(\mathrm{Spf}(W), \mathcal{M}_W)} \\ & \cong \mathcal{O}_{\mathrm{Acrys}}(U) \otimes_{R_1} \Omega^1_{(\mathrm{Spf}(R_1), \mathcal{M}_{R_1})/(\mathrm{Spf}(W), \mathcal{M}_W)}. \end{aligned}$$

We also have induced connections $\nabla_{\mathcal{O}_{\mathbb{B}_{\mathrm{crys}}^+}(U)}$ and $\nabla_{\mathcal{O}_{\mathbb{B}_{\mathrm{crys}}}(U)}$ on respective rings.

Let $\mathcal{O}_{\mathbb{B}_{\mathrm{crys}}}$ be the sheaf on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \mathrm{prok\acute{e}t}}$ associated with the presheaf given by $U \mapsto \mathcal{O}_{\mathbb{B}_{\mathrm{crys}}}(U)$. Let $\varphi_{\mathcal{O}_{\mathbb{B}_{\mathrm{crys}}}}$ be a ring map on $\mathcal{O}_{\mathbb{B}_{\mathrm{crys}}}$ induced from $\varphi_{\mathcal{O}_{\mathbb{B}_{\mathrm{crys}}}(U)}$ for each U . Connections $\nabla_{\mathcal{O}_{\mathbb{B}_{\mathrm{crys}}}(U)}$ for each U induces a connection

$$\nabla_{\mathcal{O}_{\mathbb{B}_{\mathrm{crys}}}} : \mathcal{O}_{\mathbb{B}_{\mathrm{crys}}} \rightarrow \mathcal{O}_{\mathbb{B}_{\mathrm{crys}}} \otimes_{\mathcal{O}_{\mathfrak{X}_\eta}} \Omega^1_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_\eta / K_0}.$$

Construction 5.4.10 (Semi-stable period sheaves with connections). Let U be a big log affinoid perfectoid with the associated affinoid perfectoid space $\widehat{U} = \mathrm{Spa}(S, S^+)$. We set

$$\begin{aligned} \mathcal{O}_{\mathbb{A}_{\mathrm{st}}}(U) &:= \mathcal{O}_{\mathbb{A}_{\mathrm{crys}}}(U) \otimes_{A_{\mathrm{crys}}(S^+)} A_{\mathrm{st}}(S^+) \\ \mathcal{O}_{\mathbb{B}_{\mathrm{st}}^+}(U) &:= \mathcal{O}_{\mathbb{A}_{\mathrm{st}}}(U)[1/p] \\ \mathcal{O}_{\mathbb{B}_{\mathrm{st}}}(U) &:= \mathcal{O}_{\mathbb{B}_{\mathrm{st}}^+}(U)[1/t]. \end{aligned}$$

When $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is equipped with a Frobenius lift $\varphi_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})}$, we define a Frobenius structure $\varphi_{\mathcal{O}_{\mathbb{A}_{\mathrm{st}}}(U)}$ on $\mathcal{O}_{\mathbb{A}_{\mathrm{st}}}(U)$ by

$$\varphi_{\mathcal{O}_{\mathbb{A}_{\mathrm{st}}}(U)} := \varphi_{\mathcal{O}_{\mathbb{A}_{\mathrm{crys}}}(U)} \otimes \varphi_{A_{\mathrm{st}}(S^+)}.$$

Similarly, we can define $\varphi_{\mathcal{O}_{\mathbb{B}_{\mathrm{st}}^+}(U)}$ and $\varphi_{\mathcal{O}_{\mathbb{B}_{\mathrm{st}}}(U)}$.

Let $N_{\mathcal{O}_{\mathbb{A}_{\mathrm{st}}}(U)}$ be a monodromy operator $\mathrm{id}_{\mathcal{O}_{\mathbb{A}_{\mathrm{crys}}}(U)} \otimes N_{A_{\mathrm{st}}(S^+)}$ on $\mathcal{O}_{\mathbb{A}_{\mathrm{st}}}(U)$. Let $\nabla_{\mathcal{O}_{\mathbb{A}_{\mathrm{st}}}(U)}$ be a connection $\nabla_{\mathcal{O}_{\mathbb{A}_{\mathrm{crys}}}(U)} \otimes \mathrm{id}_{A_{\mathrm{st}}(S^+)}$. Similarly, we can define $N_{\mathcal{O}_{\mathbb{B}_{\mathrm{st}}^+}(U)}$, $N_{\mathcal{O}_{\mathbb{B}_{\mathrm{st}}}(U)}$, $\nabla_{\mathcal{O}_{\mathbb{B}_{\mathrm{st}}^+}(U)}$, and $\nabla_{\mathcal{O}_{\mathbb{B}_{\mathrm{st}}}(U)}$.

Let $\mathcal{O}_{\mathbb{B}_{\mathrm{st}}}$ be the sheaf on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \mathrm{prok\acute{e}t}}$ associated with the presheaf given by $U \mapsto \mathcal{O}_{\mathbb{B}_{\mathrm{st}}}(U)$, which is equipped with a Frobenius structure $\varphi_{\mathcal{O}_{\mathbb{B}_{\mathrm{st}}}}$ and a monodromy operator $N_{\mathcal{O}_{\mathbb{B}_{\mathrm{st}}}}$ on $\mathcal{O}_{\mathbb{B}_{\mathrm{st}}}$ induced from $\varphi_{\mathcal{O}_{\mathbb{B}_{\mathrm{st}}}(U)}$ and $N_{\mathcal{O}_{\mathbb{B}_{\mathrm{st}}}(U)}$. We define a connection

$$\nabla_{\mathcal{O}_{\mathbb{B}_{\mathrm{st}}}} : \mathcal{O}_{\mathbb{B}_{\mathrm{st}}} \rightarrow \mathcal{O}_{\mathbb{B}_{\mathrm{st}}} \otimes_{R[1/p]} \Omega^1_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_\eta / K_0}$$

by $\nabla_{\mathcal{O}_{\mathbb{B}_{\mathrm{st}}}} := \nabla_{\mathcal{O}_{\mathbb{B}_{\mathrm{crys}}}} \otimes_{\mathbb{B}_{\mathrm{crys}}} \mathrm{id}_{\mathbb{B}_{\mathrm{st}}}$.

Construction 5.4.11 (cf. [AI12, 2.3.4]). Let $U \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \mathrm{prok\acute{e}t}}$ be a big log affinoid perfectoid with the associated affinoid perfectoid space $\widehat{U} = \mathrm{Spa}(S, S^+)$. Choose a factorization $U \rightarrow (\mathrm{Spf}(R), \mathcal{M}_R)_\eta \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_\eta$, where $(\mathrm{Spf}(R), \mathcal{M}_R) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is strict étale. The object U is also big with respect to a semi-stable formal model $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}} \oplus p^{\mathbb{N}})^a$ by definition. We consider the log structure $\mathcal{M}_{S^+}$ and $\mathcal{M}_{A_{\mathrm{crys}}(S^+)}$ defined by the bigness with respect to $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}} \oplus p^{\mathbb{N}})^a$. There is a natural strict closed immersion

$$(\mathrm{Spf}(S^+), \mathcal{M}_{S^+}) \hookrightarrow (\mathrm{Spf}(A_{\mathrm{crys}}(S^+)), \mathcal{M}_{A_{\mathrm{crys}}(S^+)}).$$

We set $(\mathrm{Spf}(R\langle u \rangle), \mathcal{M}_{R\langle u \rangle}) := (\mathrm{Spf}(R\langle u \rangle), \mathcal{M}_R \oplus u^{\mathbb{N}})^a$. We have a map

$$(\mathrm{Spf}(S^+), \mathcal{M}_{S^+}) \rightarrow (\mathrm{Spf}(R\langle u \rangle), \mathcal{M}_{R\langle u \rangle})$$

given by the structure map $R \rightarrow S^+$ and $u \mapsto p$. Then these two maps induce a closed immersion

$$(\mathrm{Spf}(S^+), \mathcal{M}_{S^+}) \hookrightarrow (\mathrm{Spf}(A_{\mathrm{crys}}(S^+) \widehat{\otimes}_W R\langle u \rangle), \mathcal{M}_{A_{\mathrm{crys}}(S^+) \oplus \mathcal{M}_{R\langle u \rangle}})^a.$$

Let $(\mathrm{Spf}(\widehat{\mathcal{O}}_{\widehat{\mathbb{A}}_{\mathrm{st}}}(U)), \mathcal{M}_{\widehat{\mathcal{O}}_{\widehat{\mathbb{A}}_{\mathrm{st}}}(U)})$ denote its p -completed log PD-envelope, which is independent of the choice of $(\mathrm{Spf}(R), \mathcal{M}_R)$.

By the universal property of PD-envelopes, there is a natural isomorphism

$$\widehat{\mathcal{O}}_{\widehat{\mathbb{A}}_{\mathrm{st}}}(U) \cong \mathcal{O}_{\mathbb{A}_{\mathrm{crys}}}(U) \widehat{\otimes}_{A_{\mathrm{crys}}(S^+)} \widehat{A}_{\mathrm{st}}(S^+).$$

When $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is equipped with a Frobenius lift $\varphi_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})}$, we define a Frobenius lift $\varphi_{\widehat{\mathcal{O}}_{\widehat{\mathbb{A}}_{\mathrm{st}}}(U)}$ on $\widehat{\mathcal{O}}_{\widehat{\mathbb{A}}_{\mathrm{st}}}(U)$ as the product Frobenius $\varphi_{\mathcal{O}_{\mathbb{A}_{\mathrm{crys}}}(U)} \otimes \varphi_{\widehat{A}_{\mathrm{st}}(U)}$. Let $N_{\widehat{\mathcal{O}}_{\widehat{\mathbb{A}}_{\mathrm{st}}}(U)}$ be the monodromy operator on $\widehat{\mathcal{O}}_{\widehat{\mathbb{A}}_{\mathrm{st}}}(U)$ given by $\mathrm{id}_{\mathcal{O}_{\mathbb{A}_{\mathrm{crys}}}(U)} \otimes N_{\widehat{A}_{\mathrm{st}}(U)}$.

We set

$$\begin{aligned} \widehat{\mathcal{O}}_{\widehat{\mathbb{B}}_{\mathrm{st}}}^+(U) &:= \widehat{\mathcal{O}}_{\widehat{\mathbb{A}}_{\mathrm{st}}}(U)[1/p] \\ \widehat{\mathcal{O}}_{\widehat{\mathbb{B}}_{\mathrm{st}}}(U) &:= \widehat{\mathcal{O}}_{\widehat{\mathbb{B}}_{\mathrm{st}}}^+(U)[1/t]. \end{aligned}$$

Let $\varphi_{\widehat{\mathcal{O}}_{\widehat{\mathbb{B}}_{\mathrm{st}}}^+(U)}$, $\varphi_{\widehat{\mathcal{O}}_{\widehat{\mathbb{B}}_{\mathrm{st}}}(U)}$, $N_{\widehat{\mathcal{O}}_{\widehat{\mathbb{B}}_{\mathrm{st}}}^+(U)}$, $N_{\widehat{\mathcal{O}}_{\widehat{\mathbb{B}}_{\mathrm{st}}}(U)}$ be the induced Frobenius structures and monodromy operators on respective rings.

Let $\widehat{\mathcal{O}}_{\widehat{\mathbb{B}}_{\mathrm{st}}}$ be the sheaf on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \mathrm{prok\acute{e}t}}$ associated with the presheaf given by $U \rightarrow \widehat{\mathcal{O}}_{\widehat{\mathbb{B}}_{\mathrm{st}}}(U)$. This sheaf is equipped with the induced Frobenius structure and the monodromy operator, denoted by $\varphi_{\widehat{\mathcal{O}}_{\widehat{\mathbb{B}}_{\mathrm{st}}}}$ and $N_{\widehat{\mathcal{O}}_{\widehat{\mathbb{B}}_{\mathrm{st}}}}$.

We shall give a local description of period sheaves $\mathcal{O}_{\mathbb{B}_{\mathrm{crys}}}$, $\mathcal{O}_{\mathbb{B}_{\mathrm{st}}}$, and $\widehat{\mathcal{O}}_{\widehat{\mathbb{B}}_{\mathrm{st}}}$. We define a sheaf $\mathbb{B}_{\mathrm{crys}}\{X_1 - 1, \dots, X_s - 1\}$ on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \mathrm{prok\acute{e}t}}$ by

$$U \mapsto A_{\mathrm{crys}}(S^+)\{X_1 - 1, \dots, X_s - 1\}[1/pt]$$

for a log affinoid perfectoid U with the associated affinoid perfectoid $\widehat{U} = \mathrm{Spa}(S, S^+)$. We define $\mathbb{B}_{\mathrm{crys}}\{X_1 - 1, \dots, X_s - 1, Y - 1\}$ in the same way. Furthermore, we define the sheaf $\mathbb{B}_{\mathrm{crys}}\{X_1 - 1, \dots, X_s - 1\}[\log(Y)]$ on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \mathrm{prok\acute{e}t}}$ by

$$U \mapsto A_{\mathrm{crys}}(S^+)\{X_1 - 1, \dots, X_s - 1\}[\log(Y)][1/pt],$$

where $\log(Y)$ is regarded as an independent variable.

Lemma 5.4.12. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) = (\mathrm{Spf}(R), \mathcal{M}_R)$ be a small affine log formal scheme over W with a fixed framing such that $n = 0$. Then there is the following commutative diagram of sheaves on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \mathrm{prok\acute{e}t}}/\tilde{U}_{\infty, \alpha}$:

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{B}_{\mathrm{crys}}}|_{\tilde{U}_{\infty, \alpha}} & \xrightarrow{\sim} & \mathbb{B}_{\mathrm{crys}}\{X_1 - 1, \dots, X_s - 1\}|_{\tilde{U}_{\infty, \alpha}} \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathbb{B}_{\mathrm{st}}}|_{\tilde{U}_{\infty, \alpha}} & \xrightarrow{\sim} & \mathbb{B}_{\mathrm{crys}}\{X_1 - 1, \dots, X_s - 1\}[\log(Y)]|_{\tilde{U}_{\infty, \alpha}} \\ \downarrow & & \downarrow \\ \widehat{\mathcal{O}}_{\widehat{\mathbb{B}}_{\mathrm{st}}}|_{\tilde{U}_{\infty, \alpha}} & \xrightarrow{\sim} & \mathbb{B}_{\mathrm{crys}}\{X_1 - 1, \dots, X_s - 1, Y - 1\}|_{\tilde{U}_{\infty, \alpha}}. \end{array}$$

Proof. Since we have isomorphisms

$$\mathcal{O}\mathbb{B}_{\text{st}} \cong \mathcal{O}\mathbb{B}_{\text{crys}} \otimes_{\mathbb{B}_{\text{crys}}} \mathbb{B}_{\text{st}}, \quad \widehat{\mathcal{O}\mathbb{B}}_{\text{st}} \cong \mathcal{O}\mathbb{B}_{\text{crys}} \otimes_{\mathbb{B}_{\text{crys}}} \widehat{\mathbb{B}}_{\text{st}},$$

it suffices to construct the upper horizontal isomorphism. Let U be a big log affinoid perfectoid in $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \text{prokét}}/\tilde{U}_{\infty, \alpha}$ with the associated affinoid perfectoid space $\widehat{U} = \text{Spa}(S, S^+)$. Define a map of $A_{\text{crys}}(S^+)$ -algebras

$$(5.4.1) \quad A_{\text{crys}}(S^+)\{X_1 - 1, \dots, X_s - 1\} \rightarrow \mathcal{O}\mathbb{A}_{\text{crys}}(U)$$

sending X_i to $t_i/[t_i^p]$ for each i . We shall construct the inverse map. Consider a map of W -algebras $R^0 \rightarrow A_{\text{crys}}(S^+)\{X_1 - 1, \dots, X_s - 1\}$ given by $t_i \mapsto [t_i^p]X_i$ for each i .

Since $R^0 \rightarrow R$ is p -completely étale, there exists a unique ring map $R \rightarrow A_{\text{crys}}(S^+)\{X_1 - 1, \dots, X_s - 1\}$ fitting into the following commutative diagram:

$$\begin{array}{ccc} R^0 & \longrightarrow & A_{\text{crys}}(S^+)\{X_1 - 1, \dots, X_s - 1\} \\ \downarrow & \nearrow & \downarrow \\ R & \longrightarrow & S^+. \end{array}$$

This map induces a map of $A_{\text{crys}}(S^+)$ -algebras

$$R \widehat{\otimes}_W A_{\text{crys}}(S^+) \rightarrow A_{\text{crys}}(S^+)\{X_1 - 1, \dots, X_s - 1\},$$

which induces a ring map

$$(5.4.2) \quad (R \widehat{\otimes}_W A_{\text{crys}}(S^+))[t_1/[t_1^p], \dots, t_s/[t_s^p]]^\wedge_p \rightarrow A_{\text{crys}}(S^+)\{X_1 - 1, \dots, X_s - 1\}$$

sending $t_i/[t_i^p]$ to X_i for each i . Notice that the left hand side of the map (5.4.2) is nothing but the exactification of the closed immersion

$$(\text{Spf}(S^+), \mathcal{M}_{S^+}) \hookrightarrow (\text{Spf}(R \widehat{\otimes}_W A_{\text{crys}}(S^+)), \mathcal{M}_R \oplus \mathcal{M}_{A_{\text{crys}}(S^+)})^a.$$

Since the ring map (2) is compatible with surjections to S^+ , we obtain a ring map

$$(5.4.3) \quad \mathcal{O}\mathbb{A}_{\text{crys}}(U) \rightarrow A_{\text{crys}}(S^+)\{X_1 - 1, \dots, X_s - 1\}$$

by taking PD-envelopes.

It is enough to prove the map (5.4.3) is indeed the inverse of the map (5.4.1). The composition

$$A_{\text{crys}}(S^+)\{X_1 - 1, \dots, X_s - 1\} \rightarrow \mathcal{O}\mathbb{A}_{\text{crys}}(U) \rightarrow A_{\text{crys}}(S^+)\{X_1 - 1, \dots, X_s - 1\}$$

is the identity map by definition. Hence, it suffices to show that the map $\mathcal{O}\mathbb{A}_{\text{crys}}(U) \rightarrow A_{\text{crys}}(S^+)\{X_1 - 1, \dots, X_s - 1\}$ is injective. Combining with isomorphisms in Lemma 5.4.6, we get the following commutative diagram.

$$\begin{array}{ccccc} A_{\text{crys}}(S^+)\{X_1 - 1, \dots, X_s - 1\} & \longrightarrow & \mathcal{O}\mathbb{A}_{\text{crys}}(U) & \longrightarrow & A_{\text{crys}}(S^+)\{X_1 - 1, \dots, X_s - 1\} \\ \downarrow & & \downarrow & & \downarrow \\ B_{\text{dR}}^+(S^+)[[X_1 - 1, \dots, X_s - 1]] & \xrightarrow{\sim} & \mathcal{O}\mathbb{B}_{\text{dR}}^+(U) & \xrightarrow{\sim} & B_{\text{dR}}^+(S^+)[[X_1 - 1, \dots, X_s - 1]] \end{array}$$

The middle vertical map is an isomorphism by Lemma 5.4.13 below. Therefore, we conclude that the upper right map is injective, and so the map

$$A_{\text{crys}}(S^+)\{X_1 - 1, \dots, X_s - 1\} \rightarrow \mathcal{O}\mathbb{A}_{\text{crys}}(U)$$

is an isomorphism. By inverting μ , we obtain the upper horizontal isomorphism in the diagram in the assertion. $\square$

Lemma 5.4.13. Let A be a p -torsion free p -complete ring and $I = (f_1, \dots, f_r)$ be an ideal of A . Suppose that $(\bar{f}_1, \dots, \bar{f}_r)$ is a Koszul-regular sequence in A/p and that A/I is p -torsion free. Let D denote the p -completed PD-envelope of $A \rightarrow A/I$ and B denote the I -adic completion of $A[1/p]$. Then there is a natural injective map $D \rightarrow B$.

Proof. We follow the argument of [CK19, Proposition 5.36]. Let $C := A[\{f_i^{[n]} = \frac{f_i^n}{n!}\}_{1 \leq i \leq r, n \geq 0}] \subset A[1/p]$. By [BS22, Lemma 2.38], we have $D \cong C^\wedge$, where $\wedge$ means the (classical) p -adic completion. Since a Koszul-regular sequence is quasi-regular and A/I is p -torsion free, the I^n/I^{n+1} is also p -torsion free for each $n \geq 0$. Hence, A/I^n is also p -torsion free. Since the image of the natural map $C \hookrightarrow A[1/p] \rightarrow (A/I^n)[1/p]$ is contained in an open bounded subring, this extends to a map $D \rightarrow (A/I^n)[1/p]$. Varying $n \geq 0$, we get a map $f: D \rightarrow B$.

We consider filtrations on each ring. Let $\text{Fil}^\bullet(C)$ be the PD-adic filtration on C . In other words, $\text{Fil}^n(C)$ is the ideal generated by $f_i^{[m]}$ for $1 \leq i \leq r$ and $m \geq n$. We equip D with the filtration $\text{Fil}^\bullet(D)$ such that $\text{Fil}^n(D)$ is the closure of $\text{Fil}^n(C)$. Let $\text{Fil}^\bullet(B)$ be the IB -adic filtration on B . Suppose that $\sum_{\substack{\underline{m}=(m_1, \dots, m_r) \\ m_1 + \dots + m_r = n}} a_{\underline{m}} f^{[\underline{m}]} \in \text{Fil}^{n+1}(C)$, where $a_{\underline{m}} \in A$

and $f^{[\underline{m}]} := \prod_{i=1}^r f_i^{[m_i]}$. Since $(f_1, \dots, f_r)$ is a quasi-regular sequence in $A[1/p]$, we have $a_{\underline{m}} \in I[1/p] \cap A = I$ for each $\underline{m}$. Therefore, we see that, for each $n \geq 0$, there is a natural isomorphism

$$\bigoplus_{\substack{\underline{m}=(m_1, \dots, m_r) \\ m_1 + \dots + m_r = n}} (A/I) \cdot f^{[\underline{m}]} \cong \text{gr}^n(C)$$

and the natural map $\text{gr}^n(C) \rightarrow \text{gr}^n(B)$ is injective. In particular, $\text{gr}^n(C)$ and $C/\text{Fil}^n(C)$ are p -torsion free and p -complete. Hence, the natural map $\text{Fil}^n(C)/p^m \rightarrow C/p^m$ is injective for each $n, m \geq 0$. Taking limits with respect to $m \geq 0$, we get an injection $\text{Fil}^n(C)^\wedge \rightarrow D$, which gives $\text{Fil}^n(C)^\wedge \xrightarrow{\sim} \text{Fil}^n(D)$. Furthermore, by taking the p -completion of the exact sequence

$$0 \rightarrow \text{Fil}^{n+1}(C) \rightarrow \text{Fil}^n(C) \rightarrow \text{gr}^n(C) \rightarrow 0,$$

we obtain an exact sequence

$$0 \rightarrow \text{Fil}^{n+1}(D) \rightarrow \text{Fil}^n(D) \rightarrow \text{gr}^n(C) \rightarrow 0.$$

In other words, the natural map $\text{gr}^n(C) \rightarrow \text{gr}^n(D)$ is an isomorphism. Since $\text{gr}^n(C) \rightarrow \text{gr}^n(B)$ is injective, the map $\text{gr}^n(D) \rightarrow \text{gr}^n(B)$ is also injective.

Since $\text{Fil}^\bullet(B)$ is separated, it is enough to prove that $\text{Fil}^\bullet(D)$ is separated. By taking the mod- p reduction of the pushout square in [BS22, Lemma 2.38], we get an isomorphism

$$C/p = D/p \cong A/(p, f_1^p, \dots, f_r^p)[\{X_{i,m}\}_{1 \leq i \leq r, m \geq 1}]/(X_{i,m}^p)_{1 \leq i \leq r, m \geq 1}$$

uniquely determined by $f_i^{[p^m]} \mapsto X_{i,m}$. From this, we see that $\text{Fil}^\bullet(D)/p$ is a separated filtration on D/p , which implies that $\text{Fil}^\bullet(D)$ is separated due to the p -torsion freeness of $D/\text{Fil}^n(D)$. This proves the claim. $\square$

Corollary 5.4.14. For period sheaves on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \text{prokét}}$, the following statements hold.

(1) There is the following commutative diagram:

$$\begin{array}{ccc}
 \mathbb{B}_{\text{crys}} & \xrightarrow{\sim} & \mathcal{O}\mathbb{B}_{\text{crys}}^{\nabla=0} \\
 \downarrow & & \downarrow \\
 \mathbb{B}_{\text{st}} & \xrightarrow{\sim} & \mathcal{O}\mathbb{B}_{\text{st}}^{\nabla=0} \\
 \downarrow & & \downarrow \\
 \widehat{\mathbb{B}}_{\text{st}} & \xrightarrow{\sim} & \mathcal{O}\widehat{\mathbb{B}}_{\text{st}}^{\nabla=0}.
 \end{array}$$

(2) There is a natural isomorphism

$$\mathcal{O}\mathbb{B}_{\text{crys}} \xrightarrow{\sim} \mathcal{O}\mathbb{B}_{\text{st}}^{N=0}.$$

(3) The sheaf $\mathcal{O}\mathbb{B}_{\text{st}}$ is the subsheaf of $\widehat{\mathcal{O}\mathbb{B}}_{\text{st}}$ consisting of nilpotent sections for $N_{\widehat{\mathcal{O}\mathbb{B}}_{\text{st}}}$.

Proof. Since all assertions can be checked locally on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{prokét}}$, they follow from Lemma 5.4.12 and its proof. $\square$

We can also consider crystalline and semi-stable period sheaves with connections for horizontally semi-stable log formal scheme over a not absolutely unramified base by using a fixed model over W .

Definition 5.4.15. Suppose that $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ admit a fixed unramified model $(\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})$. We define period sheaves $\mathcal{O}\mathbb{B}_{\text{crys}, (\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})}$ (resp. $\mathcal{O}\mathbb{B}_{\text{st}, (\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})}$) on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \text{prokét}}$ equipped connections $\nabla_{\mathcal{O}\mathbb{B}_{\text{crys}, (\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})}}$ (resp. $\nabla_{\mathcal{O}\mathbb{B}_{\text{st}, (\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})}}$) as the pullback of $(\mathcal{O}\mathbb{B}_{\text{crys}}, \nabla_{\mathcal{O}\mathbb{B}_{\text{crys}}})$ (resp. $(\mathcal{O}\mathbb{B}_{\text{st}}, \nabla_{\mathcal{O}\mathbb{B}_{\text{st}}})$) along a strict étale map $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta} \rightarrow (\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})_{\eta}$. When $(\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})$ is equipped with a Frobenius lift, $\mathcal{O}\mathbb{B}_{\text{crys}, (\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})}$ and $\mathcal{O}\mathbb{B}_{\text{st}, (\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})}$ are also equipped induced Frobenius structures, denoted by $\varphi_{\mathcal{O}\mathbb{B}_{\text{crys}}}$ and $\varphi_{\mathcal{O}\mathbb{B}_{\text{st}}}$ respectively.

Proposition 5.4.16. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a horizontally semi-stable log formal scheme over $\mathcal{O}_K$ with an unramified model $(\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})$. Let $\mu: (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \text{prokét}} \rightarrow \mathfrak{X}_{0, \eta, \text{ét}}$ be the natural maps of sites. Then there are natural isomorphisms

$$\mathcal{O}_{\mathfrak{X}_0}[1/p] \cong \mu_* \mathcal{O}\mathbb{B}_{\text{crys}, (\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})} \cong \mu_* \mathcal{O}\mathbb{B}_{\text{st}, (\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})}$$

that are compatible with connections. Furthermore, when $(\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})$ is equipped with a Frobenius lift, the above isomorphisms are compatible with Frobenius structures.

Proof. By Lemma 5.4.6, Proposition 5.4.8, and Lemma 5.4.12, we obtain injective maps

$$\mathcal{O}_{\mathfrak{X}_0}[1/p] \hookrightarrow \mu_* \mathcal{O}\mathbb{B}_{\text{crys}, (\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})} \hookrightarrow \mu_* \mathcal{O}\mathbb{B}_{\text{st}, (\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})} \hookrightarrow \mu_* \mathcal{O}\mathbb{B}_{\text{dR}}.$$

Taking tensor products with K , we get maps

$$(5.4.4) \quad \mathcal{O}_{\mathfrak{X}_0} \otimes_W K \hookrightarrow \mu_* \mathcal{O}\mathbb{B}_{\text{crys}, (\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})} \otimes_{K_0} K \hookrightarrow \mu_* \mathcal{O}\mathbb{B}_{\text{st}, (\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})} \otimes_{K_0} K \rightarrow \mu_* \mathcal{O}\mathbb{B}_{\text{dR}}$$

Since the natural map $\mathbb{B}_{\text{st}} \otimes_{K_0} K \rightarrow \mathbb{B}_{\text{dR}}$ is injective (for example, [Shi22, Proposition 2.27]), it follows from Lemma 5.4.6 and Lemma 5.4.12 that the map

$$\mathcal{O}\mathbb{B}_{\text{st}, (\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})} \otimes_{K_0} K \rightarrow \mathcal{O}\mathbb{B}_{\text{dR}}$$

is injective. Hence, the map

$$\mu_* \mathcal{O}\mathbb{B}_{\text{st}, (\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})} \otimes_{K_0} K \rightarrow \mu_* \mathcal{O}\mathbb{B}_{\text{dR}}$$

is also injective. Therefore, Proposition 5.4.8 implies that every map in the diagram (5.4.4) is an isomorphism. This proves the assertion. $\square$

5.4.3. Association for local systems on log adic spaces. In this subsection, we define the notion of semi-stable $\mathbb{Z}_p$ -local systems on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta}$ for a semi-stable log formal scheme $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ and study their properties. The most part of this subsection follows the argument of [GR24].

Construction 5.4.17. Let $\mathbb{B} \in \{\mathbb{B}_{\text{crys}}, \mathbb{B}_{\text{dR}}^+, \mathbb{B}_{\text{dR}}\}$. For an isocrystal $\mathcal{E}$ on $(X_0, \mathcal{M}_{X_0})$, we define $\mathbb{B}$ -module $\mathbb{B}(\mathcal{E})$ on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \text{prokét}}$ by

$$\mathbb{B}(\mathcal{E})(U) := \mathcal{E}(A_{\text{crys}}(S^+) \twoheadrightarrow S^+/p, \mathcal{M}_{A_{\text{crys}}(S^+)}) \otimes_{B_{\text{crys}}^+(S^+)} \mathbb{B}(U)$$

for a big log affinoid perfectoid U in $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \text{prokét}}$ with the associated affinoid perfectoid space $\widehat{U} = \text{Spa}(S, S^+)$. When $\mathbb{B} \in \{\mathbb{B}_{\text{dR}}^+, \mathbb{B}_{\text{dR}}\}$, there exists a natural isomorphism

$$\mathbb{B}(\mathcal{E})(U) \cong \mathcal{E}(A_{\text{crys}, K}(S^+) \twoheadrightarrow S^+/p, \mathcal{M}_{A_{\text{crys}, K}(S^+)}) \otimes_{A_{\text{crys}, K}(S^+)} \mathbb{B}(U)$$

by the crystal property of $\mathcal{E}$.

Let $\mathbb{B} \in \{\mathbb{B}_{\text{dR}}^+, \mathbb{B}_{\text{dR}}\}$. When $\mathcal{E}$ is enhanced to a filtered isocrystal $(\mathcal{E}, \text{Fil}^\bullet(E))$, we can define a descending filtration on $\mathbb{B}(\mathcal{E})$ as follows. Let U be a big log affinoid perfectoid in $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{prokét}}$ over $(\text{Spf}(\tilde{R}_{\infty, \alpha}), \mathcal{M}_{\tilde{R}_{\infty, \alpha}})_{\eta}$ and $\widehat{U} = \text{Spa}(S, S^+)$ be the associated affinoid perfectoid space. Choose a morphism $s: (\text{Spf}(A_{\text{crys}, K}(S^+)), \mathcal{M}_{A_{\text{crys}, K}(S^+)}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ commuting the following diagram:

$$\begin{array}{ccc} (\text{Spf}(S^+), \mathcal{M}_{S^+}) & \hookrightarrow & (\text{Spf}(A_{\text{crys}, K}(S^+)), \mathcal{M}_{A_{\text{crys}, K}(S^+)}) \\ \downarrow & \swarrow s & \\ (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) & & \end{array} .$$

By the crystal property of $\mathcal{E}$, the map of log PD-thickenings

$$s: (R \twoheadrightarrow R/p, \mathcal{M}_R) \rightarrow (A_{\text{crys}, K}(S^+) \twoheadrightarrow S^+/p, \mathcal{M}_{A_{\text{crys}, K}(S^+)})$$

gives an isomorphism

$$\mathbb{B}(\mathcal{E})(U) \cong E \otimes_{R[1/p], s} \mathbb{B}(U).$$

The right hand side is equipped with the product filtration $\text{Fil}^\bullet(E) \otimes \text{Fil}^\bullet(\mathbb{B}(U))$, which induced a filtration on the left side.

To justify this procedure, we need to check that such a map s indeed exists and that the resulting filtration on $\mathbb{B}(\mathcal{E})(U)$ is independent of the choice of s . First, we shall prove the existence of s . Take a factorization

$$U \rightarrow (\text{Spf}(\tilde{R}_{\infty, \alpha}), \mathcal{M}_{\tilde{R}_{\infty, \alpha}})_{\eta} \rightarrow (\text{Spf}(R), \mathcal{M}_R)_{\eta} \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta}$$

as in Definition 5.4.4. Let $(A_{\text{crys}, K}(S^+), \mathbb{Q}_{\geq 0}^r \oplus \mathbb{Z} \cdot e)$ be a prelog ring defined by $\mathbb{Q}_{\geq 0}^r \rightarrow A_{\text{inf}}(S^+) \rightarrow A_{\text{crys}, K}(S^+)$ and $e \mapsto \pi/[\pi^b]$. Note that we have a natural isomorphism

$$(\text{Spf}(A_{\text{crys}, K}(S^+)), \mathbb{Q}_{\geq 0}^r \oplus \mathbb{Z} \cdot e)^a \cong (\text{Spf}(A_{\text{crys}, K}(S^+)), \mathcal{M}_{A_{\text{crys}, K}(S^+)}).$$

Consider a map of prelog rings

$$(R^0, \mathbb{N}^r) \rightarrow (A_{\text{crys}, K}(S^+), \mathbb{Q}_{\geq 0}^r \oplus \mathbb{Z} \cdot e)$$

defined by

$$\begin{aligned} x_i &\mapsto [x_i^b] \quad (1 \leq i \leq l) \\ y_j &\mapsto [y_j^b] \quad (1 \leq j \leq m) \\ z_k &\mapsto [z_k^b] \quad (1 \leq k \leq n-1) \\ z_n &\mapsto \pi\left(\prod_{k=1}^{n-1} [z_k^b]\right)^{-1} \end{aligned}$$

and the monoid map $\mathbb{N}^r \rightarrow \mathbb{Q}_{\geq 0}^r \oplus \mathbb{Z} \cdot e$ defined by

$$e_i \mapsto \begin{cases} e_i & (1 \leq i \leq r-1) \\ e_n + e & (i = n). \end{cases}$$

Then, passing to the associated log structures, we obtain a desired map s .

To check that the resulting filtration on $\mathbb{B}(\mathcal{E})(U)$ is independent of the choice of s , it suffices to prove that, for another choice

$$s': (\mathrm{Spf}(A_{\mathrm{crys},K}(S^+)), \mathcal{M}_{A_{\mathrm{crys},K}(S^+)}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}),$$

the isomorphism

$$E \otimes_{\mathcal{O}_{\mathfrak{X}_\eta}, s} \mathbb{B}(U) \cong \mathbb{B}(\mathcal{E})(U) \cong E \otimes_{\mathcal{O}_{\mathfrak{X}_\eta}, s'} \mathbb{B}(U)$$

is a filtered isomorphism. This isomorphism can be described by using the connection ∇_E as in Construction 5.3.12, and the claim follows from this description and Griffiths transversality (cf. [GR24, Construction 2.28], [DLMS24, Construction 3.44]).

Definition 5.4.18. Let $\mathbb{L}$ be a Kummer étale $\mathbb{Z}_p$ -local system on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_\eta$.

- (1) We say that $\mathbb{L}$ is *weakly semi-stable* if, for some locally free F -isocrystal $\mathcal{E}$ on $(X, \mathcal{M}_X)$, there exists an isomorphism

$$\vartheta: \mathbb{L} \otimes_{\mathbb{Z}_p} \mathbb{B}_{\mathrm{crys}} \cong \mathbb{B}_{\mathrm{crys}}(\mathcal{E})$$

which is compatible with Frobenius isomorphisms. In this case, we say that $\mathbb{L}$ and $\mathcal{E}$ are *associated*.

- (2) We say that $\mathbb{L}$ is *semi-stable* if, for some filtered F -isocrystal $(\mathcal{E}, \mathrm{Fil}^\bullet(E))$ on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$, there exists an isomorphism

$$\vartheta: \mathbb{L} \otimes_{\mathbb{Z}_p} \mathbb{B}_{\mathrm{crys}} \cong \mathbb{B}_{\mathrm{crys}}(\mathcal{E})$$

which is compatible with Frobenius isomorphisms and filtered after taking the base change along $\mathbb{B}_{\mathrm{crys}} \rightarrow \mathbb{B}_{\mathrm{dR}}$. Here, the filtration on $\mathbb{B}_{\mathrm{dR}}(\mathcal{E})$ is the one constructed in Construction 5.4.17. In this case, we say that $\mathbb{L}$ and $(\mathcal{E}, \mathrm{Fil}^\bullet(E))$ are *associated*.

Proposition 5.4.19. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a semi-stable log formal scheme over $\mathcal{O}_K$ and let $\mathcal{E}$ be an isocrystal on $(X_0, \mathcal{M}_{X_0})$. Let (E, ∇_E) be the corresponding vector bundle with an integrable connection on $\mathfrak{X}_\eta$.

- (1) Let ∇' be the product connection ∇_E and $\nabla_{\mathcal{O}_{\mathbb{B}_{\mathrm{dR}}^+}}$ on $\mathcal{O}_{\mathbb{B}_{\mathrm{dR}}^+}$ -vector bundle $E \otimes_{\mathcal{O}_{\mathfrak{X}_\eta}} \mathcal{O}_{\mathbb{B}_{\mathrm{dR}}^+}$. Then $(E \otimes_{\mathcal{O}_{\mathfrak{X}_\eta}} \mathcal{O}_{\mathbb{B}_{\mathrm{dR}}^+}, \nabla')$ is naturally isomorphic to $(\mathbb{B}_{\mathrm{dR}}^+(\mathcal{E}) \otimes_{\mathbb{B}_{\mathrm{dR}}^+} \mathcal{O}_{\mathbb{B}_{\mathrm{dR}}^+}, \mathrm{id} \otimes \nabla_{\mathcal{O}_{\mathbb{B}_{\mathrm{dR}}^+}})$.
- (2) Assume that $\mathcal{E}$ underlies a filtered isocrystal $(\mathcal{E}, \mathrm{Fil}^\bullet(E))$, and let $\mathrm{Fil}^\bullet \mathbb{B}_{\mathrm{dR}}^+(\mathcal{E})$ be the induced filtration on $\mathbb{B}_{\mathrm{dR}}^+(\mathcal{E})$. Then the product filtration $\mathrm{Fil}^\bullet E \otimes_{\mathcal{O}_{\mathfrak{X}_\eta}} \mathrm{Fil}^\bullet \mathcal{O}_{\mathbb{B}_{\mathrm{dR}}^+, \log}$ is naturally isomorphic to the product filtration $\mathrm{Fil}^\bullet \mathbb{B}_{\mathrm{dR}}^+(\mathcal{E}) \otimes_{\mathbb{B}_{\mathrm{dR}}^+} \mathrm{Fil}^\bullet \mathcal{O}_{\mathbb{B}_{\mathrm{dR}}^+, \log}$.

Proof. We follow the argument of [GR24, Proposition 2.36]. By working étale locally on $\mathfrak{X}$, we may assume that $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) = (\mathrm{Spf}(R), \mathcal{M}_R)$ is a small affine log formal scheme with a fixed framing. Let U be a big log affinoid perfectoid in $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \mathrm{prokét}}$ with the associated affinoid perfectoid space $\widehat{U} = \mathrm{Spa}(S, S^+)$. We define $\mathcal{O}\mathbb{A}_{\mathrm{crys}}(S^+)$, $\mathcal{O}\mathbb{A}_{\mathrm{crys}}(1)(S^+)$, and $D^+(1)$ to be the p -completed log PD-envelope of

$$\begin{aligned} (R\widehat{\otimes}_W A_{\mathrm{inf}}(S^+), \mathcal{M}_R \oplus \mathcal{M}_{A_{\mathrm{inf}}(S^+)})^a &\rightarrow (S^+, \mathcal{M}_{S^+}), \\ (R\widehat{\otimes}_{\mathcal{O}_K} R\widehat{\otimes}_W A_{\mathrm{inf}}(S^+), \mathcal{M}_R \oplus \mathcal{M}_R \oplus \mathcal{M}_{A_{\mathrm{inf}}(S^+)})^a &\rightarrow (S^+, \mathcal{M}_{S^+}), \\ (R\widehat{\otimes}_{\mathcal{O}_K} R, \mathcal{M}_R \oplus \mathcal{M}_R)^a &\rightarrow (R, \mathcal{M}_R), \end{aligned}$$

respectively. They fit into the following diagram of p -adic PD-thickenings in $(X_0, \mathcal{M}_{X_0})_{\mathrm{crys}}$:

$$\begin{array}{ccc} D^+(1) & \longrightarrow & \mathcal{O}\mathbb{A}_{\mathrm{crys}}(1)(S^+) \\ p_1 \uparrow & & \uparrow q_1 \\ & & \uparrow q_2 \\ R & \longrightarrow & \mathcal{O}\mathbb{A}_{\mathrm{crys}}(S^+) \\ & & \uparrow \\ & & A_{\mathrm{crys}}(S^+) \end{array}$$

Fix a factorization $U \rightarrow (\mathrm{Spf}(\widetilde{R}_{\infty, \alpha}), \mathcal{M}_{\widetilde{R}_{\infty, \alpha}})_{\eta} \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta}$, which induces charts $\mathbb{Q}_{\geq 0}^r \rightarrow \mathcal{M}_{S^+}$ and $\mathbb{Q}_{\geq 0}^r \rightarrow \mathcal{M}_{A_{\mathrm{inf}}(S^+)}$. We use the notation in Remark 5.2.1 and Construction 5.3.12. Analogously, the above diagram admits maps to the following diagram:

$$\begin{array}{ccc} R[1/p][[\xi_1, \dots, \xi_s]] \cong D(1) & \longrightarrow & \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+(1)(S^+) \cong \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+(S^+)[[\xi_1, \dots, \xi_s]] \\ p_1 \uparrow & & \uparrow q_1 \\ & & \uparrow q_2 \\ R[1/p] & \longrightarrow & \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+(S^+) \\ & & \uparrow \\ & & B_{\mathrm{dR}}^+(S^+) \end{array}$$

where $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+(S^+)$, $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+(1)(S^+)$, and $D(1)$ is the formal completion of the exactification of surjections of prelog rings

$$\begin{aligned} ((R\widehat{\otimes}_W A_{\mathrm{inf}}(S^+))[1/p], \mathbb{N}^r \oplus \mathbb{Q}_{\geq 0}^r) &\rightarrow (S^+, \mathbb{Q}_{\geq 0}^r), \\ ((R\widehat{\otimes}_{\mathcal{O}_K} R\widehat{\otimes}_W A_{\mathrm{inf}}(S^+))[1/p], \mathbb{N}^r \oplus \mathbb{N}^r \oplus \mathbb{Q}_{\geq 0}^r) &\rightarrow (S^+, \mathbb{Q}_{\geq 0}^r), \\ ((R\widehat{\otimes}_{\mathcal{O}_K} R)[1/p], \mathbb{N}^r \oplus \mathbb{N}^r) &\rightarrow (R, \mathbb{N}^r). \end{aligned}$$

Note that $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+(S^+)$ coincides with $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+(U)$ by Construction 5.4.5. By extension of scalars, we can evaluate $\mathcal{E}$ at each term in the above diagram.

(1): The crystal property of $\mathcal{E}$ for the middle horizontal map and the right bottom vertical map induces a natural isomorphism

$$(E \otimes_{R[1/p]} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+)(U) \cong (\mathbb{B}_{\mathrm{dR}}^+(\mathcal{E}) \otimes_{\mathbb{B}_{\mathrm{dR}}^+} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+)(U).$$

It suffices to prove that this isomorphism is compatible with connections. The maps $p_i: R[1/p] \rightarrow D(1)$ and $q_i: \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+(S^+) \rightarrow \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+(1)(S^+)$ induce $p_i: E \rightarrow \mathcal{E}(D(1))$ and

$q_i: \mathcal{E}(\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+(S^+)) \rightarrow \mathcal{E}(\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+(1)(S^+))$ for $i = 1, 2$, and we have equations

$$\begin{aligned} & q_1(e \otimes f) - q_2(e \otimes f) \\ &= q_1(f)p_1(e) - q_2(f)p_2(e) \\ &= q_1(f)(p_1(e) - p_2(e)) + (q_1(f) - q_2(f))p_2(e) \\ &= q_1(f) \left(\sum_{(n_1, \dots, n_s) \in \mathbb{N}^s \setminus \{0\}} \left(\prod_{i=1}^s \nabla_{E, t_i^{n_i} \frac{\partial^{n_i}}{\partial t_i^{n_i}}} \right)(e) \left(\prod_{i=1}^s \frac{\xi_i^{n_i}}{n_i!} \right) \right) \\ &+ \left(\sum_{(n_1, \dots, n_s) \in \mathbb{N}^s \setminus \{0\}} \left(\prod_{i=1}^s \nabla_{\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+, \log, t_i^{n_i} \frac{\partial^{n_i}}{\partial t_i^{n_i}}} \right)(f) \left(\prod_{i=1}^s \frac{\xi_i^{n_i}}{n_i!} \right) \right) p_2(e) \end{aligned}$$

for $e \otimes f \in E \otimes_{R[1/p]} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+(S^+) \cong \mathcal{E}(\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+(S^+))$. Hence, the coefficient of ξ_i in the Taylor expansion of $q_1(x) - q_2(x)$ is $\nabla'_{t_i \frac{\partial}{\partial t_i}}(x)$ for every $x \in \mathcal{E}(\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+(S^+))$. Therefore, applying the crystal property of $\mathcal{E}$ to the right column in the above diagram, we see that ∇' is horizontal on $\mathcal{E}(\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+(S^+))$. This proves the assertion.

(2): By Construction 5.4.17, the filtration on $\mathbb{B}_{\mathrm{dR}}^+(\mathcal{E}) \otimes_{\mathbb{B}_{\mathrm{dR}}^+} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$ is computed by choosing a section

$$s: (R, \mathcal{M}_R) \rightarrow (A_{\mathrm{crys}, K}(S^+), \mathcal{M}_{A_{\mathrm{crys}, K}(S^+)}).$$

There exists a unique map β of p -adic log PD-thickenings fitting into the following diagram:

$$(R, \mathcal{M}_R) \xrightarrow[p_2]{p_1} (D^+(1), \mathcal{M}_{D^+(1)}) \xrightarrow{\beta} (\mathcal{O}\mathbb{A}_{\mathrm{crys}}(S^+), \mathcal{M}_{\mathcal{O}\mathbb{A}_{\mathrm{crys}}(S^+)})$$

such that $\beta \circ p_1$ is a natural map and $\beta \circ p_2$ is the composition map

$$(R, \mathcal{M}_R) \xrightarrow{s} (A_{\mathrm{crys}, K}(S^+), \mathcal{M}_{A_{\mathrm{crys}, K}(S^+)}) \rightarrow (\mathcal{O}\mathbb{A}_{\mathrm{crys}}(S^+), \mathcal{M}_{\mathcal{O}\mathbb{A}_{\mathrm{crys}}(S^+)}).$$

The above diagram admits a map to the following diagram:

$$R[1/p] \xrightarrow[p_2]{p_1} D(1) \xrightarrow{\beta} \mathcal{O}\mathbb{B}_{\mathrm{dR}}(S^+).$$

Therefore, the isomorphism

$$E \otimes_{R[1/p]} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+(S^+) \xrightarrow{\sim} (\mathbb{B}_{\mathrm{dR}}^+(\mathcal{E}) \otimes_{\mathbb{B}_{\mathrm{dR}}^+} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+)(U) \xrightarrow{\sim} E \otimes_{R[1/p], \beta \circ p_2} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+(S^+)$$

is given by

$$e \otimes 1 \mapsto \sum_{(n_1, \dots, n_s) \in \mathbb{N}^s} \left(\prod_{i=1}^s \nabla_{E, t_i^{n_i} \frac{\partial^{n_i}}{\partial t_i^{n_i}}} \right)(e) \otimes \left(\prod_{i=1}^s \frac{\beta(\xi_i)^{n_i}}{n_i!} \right)$$

for $e \in E$, and the Griffiths transversality implies that this is a filtered isomorphism. $\square$

Proposition 5.4.20. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a semi-stable log formal scheme over $\mathcal{O}_K$. Let $\mathbb{L}$ be a weakly semi-stable Kummer étale local system on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta}$, and take a F -isocrystal $\mathcal{E}$ on $(X, \mathcal{M}_X)$ and an isomorphism $\vartheta: \mathbb{L} \otimes_{\mathbb{Z}_p} \mathbb{B}_{\mathrm{crys}} \cong \mathbb{B}_{\mathrm{crys}}(\mathcal{E})$ as in Definition 5.4.18 (1). Then there exists a unique filtration $\mathrm{Fil}_{\mathrm{dR}}^{\bullet}(E)$ on $E := \mathcal{E}_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})}$ satisfying the following conditions:

- (1) $(\mathcal{E}, \mathrm{Fil}_{\mathrm{dR}}^{\bullet}(E))$ is a filtered F -isocrystal on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$;

(2) the isomorphism

$$\vartheta \otimes_{\mathbb{B}_{\text{crys}}} \mathbb{B}_{\text{dR}} : \mathbb{L} \otimes_{\mathbb{Z}_p} \mathbb{B}_{\text{dR}} \cong \mathbb{B}_{\text{dR}}(\mathcal{E})$$

is a filtered isomorphism, where the filtration on $\mathbb{B}_{\text{dR}}(\mathcal{E})$ is the one constructed in Construction 5.4.17.

Proof. Let $\mathfrak{U}$ be the open subset of $\mathfrak{X}$ on which the log structure $\mathcal{M}_{\mathfrak{X}}$ is trivial and U_0 be the mod- p fiber of $\mathfrak{U}$. Then $\mathbb{L}|_{\mathfrak{U}_\eta}$ is weakly crystalline (with respect to $\mathfrak{U}$) in the sense of [GR24, Definition 2.31], and $\mathbb{L}|_{\mathfrak{U}_\eta}$ and $\mathcal{E}|_{U_0}$ are associated. By [GR24, Proposition 2.38], $\mathbb{L}|_{\mathfrak{U}_\eta}$ is de Rham in the sense of [Sch13, Definition 8.3]. Let U denote the open subset of $\mathfrak{X}_\eta$ on which the log structure $\mathcal{M}_{\mathfrak{X}_\eta}$ is trivial. Then U contains $\mathfrak{U}_\eta$ and $\mathbb{L}|_U$ is de Rham by [LZ17, Theorem 3.9 (iv)].

Set $\mathcal{O}\mathbb{B}_{\text{dR},\log}(\mathcal{E}) := \mathbb{B}_{\text{dR}}(\mathcal{E}) \otimes_{\mathbb{B}_{\text{dR}}} \mathcal{O}\mathbb{B}_{\text{dR},\log}$, which is isomorphic to $E \otimes_{\mathcal{O}_{\mathfrak{X}_\eta}} \mathcal{O}\mathbb{B}_{\text{dR},\log}$ as $\mathcal{O}\mathbb{B}_{\text{dR},\log}$ -vector bundles with connection by Proposition 5.4.19. Let $\mu : (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \text{prokét}} \rightarrow \mathfrak{X}_{\eta, \text{an}}$ be the canonical map of sites. We have isomorphisms

$$\mu_*(\mathbb{L} \otimes_{\mathbb{Z}_p} \mathcal{O}\mathbb{B}_{\text{dR},\log}) \cong \mu_*(\mathbb{B}_{\text{dR}}(\mathcal{E}) \otimes_{\mathbb{B}_{\text{dR}}} \mathcal{O}\mathbb{B}_{\text{dR},\log}) \cong \mu_*(E \otimes_{\mathcal{O}_{\mathfrak{X}_\eta}} \mathcal{O}\mathbb{B}_{\text{dR},\log}) \cong E.$$

By [DLLZ23a, Theorem 3.2.7 (1)(2)] and what we proved in the previous paragraph, the direct image of the filtration $\mathbb{L} \otimes_{\mathbb{Z}_p} \text{Fil}^\bullet \mathcal{O}\mathbb{B}_{\text{dR},\log}$ gives a filtration $\text{Fil}^\bullet(E)$ on E with locally free grading pieces such that $(\mathcal{E}, \text{Fil}^\bullet(E))$ is a filtered F -isocrystal, and the given isomorphism is refined to filtered isomorphisms

$$\begin{aligned} & (\mathbb{L} \otimes_{\mathbb{Z}_p} \mathcal{O}\mathbb{B}_{\text{dR},\log}, \mathbb{L} \otimes_{\mathbb{Z}_p} \text{Fil}^\bullet \mathcal{O}\mathbb{B}_{\text{dR},\log}) \\ & \cong (E \otimes_{\mathcal{O}_{\mathfrak{X}_\eta}} \mathcal{O}\mathbb{B}_{\text{dR},\log}, \text{Fil}^\bullet E \otimes_{\mathcal{O}_{\mathfrak{X}_\eta}} \text{Fil}^\bullet \mathcal{O}\mathbb{B}_{\text{dR},\log}) \\ & \cong (\mathbb{B}_{\text{dR}}(\mathcal{E}) \otimes_{\mathbb{B}_{\text{dR}}} \mathcal{O}\mathbb{B}_{\text{dR},\log}, \text{Fil}^\bullet \mathbb{B}_{\text{dR}}(\mathcal{E}) \otimes_{\mathbb{B}_{\text{dR}}} \text{Fil}^\bullet \mathcal{O}\mathbb{B}_{\text{dR},\log}) \end{aligned}$$

that are compatible with connections. Taking horizontal sections in both sides, we see that $\vartheta \otimes_{\mathbb{B}_{\text{crys}}} \mathbb{B}_{\text{dR}}$ gives a filtered isomorphism

$$(\mathbb{L} \otimes_{\mathbb{Z}_p} \mathbb{B}_{\text{dR}}, \mathbb{L} \otimes_{\mathbb{Z}_p} \text{Fil}^\bullet \mathbb{B}_{\text{dR}}) \cong (\mathbb{B}_{\text{dR}}(\mathcal{E}), \text{Fil}^\bullet \mathbb{B}_{\text{dR}}(\mathcal{E}))$$

by [DLLZ23a, Corollary 2.4.2 (2)]. This proves the assertion for the existence of the desired filtration, and the uniqueness of it follows from the above argument. $\square$

Proposition 5.4.21. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) = (\text{Spf}(R), \mathcal{M}_R)$ be a horizontally semi-stable log formal scheme over $\mathcal{O}_K$ with a fixed unramified model $(\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0}) = (\text{Spf}(R_0), \mathcal{M}_{R_0})$. Let $\mathcal{E}$ be an object of $\text{Isoc}_{\text{Iff}}^\varphi((X_k, \mathcal{M}_{X_k} \oplus 0^{\mathbb{N}})_{\text{crys}}^a)$ and $(E_0, \nabla_{E_0}, \varphi_{E_0}, N_{E_0})$ be the object of the category $\text{Vect}^{\varphi, \nabla, N}(R_0)$ associated with $\text{Res}(\mathcal{E})$ (see Proposition 5.3.36). Then there is a natural isomorphism of the following $\mathcal{O}\mathbb{B}_{\text{st}}$ -modules with connection, Frobenius structures, and monodromy operators:

- $(\mathbb{B}_{\text{crys}}(\mathcal{E}) \otimes_{\mathbb{B}_{\text{crys}}} \mathcal{O}\mathbb{B}_{\text{st},(\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})}, \text{id} \otimes \nabla_{\mathcal{O}\mathbb{B}_{\text{st},(\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})}}, \varphi_{\mathbb{B}_{\text{crys}}(\mathcal{E})} \otimes \varphi_{\mathcal{O}\mathbb{B}_{\text{st}}}, \text{id} \otimes N_{\mathcal{O}\mathbb{B}_{\text{st},(\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})}})$
- $(E_0 \otimes_{R_0[1/p]} \mathcal{O}\mathbb{B}_{\text{st},(\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})}, \varphi_{E_0} \otimes \varphi_{\mathcal{O}\mathbb{B}_{\text{st},(\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})}}, \nabla_{E_0} \otimes \nabla_{\mathcal{O}\mathbb{B}_{\text{st},(\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})}}, N_{E_0} \otimes \text{id} + \text{id} \otimes N_{\mathcal{O}\mathbb{B}_{\text{st},(\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})}}).$

Proof. Since the two $\widehat{\mathcal{O}\mathbb{B}_{\text{st},(\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})}}$ -modules on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \text{prokét}}$ in the assertion is obtained as the restriction of the corresponding $\widehat{\mathcal{O}\mathbb{B}_{\text{st}}}$ -modules on $(\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})_{\eta, \text{prokét}}$, we may assume that $\mathcal{O}_K = W$. It is enough to prove that there is natural isomorphisms of the following $\widehat{\mathcal{O}\mathbb{B}_{\text{st}}}$ -modules with connection, Frobenius structures, and monodromy operators:

- $(\mathbb{B}_{\text{crys}}(\mathcal{E}) \otimes_{\mathbb{B}_{\text{crys}}} \widehat{\mathcal{O}\mathbb{B}_{\text{st}}}, \text{id} \otimes \nabla_{\widehat{\mathcal{O}\mathbb{B}_{\text{st}}}}, \varphi_{\mathbb{B}_{\text{crys}}(\mathcal{E})} \otimes \varphi_{\widehat{\mathcal{O}\mathbb{B}_{\text{st}}}}, \text{id} \otimes N_{\widehat{\mathcal{O}\mathbb{B}_{\text{st}}}})$

$$\bullet (E_0 \otimes_{R_0[1/p]} \mathcal{O}_{\widehat{\mathbb{B}}_{\text{st}}}, \varphi_{E_0} \otimes \varphi_{\mathcal{O}_{\widehat{\mathbb{B}}_{\text{st}}}}, \nabla_{E_0} \otimes \nabla_{\mathcal{O}_{\widehat{\mathbb{B}}_{\text{st}}}}, N_{E_0} \otimes \text{id} + \text{id} \otimes N_{\mathcal{O}_{\widehat{\mathbb{B}}_{\text{st}}}}).$$

Indeed, Lemma 5.3.31 and Corollary 5.4.14(3) imply that taking nilpotent sections for monodromy operators on both sides provides the desired isomorphism.

Let U be a big log affinoid perfectoid in $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \text{prokét}}$ with the associated affinoid perfectoid space $\widehat{U} = \text{Spa}(S, S^+)$. Let $\mathcal{M}_{S^+}$, $\mathcal{M}_{A_{\text{crys}}(S^+)}$, and $(\text{Spf}(R\langle u \rangle), \mathcal{M}_{R\langle u \rangle})$ be as in Construction 5.4.11. Let $\mathcal{M}_{R\{u\}}$ be the pullback log structure of $\mathcal{M}_{R\langle u \rangle}$. The strict closed immersion

$$(X_k, \mathcal{M}_{X_k} \oplus 0^{\mathbb{N}})^a \rightarrow (\text{Spf}(R\{u\}), \mathcal{M}_{R\{u\}})$$

defined by $u \mapsto 0$ belongs to $(X_k, \mathcal{M}_{X_k} \oplus 0^{\mathbb{N}})_{\text{crys}}^a$. Let $(\text{Spf}(D^+(1)), \mathcal{M}_{D^+(1)})$ be the p -completed log PD-envelope of the diagonal map

$$(\text{Spf}(R/p), \mathcal{M}_{R/p} \oplus 0^{\mathbb{N}})^a \rightarrow (\text{Spf}(R\langle u_1 \rangle), \mathcal{M}_{R\langle u_1 \rangle}) \times_W (\text{Spf}(R\langle u_2 \rangle), \mathcal{M}_{R\langle u_2 \rangle}),$$

given by $u_1, u_2 \mapsto 0$. We define $(\text{Spf}(\mathcal{O}_{\widehat{\mathbb{A}}_{\text{st}}}(1)(U)), \mathcal{M}_{\mathcal{O}_{\widehat{\mathbb{A}}_{\text{st}}}(1)(U)})$ as the p -completed log PD-envelope of the closed immersion

$$(\text{Spf}(S^+/p), \mathcal{M}_{S^+/p}) \hookrightarrow (\text{Spf}(A_{\text{crys}}(S^+) \widehat{\otimes}_W R\langle u_1 \rangle \widehat{\otimes}_W R\langle u_2 \rangle), \mathcal{M}_{A_{\text{crys}}(S^+) \oplus \mathcal{M}_{R\langle u_1 \rangle} \oplus \mathcal{M}_{R\langle u_2 \rangle}})^a$$

Then we have the following commutative diagram of p -adic log PD-thickenings equipped with Frobenius lists in $(X_k, \mathcal{M}_{X_k})_{\text{crys}}$. (We omit log structures and PD-ideals.)

$$\begin{array}{ccc} D^+(1) & \longrightarrow & \mathcal{O}_{\widehat{\mathbb{A}}_{\text{st}}}(1)(U) \\ p_1 \uparrow & & \uparrow q_1 \\ & & \uparrow q_2 \\ R\{u\} & \longrightarrow & \mathcal{O}_{\widehat{\mathbb{A}}_{\text{st}}}(U) \\ & & \uparrow \\ & & A_{\text{crys}}(S^+) \end{array}$$

The crystal property of $\mathcal{E}$ gives isomorphisms of $\mathcal{O}_{\widehat{\mathbb{B}}_{\text{st}}}^+(U)$ -modules with Frobenius structures

$$\mathcal{E}_{A_{\text{crys}}(S^+)} \otimes \mathcal{O}_{\widehat{\mathbb{A}}_{\text{st}}}(U) \cong \mathcal{E}_{\mathcal{O}_{\widehat{\mathbb{A}}_{\text{st}}}(U)} \cong \mathcal{E}_{R\langle u \rangle} \otimes_{R\langle u \rangle} \mathcal{O}_{\widehat{\mathbb{A}}_{\text{st}}}(U).$$

By Lemma 5.3.35, we have an isomorphism

$$\begin{aligned} & (\mathcal{E}_{R\{u\}}, \varphi_{\mathcal{E}_{R\{u\}}}, \nabla_{\mathcal{E}_{R\{u\}}}^{\text{hol}}, N_{\mathcal{E}_{R\{u\}}}) \\ & \cong (E_0 \widehat{\otimes}_W W\{u\}, \varphi_{E_0} \otimes \varphi_{W\{u\}}, \nabla_{E_0} \otimes \text{id}, N_{E_0} \otimes \text{id} + \text{id} \otimes N_{W\{u\}}). \end{aligned}$$

Combining these isomorphisms gives an isomorphism of $\mathcal{O}_{\widehat{\mathbb{B}}_{\text{st}}}^+(U)$ -modules with Frobenius structures

$$\mathcal{E}_{A_{\text{crys}}(S^+)} \otimes_{A_{\text{crys}}(S^+)} \mathcal{O}_{\widehat{\mathbb{A}}_{\text{st}}}(U) \cong E_0 \otimes_R \mathcal{O}_{\widehat{\mathbb{A}}_{\text{st}}}(U).$$

This gives a desired isomorphism except for the compatibility with connections and monodromy operators.

To prove the remaining, it is enough to show that the connection and the monodromy operator on $\mathcal{E}_{\widehat{\mathbb{A}}_{\text{st}}(U)}$ induced from $\nabla_{E_0} \otimes \nabla_{\mathcal{O}_{\widehat{\mathbb{A}}_{\text{st}}}(U)}$ and $N_{E_0} \otimes \text{id} + \text{id} \otimes N_{\mathcal{O}_{\widehat{\mathbb{A}}_{\text{st}}}(U)}$, denoted by ∇' and N' , kills $\mathcal{E}_{A_{\text{crys}}(S^+)}$. When we fix a framing on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$, there is an isomorphism

$$R\{u\}\{\xi_1, \dots, \xi_s, \xi\} \cong D^+(1)$$

given by $\xi \mapsto p_1(t_i)p_2(t_i)^{-1} - 1$ and $\xi \mapsto p_1(u)p_2(u)^{-1} - 1$, and we have an equation

$$\begin{aligned} p_1^*(x) - p_2^*(x) &= \sum_{i=1}^s \nabla_{\mathcal{E}_{R\{u\}, t_i} \frac{\partial}{\partial t_i}}(x) \xi_i + N_{\mathcal{E}_{R\{u\}}}(x) \xi \\ &= \sum_{i=1}^s \nabla_{\mathcal{E}_0, t_i \frac{\partial}{\partial t_i}}(y) \xi_i + N_{E_0}(y) \xi \end{aligned}$$

for $x = y \otimes 1 \in \mathcal{E}_{R\{u\}} = E_0 \widehat{\otimes}_W W\{u\}$ after taking modulo $(\xi_1, \dots, \xi_s, \xi)^{[\geq 2]}$. The square in the above diagram is cocartesian, and there is an induced isomorphism

$$\mathcal{O}_{\widehat{\mathbb{A}}_{\text{st}}}(U)\{\xi_1, \dots, \xi_s, \xi\} \cong \mathcal{O}_{\widehat{\mathbb{A}}_{\text{st}}}(U)(1).$$

Hence, we obtain the following equation in $\mathcal{E}_{\mathcal{O}_{\widehat{\mathbb{A}}_{\text{st}}}(U)} \otimes_{\mathcal{O}_{\widehat{\mathbb{A}}_{\text{st}}}(U), q_2} \mathcal{O}_{\widehat{\mathbb{A}}_{\text{st}}}(U)(1)/(\xi_1, \dots, \xi)^{[\geq 2]}$:

$$\begin{aligned} q_1^*(x) - q_2^*(x) &= (q_1(f) - q_2(f))y + f(p_1^*(y \otimes 1) - p_2^*(y \otimes 1)) \\ &= \left(\sum_{i=1}^s \nabla_{\mathcal{O}_{\widehat{\mathbb{A}}_{\text{st}}}(U), t_i \frac{\partial}{\partial t_i}}(f) \xi_i + N_{\mathcal{O}_{\widehat{\mathbb{A}}_{\text{st}}}(U)}(f) \xi \right) \cdot y \\ &\quad + f \cdot \left(\sum_{i=1}^s \nabla_{\mathcal{E}_0, t_i \frac{\partial}{\partial t_i}}(y) \xi_i + N_{E_0}(y) \xi \right) \end{aligned}$$

for $x = y \otimes f \in \mathcal{E}_{\mathcal{O}_{\widehat{\mathbb{A}}_{\text{st}}}(U)} = E_0 \otimes_R \mathcal{O}_{\widehat{\mathbb{A}}_{\text{st}}}(U)$ after taking modulo $(\xi_1, \dots, \xi_s, \xi)^{[\geq 2]}$, which implies that $\nabla'(x)$ and $N'(x)$ occur in coefficients of the Taylor expansion of $q_1^*(x) - q_2^*(x)$. Therefore, ∇' and N' kill $\mathcal{E}_{A_{\text{crys}}(S^+)}$ by the crystal property of $\mathcal{E}$ for the right column in the above diagram. $\square$

Proposition 5.4.22. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a semi-stable log formal scheme over $\mathcal{O}_K$ which admits framings with $n \leq 1$ étale locally. Then the functor

$$\text{Isoc}_{\text{lftf}}^{\varphi}((X_{p=0}, \mathcal{M}_{X_{p=0}})_{\text{crys}}) \rightarrow \text{Vect}^{\varphi}(\mathbb{B}_{\text{crys}})$$

sending $\mathcal{E}$ to $\mathbb{B}_{\text{crys}}(\mathcal{E})$ is fully faithful.

Proof. By working étale locally on $\mathfrak{X}$, we may assume that $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) = (\text{Spf}(R), \mathcal{M}_R)$ is a small affine log formal scheme over $\mathcal{O}_K$ with a fixed framing such that $n \leq 1$. Let $(\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0}) = (\text{Spf}(R_0), \mathcal{M}_{R_0})$ denote the unramified model of $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ defined by the fixed framing.

We may assume that $n = 1$ because, for a horizontally semi-stable log formal scheme $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ over $\mathcal{O}_K$, the natural functor

$$\text{Isoc}^{\varphi}((Y_{p=0}, \mathcal{M}_{Y_{p=0}})_{\text{crys}}) \rightarrow \text{Isoc}^{\varphi}((Y_{p=0}, \mathcal{M}_{Y_{p=0}} \oplus \pi^{\mathbb{N}})_{\text{crys}}^a)$$

is fully faithful by Lemma 5.3.7 (1) and Corollary 5.3.37. In this case, Proposition 5.4.21 gives the following commutative diagram:

$$\begin{array}{ccc} \text{Vect}^{\varphi}(\mathbb{B}_{\text{crys}}) & \xrightarrow{-\otimes_{\mathbb{B}_{\text{crys}}} \mathcal{O}_{\mathbb{B}_{\text{st}}, (\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})}} & \text{Vect}^{\varphi, \nabla, N}(\mathcal{O}_{\mathbb{B}_{\text{st}}, (\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})}) \\ \mathcal{E} \mapsto \mathbb{B}_{\text{crys}}(\mathcal{E}) \uparrow & & \uparrow -\otimes_{R_0} \mathcal{O}_{\mathbb{B}_{\text{st}}, (\mathfrak{X}_0, \mathcal{M}_{\mathfrak{X}_0})} \\ \text{Isoc}_{\text{lftf}}^{\varphi}((X_{p=0}, \mathcal{M}_{X_{p=0}})_{\text{crys}}) & \xrightarrow{\sim} & \text{Vect}^{\varphi, \nabla, N}(R_0) \end{array}$$

The bottom horizontal functor is the composition of the equivalence in Proposition 5.3.36 with the equivalence $\mathrm{Vect}^{\varphi, \nabla, N}(R_0) \simeq \mathrm{Isoc}_{\mathrm{lf}}^{\varphi, N}((X_k, \mathcal{M}_{X_k})_{\mathrm{crys}})$. Since a problem on morphisms between vector bundles can be reduced to one on global sections of them by taking internal homomorphisms, the right vertical functor is fully faithful by Proposition 5.4.16, and the upper horizontal functor is fully faithful by $\mathbb{B}_{\mathrm{crys}} = \mathcal{O}_{\mathbb{B}_{\mathrm{st}}}^{\nabla=N=0}$ (Corollary 5.4.14 (1)(2)). Therefore, the left vertical functor is also fully faithful. $\square$

Remark 5.4.23. When $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is horizontally semi-stable, we can use the method of [GR24, Proposition 2.29], and the above proposition is proved without F -structures. However, we do not use it in this thesis.

5.5. Log prismatic crystals

First, let us review some basics of prisms introduced by [BS22]. A δ -ring is a ring A with a map $\delta : A \rightarrow A$ satisfying the following conditions:

- (1) $\delta(1) = 0$
- (2) $\delta(xy) = \delta(x)y^p + x^p\delta(y) + p\delta(x)\delta(y)$;
- (3) $\delta(x+y) = \delta(x) + \delta(y) - \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} x^i y^{p-i}$.

For a δ -ring A , a map $\phi_A : A \rightarrow A$ mapping a to $a^p + p\delta(a)$ is a ring map lifting the Frobenius map on A/p . A *prism* is a pair (A, I) consisting of a derived (p, I) -complete ring A and a locally principal ideal I of A such that $p \in I + \phi_A(I)A$. We say that a prism (A, I) is *bounded* if A/I has bounded p^∞ -torsion. A prism (A, I) is called *orientable* if I is a principal ideal. Throughout this thesis, we use the following lemma with no reference.

Lemma 5.5.1. Let (A, I) be a bounded prism and B be a (p, I) -adically complete A -algebra. Then the map $A \rightarrow B$ is (p, I) -completely (faithfully) flat if and only if it is (p, I) -adically (faithfully) flat.

Proof. This follows from results of Yektieli. For the proof, see [IKY24, Lemma 1.3] for example. $\square$

Let $\mathfrak{X}$ be a bounded p -adic formal scheme.

Definition 5.5.2. The *absolute prismatic site* $\mathfrak{X}_{\Delta}$ is the category of prisms (A, I) equipped with maps $\mathrm{Spf}(A/I) \rightarrow \mathfrak{X}$. We equip $\mathfrak{X}_{\Delta}$ with completely flat topology. We simply write R_{Δ} for $\mathrm{Spf}(R)_{\Delta}$. Let $\mathcal{O}_{\Delta}$ (resp. $\mathcal{I}_{\Delta}$) (resp. $\tilde{\mathcal{O}}_{\Delta}$) denote the sheaf on $\mathfrak{X}_{\Delta}$ given by $(A, I) \mapsto A$ (resp. I) (resp. $A/(I, p)$). The Frobenius lift ϕ_A for a prism (A, I) induces a map $\phi : \mathcal{O}_{\Delta} \rightarrow \mathcal{O}_{\Delta}$.

Definition 5.5.3. A *prismatic crystal* on $\mathfrak{X}$ is a vector bundle on the ringed site $(\mathfrak{X}_{\Delta}, \mathcal{O}_{\Delta})$, and a *prismatic F -crystal* on $\mathfrak{X}$ is a pair $(\mathcal{E}, \varphi_{\mathcal{E}})$ consisting of a prismatic crystal $\mathcal{E}$ and an isomorphism $\varphi_{\mathcal{E}} : (\phi^*\mathcal{E})[1/\mathcal{I}_{\Delta}] \xrightarrow{\sim} \mathcal{E}[1/\mathcal{I}_{\Delta}]$. When no confusion occurs, we simply write $\mathcal{E}$ for $(\mathcal{E}, \varphi_{\mathcal{E}})$. Let $\mathrm{Vect}(\mathfrak{X}_{\Delta})$ (resp. $\mathrm{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$) denote the category of prismatic crystals (resp. prismatic F -crystals) on $\mathfrak{X}$. For integers $a \leq b$, we let $\mathrm{Vect}_{[a, b]}^{\varphi}(\mathfrak{X}_{\Delta})$ denote the full subcategory of $\mathrm{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$ consisting of objects $(\mathcal{E}, \varphi_{\mathcal{E}})$ with

$$\mathcal{I}_{\Delta}^b \mathcal{E} \subset \varphi_{\mathcal{E}}(\phi^* \mathcal{E}) \subset \mathcal{I}_{\Delta}^a \mathcal{E}.$$

In particular, we write $\mathrm{DM}(\mathfrak{X}_{\Delta})$ for the category $\mathrm{Vect}_{[0, 1]}^{\varphi}(\mathfrak{X}_{\Delta})$. Objects of $\mathrm{DM}(\mathfrak{X}_{\Delta})$ are called *prismatic Dieudonné crystals*. Finally, we let $\mathrm{Vect}(\mathfrak{X}_{\Delta}, \tilde{\mathcal{O}}_{\Delta})$ denote the category of vector bundles on the ringed site $(\mathfrak{X}_{\Delta}, \tilde{\mathcal{O}}_{\Delta})$.

Lemma 5.5.4 (Quasi-syntomic descent for prismatic crystals). Let $\mathfrak{Y} \rightarrow \mathfrak{X}$ be a quasi-compact quasi-syntomic surjection of bounded p -adic formal schemes and $\mathfrak{Y}^{(\bullet)}$ be the Čech nerve of $\mathfrak{Y} \rightarrow \mathfrak{X}$. Then the natural functor

$$\mathrm{Vect}(\mathfrak{X}_{\Delta}) \rightarrow \varprojlim_{\bullet \in \Delta} \mathrm{Vect}(\mathfrak{Y}_{\Delta}^{(\bullet)})$$

is bi-exact equivalent. An analogous assertion for prismatic F -crystals also holds.

Proof. This follows from [BS22, Proposition 7.11]. $\square$

Now, we recall some basics of log prisms introduced in [Kos22]. A $\delta_{\log}$ -ring is a prelog ring (A, M) with a δ -ring structure on A and a map $\delta_{\log} : M \rightarrow A$ satisfying the following conditions:

- (1) $\delta_{\log}(e) = 0$;
- (2) $\alpha(m)^p \delta_{\log}(m) = \delta(\alpha(m))$;
- (3) $\delta_{\log}(mm') = \delta_{\log}(m) + \delta_{\log}(m') + p\delta_{\log}(m)\delta_{\log}(m')$,

where $\alpha : M \rightarrow A$ is the structure map. A *bounded prelog prism* (A, I, M) (resp. *bounded log prism* $(A, I, \mathcal{M}_A)$) is a bounded prism (A, I) equipped with a prelog structure $M \rightarrow A$ (resp. log structure $\mathcal{M}_A$ on $\mathrm{Spf}(A)$) and a $\delta_{\log}$ -structure on the prelog ring (A, M) (resp. the log formal scheme $(\mathrm{Spf}(A), \mathcal{M}_A)$). Here, a $\delta_{\log}$ -structure on the log formal scheme $(\mathrm{Spf}(A), \mathcal{M}_A)$ is a sheaf map $\mathcal{M}_A \rightarrow \mathcal{O}_{\mathrm{Spf}(A)}$ that defines a $\delta_{\log}$ -structure on each section. We can attach to a prelog prism (A, I, M) a log prism $(A, I, M)^a := (A, I, \mathcal{M}_A := M^a)$. For a log prism $(A, I, \mathcal{M}_A)$, the ring map ϕ_A and the map of log structures $\phi_A^* \mathcal{M}_A \rightarrow \mathcal{M}_A$ given by $m \mapsto m^p(1 + p\delta_{\log}(m))$ for $m \in \phi_A^{-1} \mathcal{M}_A$ (here, $1 + p\delta_{\log}(m) \in \mathcal{O}_{\mathrm{Spf}(A)}^\times$ is regarded as a section of $\mathcal{M}_A$) define a morphism of log formal schemes $\phi_{(A, \mathcal{M}_A)} : (\mathrm{Spf}(A), \mathcal{M}_A) \rightarrow (\mathrm{Spf}(A), \mathcal{M}_A)$. When no confusion occurs, we write simply ϕ_A for $\phi_{(A, \mathcal{M}_A)}$.

Lemma 5.5.5. Let (A, I, M) be a bounded prelog prism. For $m \in M$ admitting a p^n -th root for every $n \geq 1$, we have $\delta_{\log}(m) = 0$.

Proof. For $x \in M$, we have a formula

$$\delta_{\log}(x^p) = \sum_{i=1}^p \binom{p}{i} p^{i-1} \delta_{\log}(x)^i,$$

which can be checked by reducing the problem to the case that A is a free $\delta_{\log}$ -ring (cf. [Kos22, Lemma 2.11]). By this formula, $\delta_{\log}(x) \in p^i A$ implies $\delta_{\log}(x^p) \in p^{i+1} A$. Hence, $\delta_{\log}(m)$ is contained in $\bigcap_{i \geq 1} p^i A$ by the assumption. Since A is p -adically separated by the boundedness, we obtain $\delta_{\log}(m) = 0$. $\square$

Definition 5.5.6 (Absolute log prismatic sites, [Kos22, Remark 4.6]). Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a bounded p -adic log formal scheme. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}$ be the site with objects described as follows: an object consists of

- a log prism $(A, I, \mathcal{M}_A)$ such that $(\mathrm{Spf}(A), \mathcal{M}_A)$ is quasi-coherent,
- a map of log formal schemes $(\mathrm{Spf}(A/I), \mathcal{M}_{A/I}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$, where $\mathcal{M}_{A/I}$ is the restriction of $\mathcal{M}_A$.

We simply write $(A, I, \mathcal{M}_A)$ for an object when there exists no confusion. A morphism $(A, I, \mathcal{M}_A) \rightarrow (B, J, \mathcal{M}_B)$ is the following commutative diagram:

$$\begin{array}{ccccc} (\mathrm{Spf}(B), \mathcal{M}_B) & \longleftarrow & (\mathrm{Spf}(B/J), \mathcal{M}_{B/J}) & & \\ \downarrow & & \downarrow & \searrow & \\ (\mathrm{Spf}(A), \mathcal{M}_A) & \longleftarrow & (\mathrm{Spf}(A/I), \mathcal{M}_{A/I}) & \longrightarrow & (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}). \end{array}$$

A morphism $(A, I, \mathcal{M}_A) \rightarrow (B, J, \mathcal{M}_B)$ in $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}$ is a covering if the corresponding map $(\mathrm{Spf}(B), \mathcal{M}_B) \rightarrow (\mathrm{Spf}(A), \mathcal{M}_A)$ is strict (p, I) -completely faithfully flat. The resulting site $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}$ is called the *(absolute) log prismatic site* of $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$.

The structure sheaf $\mathcal{O}_{\Delta}$ is defined by $(A, I, \mathcal{M}_A) \mapsto A$. Similarly, the sheaf $\overline{\mathcal{O}}_{\Delta}$ (resp. $\mathcal{I}_{\Delta}$) (resp. $\tilde{\mathcal{O}}_{\Delta}$) is defined by $(A, I, \mathcal{M}_A) \mapsto A/I$ (resp. I) (resp. $A/(I, p)$). The ring map $\mathcal{O}_{\Delta} \rightarrow \overline{\mathcal{O}}_{\Delta}$ induced by ϕ_A for log prisms $(A, I, \mathcal{M}_A)$ is denoted by ϕ .

Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}$ denote the full subcategory of $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}$ consisting of objects whose structure morphisms $(\mathrm{Spf}(A/I), \mathcal{M}_{A/I}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ are strict. The site $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}$ is called the *(absolute) strict log prismatic site* of $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$.

Remark 5.5.7. The definition of log prismatic sites of [Kos22, Remark 4.6] is different from the one given above in that the following two assumptions are added in *loc.cit.*:

- a log prism $(A, I, \mathcal{M}_A)$ admits a global chart;
- the structure morphism $(\mathrm{Spf}(A/I), \mathcal{M}_{A/I}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ admits a chart étale locally.

The first additional assumption does not change the associated topoi. On the other hand, the second additional one changes the associated topoi. However, the category of prismatic crystals is unchanged under certain mild assumptions (see Lemma 5.5.12).

Remark 5.5.8. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a bounded p -adic quasi-coherent and integral log formal scheme. For the definition of $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}$, the condition that $(A, \mathcal{M}_A)$ is quasi-coherent is redundant. To check this, it suffices to show that, for a log prism $(A, I, \mathcal{M}_A)$ with a strict morphism $(\mathrm{Spf}(A/I), \mathcal{M}_{A/I}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$, the log formal scheme $(\mathrm{Spf}(A), \mathcal{M}_A)$ is quasi-coherent. By working étale locally on $\mathrm{Spf}(A)$, we may assume that there exists an integral chart $P \rightarrow \mathcal{M}_{\mathfrak{X}}$. We put

$$\tilde{P} := P \times_{\Gamma(\mathrm{Spf}(A/I), \mathcal{M}_{A/I})} \Gamma(\mathrm{Spf}(A), \mathcal{M}_A).$$

Since $\mathcal{M}_A$ is integral, $\tilde{P} \rightarrow P$ is a $1+I$ -torsor by [Kos22, Lemma 3.8]. Therefore, a natural map $\tilde{P} \rightarrow \mathcal{M}_A$ is a chart.

Definition 5.5.9 (Log prismatic F -crystals). Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a bounded p -adic log formal scheme. We let $\mathrm{Vect}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ (resp. $\mathrm{Vect}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}})$) denote the category of vector bundles on the ringed site on $((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}, \mathcal{O}_{\Delta})$ (resp. $((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}, \mathcal{O}_{\Delta})$). Objects of $\mathrm{Vect}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ are called *log prismatic crystals*.

We let $\mathrm{Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ denote the category of pairs $(\mathcal{E}, \varphi_{\mathcal{E}})$ where $\mathcal{E}$ is an object of $\mathrm{Vect}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ and $\varphi_{\mathcal{E}}$ is an isomorphism $(\phi^* \mathcal{E})[1/\mathcal{I}_{\Delta}] \xrightarrow{\sim} \mathcal{E}[1/\mathcal{I}_{\Delta}]$. Objects of $\mathrm{Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ are called *log prismatic F -crystals*. For integers $a \leq b$, we let $\mathrm{Vect}_{[a,b]}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ denote the full subcategory of $\mathrm{Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ consisting of objects $(\mathcal{E}, \varphi_{\mathcal{E}})$ with

$$\mathcal{I}_{\Delta}^b \mathcal{E} \subset \varphi_{\mathcal{E}}(\phi^* \mathcal{E}) \subset \mathcal{I}_{\Delta}^a \mathcal{E}.$$

In particular, let $\mathrm{DM}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) := \mathrm{Vect}_{[0,1]}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$, whose objects are called *log prismatic Dieudonné crystals*. Finally, we let $\mathrm{Vect}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}, \tilde{\mathcal{O}}_{\Delta})$ denote the category of vector bundles on the ringed site $((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}, \tilde{\mathcal{O}}_{\Delta})$. Similarly, we define variants

$$\mathrm{Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}), \mathrm{Vect}_{[a,b]}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}), \mathrm{DM}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}), \mathrm{Vect}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}, \tilde{\mathcal{O}}_{\Delta}).$$

All of these categories are canonically equipped with the structure of an exact category.

Remark 5.5.10. For $\star \in \{\mathrm{str}, \emptyset\}$, we have the following canonical bi-exact equivalences;

$$\begin{aligned} \mathrm{Vect}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\star}) &\xrightarrow{\sim} \varprojlim_{(A,I,\mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\star}} \mathrm{Vect}(A), \\ \mathrm{Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\star}) &\xrightarrow{\sim} \varprojlim_{(A,I,\mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\star}} \mathrm{Vect}^{\varphi}(A, I), \\ \mathrm{Vect}_{[a,b]}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\star}) &\xrightarrow{\sim} \varprojlim_{(A,I,\mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\star}} \mathrm{Vect}_{[a,b]}^{\varphi}(A, I), \\ \mathrm{Vect}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\star}, \tilde{\mathcal{O}}_{\Delta}) &\xrightarrow{\sim} \varprojlim_{(A,I,\mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\star}} \mathrm{Vect}(A/(p, I)). \end{aligned}$$

Here, $\mathrm{Vect}(R)$ is the category of finite projective R -modules for a ring R . For a prism (A, I) , $\mathrm{Vect}^{\varphi}(A, I)$ is the category of pairs (N, φ_N) of a finite projective A -module N and an isomorphism $\varphi_N: (\phi_A^* N)[1/I] \xrightarrow{\sim} N[1/I]$, and $\mathrm{Vect}_{[a,b]}^{\varphi}(A, I)$ is the full subcategory of $\mathrm{Vect}^{\varphi}(A, I)$ consisting of objects (N, φ_N) such that $I^b N \subset \varphi_N(\phi_A^* N) \subset I^a N$. For an object $\mathcal{E}$ of the source category, the image of $\mathcal{E}$ via the projection with respect to $(A, I, \mathcal{M}_A)$ is denoted by $\mathcal{E}_{(A,I,\mathcal{M}_A)}$. When no confusion occurs, we simply write $\mathcal{E}_A$ for this.

Definition 5.5.11 (Analytic log prismatic F -crystals, [GR24, DLMS24]).

Let (A, I) be a prism. We let $\mathrm{Vect}^{\mathrm{an}}(A, I)$ denote the exact category of vector bundles on a scheme $U(A, I) := \mathrm{Spec}(A) \setminus V(p, I)$, and $\mathrm{Vect}^{\mathrm{an}, \varphi}(A, I)$ denote the exact category of pairs $(\mathcal{E}, \varphi_{\mathcal{E}})$ consisting of $\mathcal{E} \in \mathrm{Vect}^{\mathrm{an}}(A, I)$ and an isomorphism $\varphi_{\mathcal{E}}: (\phi_A^* \mathcal{E})[1/I] \xrightarrow{\sim} \mathcal{E}[1/I]$.

For a bounded p -adic formal scheme $\mathfrak{X}$, we define the exact categories $\mathrm{Vect}^{\mathrm{an}}(\mathfrak{X}_{\Delta})$ and $\mathrm{Vect}^{\mathrm{an}, \varphi}(\mathfrak{X}_{\Delta})$ by

$$\begin{aligned} \mathrm{Vect}^{\mathrm{an}}(\mathfrak{X}_{\Delta}) &:= \varprojlim_{(A,I,\mathcal{M}_A) \in \mathfrak{X}_{\Delta}} \mathrm{Vect}^{\mathrm{an}}(A, I), \\ \mathrm{Vect}^{\mathrm{an}, \varphi}(\mathfrak{X}_{\Delta}) &:= \varprojlim_{(A,I,\mathcal{M}_A) \in \mathfrak{X}_{\Delta}} \mathrm{Vect}^{\mathrm{an}, \varphi}(A, I). \end{aligned}$$

For a bounded p -adic log formal scheme $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$, we define the exact categories $\mathrm{Vect}^{\mathrm{an}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ and $\mathrm{Vect}^{\mathrm{an}, \varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ by

$$\begin{aligned} \mathrm{Vect}^{\mathrm{an}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) &:= \varprojlim_{(A,I,\mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}} \mathrm{Vect}^{\mathrm{an}}(A, I), \\ \mathrm{Vect}^{\mathrm{an}, \varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) &:= \varprojlim_{(A,I,\mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}} \mathrm{Vect}^{\mathrm{an}, \varphi}(A, I). \end{aligned}$$

Similarly, we define the exact categories $\mathrm{Vect}^{\mathrm{an}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}})$ and $\mathrm{Vect}^{\mathrm{an}, \varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}})$ by

$$\begin{aligned} \mathrm{Vect}^{\mathrm{an}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}) &:= \varprojlim_{(A, I, \mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}} \mathrm{Vect}^{\mathrm{an}}(A, I), \\ \mathrm{Vect}^{\mathrm{an}, \varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}) &:= \varprojlim_{(A, I, \mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}} \mathrm{Vect}^{\mathrm{an}, \varphi}(A, I). \end{aligned}$$

Objects of the category $\mathrm{Vect}^{\mathrm{an}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ (resp. $\mathrm{Vect}^{\mathrm{an}, \varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$) are called *analytic log prismatic crystals* (resp. *analytic log prismatic F -crystals*) on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$.

Lemma 5.5.12. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a quasi-coherent and integral bounded p -adic formal scheme. Then the inclusion functor $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}} \hookrightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}$ has a right adjoint functor

$$(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta} \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}} \quad ((A, I, \mathcal{M}_A) \mapsto (A, I, \mathcal{N}_A)).$$

In particular, $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}$ is cofinal in $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}$, and, for any $\star \in \{\emptyset, \varphi, \mathrm{an}, (\mathrm{an}, \varphi)\}$, the following restriction functors give bi-exact equivalences:

$$\begin{aligned} \mathrm{Vect}^{\star}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) &\rightarrow \mathrm{Vect}^{\star}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}), \\ \mathrm{Vect}_{[a, b]}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) &\rightarrow \mathrm{Vect}_{[a, b]}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}), \\ \mathrm{Vect}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}, \tilde{\mathcal{O}}_{\Delta}) &\rightarrow \mathrm{Vect}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}, \tilde{\mathcal{O}}_{\Delta}). \end{aligned}$$

Proof. Let $(A, I, \mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}$. We let $f: (\mathrm{Spf}(A/I), \mathcal{M}_{A/I}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ denote the structure morphism. Under the identification $\mathrm{Spf}(A)_{\mathrm{\acute{e}t}} \simeq \mathrm{Spf}(A/I)_{\mathrm{\acute{e}t}}$, we define the log structure $\mathcal{N}_A$ on $\mathrm{Spf}(A)$ by $\mathcal{N}_A := \mathcal{M}_A \times_{\mathcal{M}_{A/I}} f^* \mathcal{M}_{\mathfrak{X}}$. By restricting the $\delta_{\log}$ -structure to $\mathcal{N}_A$, we obtain a log prism $(A, I, \mathcal{N}_A)$, which belongs to $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}$ by Remark 5.5.8. Then the functor $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta} \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}$ sending $(A, I, \mathcal{M}_A)$ to $(A, I, \mathcal{N}_A)$ is the desired one by construction. $\square$

Lemma 5.5.13. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a bounded p -adic integral log formal scheme. Suppose that $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ admits a chart $M \rightarrow \mathcal{M}_{\mathfrak{X}}$ where M is uniquely p -divisible (see Definition 5.A.2 for the definition). Then the functor forgetting log structures

$$(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}} \rightarrow \mathfrak{X}_{\Delta}$$

sending $(A, I, \mathcal{M}_A)$ to (A, I) is an equivalence.

In particular, for $\star \in \{\emptyset, \varphi, \mathrm{an}, (\mathrm{an}, \varphi)\}$, the following natural functors give bi-exact equivalences:

$$\begin{aligned} \mathrm{Vect}^{\star}(\mathfrak{X}_{\Delta}) &\rightarrow \mathrm{Vect}^{\star}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}), \\ \mathrm{Vect}_{[a, b]}^{\varphi}(\mathfrak{X}_{\Delta}) &\rightarrow \mathrm{Vect}_{[a, b]}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}), \\ \mathrm{Vect}(\mathfrak{X}_{\Delta}, \tilde{\mathcal{O}}_{\Delta}) &\rightarrow \mathrm{Vect}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}, \tilde{\mathcal{O}}_{\Delta}). \end{aligned}$$

Proof. The latter assertion follows from the former and Lemma 5.5.12.

We shall prove the former. We construct a quasi-inverse functor. Let $(A, I) \in \mathfrak{X}_{\Delta}$. We let f denote the map

$$M \rightarrow \Gamma(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow \Gamma(\mathrm{Spf}(A/I), \mathcal{M}_{A/I}) \rightarrow A/I.$$

Since M is uniquely p -divisible, there exists a unique lift $\tilde{f}: M \rightarrow A$ of f by Lemma 5.A.3. By setting $\delta_{\log} = 0$, the map $\tilde{f}$ defines a prelog prism (A, I, M) , and $(A, I, M)^a$ belongs to $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}$.

We shall check the functor sending (A, I) to $(A, I, M)^a$ constructed above is indeed a quasi-inverse functor. As a non-trivial direction, it suffices to prove that, for $(A, I, \mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\text{str}}$, there exists a natural isomorphism $(A, I, M)^a \xrightarrow{\sim} (A, I, \mathcal{M}_A)$, where (A, I, M) is a prelog prism defined above. By Lemma 5.A.5 (1), we have a unique map of log structure $M^a \rightarrow \mathcal{M}_A$ lifting the identity map on $\mathcal{M}_{A/I}$. This map $M^a \rightarrow \mathcal{M}_A$ is an isomorphism because so is the restriction to $\text{Spf}(A/I)$. It follows from Lemma 5.5.5 that this map is compatible with $\delta_{\log}$ -structures. $\square$

Construction 5.5.14 (Breuil-Kisin log prisms). Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) = (\text{Spf}(R), \mathcal{M}_R)$ be a small affine log formal scheme over $\mathcal{O}_K$ and fix a framing t . We shall construct a log prism $(\mathfrak{S}_{R,t}, (E), \mathcal{M}_{\mathfrak{S}_{R,t}}) = (\mathfrak{S}_{R,t}, (E), \mathbb{N}^r)^a \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\text{str}}$. Let

$$\mathfrak{S}_{R^0} := W(k)\langle x_1^{\pm 1}, \dots, x_l^{\pm 1}, y_1, \dots, y_m, z_1, \dots, z_n \rangle[[t]] / \left(\prod_{k=1}^n z_k - t \right).$$

We define a prelog structure $\mathbb{N}^r \rightarrow \mathfrak{S}_{R^0}$ by

$$e_i \mapsto \begin{cases} y_i & (1 \leq i \leq m) \\ z_{i-m} & (m+1 \leq i \leq r), \end{cases}$$

and define a $\delta_{\log}$ -structure on this prelog ring by $\delta(x_i) = \delta(y_i) = \delta(z_i) = 0$ and $\delta_{\log}(e_i) = 0$ for all i . Then we have an isomorphism $\mathfrak{S}_{R^0}/(E) \cong R^0$, and a log prism $(\mathfrak{S}_{R^0}, (E), \mathbb{N}^r)^a$ is an object of $((\text{Spf}(R^0), \mathbb{N}^r)^a)_{\Delta}^{\text{str}}$. The p -adically étale R^0 -algebra R lifts uniquely to a (p, E) -adically étale $\mathfrak{S}_{R^0}$ -algebra $\mathfrak{S}_{R,t}$. By [Kos22, Lemma 2.13], there exists a unique prelog prism structure on $(\mathfrak{S}_{R,t}, (E), \mathbb{N}^r)$ such that $(\mathfrak{S}_{R^0}, (E), \mathbb{N}^r) \rightarrow (\mathfrak{S}_{R,t}, (E), \mathbb{N}^r)$ is a map of prelog prisms. Then $(\mathfrak{S}_{R,t}, (E), \mathcal{M}_{\mathfrak{S}_{R,t}}) := (\mathfrak{S}_{R,t}, (E), \mathbb{N}^r)^a$ is an object of $(\text{Spf}(R), \mathcal{M}_R)_{\Delta}^{\text{str}}$. When no confusion occurs, we simply write $(\mathfrak{S}_R, (E), \mathcal{M}_{\mathfrak{S}_R})$ for $(\mathfrak{S}_{R,t}, (E), \mathcal{M}_{\mathfrak{S}_{R,t}})$.

By Lemma 5.5.13, there exists a unique log structure $\mathcal{M}_{A_{\text{inf}}(\tilde{R}_{\infty, \alpha})}$ on $\text{Spf}(A_{\text{inf}}(\tilde{R}_{\infty, \alpha}))$ such that $(A_{\text{inf}}(\tilde{R}_{\infty, \alpha}), (\xi), \mathcal{M}_{A_{\text{inf}}(\tilde{R}_{\infty, \alpha})})$ is a log prism in $(\text{Spf}(\tilde{R}_{\infty, \alpha}), \mathcal{M}_{\tilde{R}_{\infty, \alpha}})_{\Delta}^{\text{str}}$ up to a unique isomorphism. By the proof of *loc. cit.*, $\mathcal{M}_{A_{\text{inf}}(\tilde{R}_{\infty, \alpha})}$ is isomorphic to the associated log structure with the prelog structure $\mathbb{Q}_{\geq 0}^r \rightarrow A_{\text{inf}}(\tilde{R}_{\infty, \alpha})$ defined by

$$se_i \mapsto \begin{cases} [(y_i^s)^b] & (1 \leq i \leq m) \\ [(z_{i-m}^s)^b] & (m+1 \leq i \leq r). \end{cases}$$

We can define a log prism map $(\mathfrak{S}_R, (E), \mathcal{M}_{\mathfrak{S}_R}) \rightarrow (A_{\text{inf}}(\tilde{R}_{\infty, \alpha}), (\xi), \mathcal{M}_{A_{\text{inf}}(\tilde{R}_{\infty, \alpha})})$ as follows. Consider a map of prelog rings $(\mathfrak{S}_{R^0}, \mathbb{N}^r) \rightarrow (A_{\text{inf}}(\tilde{R}_{\infty, \alpha}), \mathbb{Q}_{\geq 0}^r)$ defined by

$$\begin{aligned} x_i &\mapsto [x_i^b] \\ y_j &\mapsto [y_j^b] \\ z_k &\mapsto [z_k^b] \end{aligned}$$

and the natural inclusion $\mathbb{N}^r \rightarrow \mathbb{Q}_{\geq 0}^r$. Since $\mathfrak{S}_{R^0} \rightarrow \mathfrak{S}_R$ is étale, the above map lifts uniquely to a map of prelog rings $(\mathfrak{S}_R, \mathbb{N}^r) \rightarrow (A_{\text{inf}}(\tilde{R}_{\infty, \alpha}), \mathbb{Q}_{\geq 0}^r)$, which gives a log prism map

$$(\mathfrak{S}_R, (E), \mathcal{M}_{\mathfrak{S}_R}) \rightarrow (A_{\text{inf}}(\tilde{R}_{\infty, \alpha}), (\xi), \mathcal{M}_{A_{\text{inf}}(\tilde{R}_{\infty, \alpha})})$$

by taking the associated log structures.

Remark 5.5.15. Let $(\mathrm{Spf}(R), \mathcal{M}_R)$ be a small affine log formal scheme over $\mathcal{O}_K$ admitting a fixed framing with $n = 0$. Then $(\mathrm{Spf}(R), \mathcal{M}_R \oplus \pi^{\mathbb{N}})^a$ is also small affine and admits an induced framing with $n = 1$. Let $(\mathrm{Spf}(R_0), \mathcal{M}_{R_0})$ be the unramified model of $(\mathrm{Spf}(R), \mathcal{M}_R)$ defined by the fixed framing and φ_{R_0} be the Frobenius lift on $(\mathrm{Spf}(R_0), \mathcal{M}_{R_0})$ defined in Definition 5.2.5.

Consider a log formal scheme $(\mathrm{Spf}(R_0[[t]]), \mathcal{M}_{R_0[[t]]}) := (\mathrm{Spf}(R_0[[t]]), \mathcal{M}_R \oplus t^{\mathbb{N}})^a$. The map φ_{R_0} induces a Frobenius lift $\varphi_{R_0[[t]]}$ on $(\mathrm{Spf}(R_0[[t]]), \mathcal{M}_{R_0[[t]]})$ given by $t \mapsto t^p$, which induces a δ -ring structure on $R_0[[t]]$. This is refined to a $\delta_{\log}$ -ring structure on $(\mathrm{Spf}(R_0[[t]]), \mathcal{M}_{R_0[[t]]})$ defined by $\delta_{\log}(y_j) = \delta_{\log}(t) = 0$. Then a triple $(R_0[[t]], (E_t), \mathcal{M}_{R_0[[t]]})$ is a log prism which is isomorphic to the Breuil-Kisin log prism $(\mathfrak{S}_R, (E), \mathcal{M}_{\mathfrak{S}_R})$ associated with $(\mathrm{Spf}(R), \mathcal{M}_R \oplus \pi^{\mathbb{N}})^a$. If we replace $\mathcal{M}_R \oplus t^{\mathbb{N}}$ with $\mathcal{M}_R$ at the first point, we can obtain the Breuil-Kisin log prism associated with $(\mathrm{Spf}(R), \mathcal{M}_R)$.

Lemma 5.5.16 (cf. [DLMS24, Lemma 2.7 and Lemma 2.8]). Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) = (\mathrm{Spf}(R), \mathcal{M}_R)$ be a small affine log formal scheme over $\mathcal{O}_K$ with a fixed framing, and $(A, I, \mathcal{M}_A)$ be an orientable log prism in $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}$. Then there exists a coproduct in $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}$

$$(B, J, \mathcal{M}_B) := (\mathfrak{S}_R, (E), \mathcal{M}_{\mathfrak{S}_R}) \sqcup (A, I, \mathcal{M}_A)$$

belonging to $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}$, and a map $(A, I, \mathcal{M}_A) \rightarrow (B, J, \mathcal{M}_B)$ is a flat cover. In particular, $(\mathfrak{S}_R, (E), \mathcal{M}_{\mathfrak{S}_R})$ covers the final object of the topos associated to $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}$.

Proof. By [Kos22, Lemma 3.8], $\Gamma(\mathrm{Spf}(A), \mathcal{M}_A) \rightarrow \Gamma(\mathrm{Spf}(A/I), \mathcal{M}_{A/I})$ is surjective. Hence, a map $\mathbb{N}^r \rightarrow \Gamma(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow \Gamma(\mathrm{Spf}(A/I), \mathcal{M}_{A/I})$ admits a lift $\mathbb{N}^r \rightarrow \Gamma(\mathrm{Spf}(A), \mathcal{M}_A)$. Fix such a lift and let (A, M_A) denote the prelog ring associated with the fixed lift. Here, $M_A = \mathbb{N}^r$ as monoids. Then $(A, I, \mathcal{M}_A) = (A, I, M_A)^a$. Let

$$C := A[s_1^{\pm}, \dots, s_l^{\pm}, t_1, \dots, t_m, u_1, \dots, u_n][v] / \left(\prod_{k=1}^n u_k - v \right)$$

Let (C, M_C) the associated prelog ring where $M_C := M_A \oplus \mathbb{N}^r \rightarrow C$ is given by $M_A \rightarrow A \rightarrow C$ and

$$e_i \mapsto \begin{cases} t_i & (1 \leq i \leq m) \\ u_{i-m} & (m+1 \leq i \leq r). \end{cases}$$

The prelog ring (C, M_C) is equipped the unique $\delta_{\log}$ -structure such that the natural map $(A, M_A) \rightarrow (C, M_C)$ is compatible with $\delta_{\log}$ -structures and $\delta(s_i) = \delta(t_j) = \delta(u_k) = \delta_{\log}(e_i) = 0$ for every i, j, k . Consider the surjection $(C, M_C) \rightarrow (A/I, M_A)$ induced by $s_i \mapsto x_i$, $t_j \mapsto y_j$, and $u_k \mapsto z_k$, and let

$$(C, M_C) \rightarrow (C', M_{C'}) \rightarrow (A/I, M_A)$$

be the exactification of it. Explicitly, we have

$$\begin{aligned} C' &:= C[(\tilde{y}_1/t_1)^{\pm 1}, \dots, (\tilde{y}_m/t_m)^{\pm 1}, (\tilde{z}_1/u_1)^{\pm 1}, (\tilde{z}_n/u_n)^{\pm 1}] \\ &:= C[T_1^{\pm 1}, \dots, T_m^{\pm 1}, U_1^{\pm 1}, \dots, U_n^{\pm 1}] / (t_j T_j - \tilde{y}_j, u_k U_k - \tilde{z}_k)_{j,k}, \end{aligned}$$

where $\tilde{y}_j$ (resp. $\tilde{z}_k$) is the image of e_j (resp. e_{m+k}) via $\mathbb{N}^r = M_A \rightarrow A$ for every j, k . The $\delta_{\log}$ -structure on (C, M_C) uniquely extends to $(C', M_{C'})$ by [Kos22, Proposition 2.16]. Fix a lift $\tilde{x}_i \in A$ of $x_i \in A/I$ for each i . Let J denote the kernel of the surjection $C' \rightarrow A/I$. Then J is generated by I and $(s_i - \tilde{x}_i, \tilde{y}_j/t_j - 1, \tilde{z}_k/u_k - 1)_{i,j,k}$. Let $(C')^{\wedge}$ denote the derived (p, I) -adic completion of C' , which coincides with the classical (p, I) -completion because C' is flat over A (cf. the proof of [Kos22, Proposition 3.9]), and (B, IB) denote the prismatic

envelope of the δ -pair $((C')^\wedge, J(C')^\wedge)$ over (A, I) . Since $(s_i - \tilde{x}_i, \tilde{y}_j/t_j - 1, \tilde{z}_k/u_k - 1)_{i,j,k}$ is a regular sequence in $(C')^\wedge$ and the natural map of derived quotient rings

$$A/\mathbb{L}(p, d) \rightarrow (C')^\wedge/\mathbb{L}(p, d, s_i - \tilde{x}_i, \tilde{y}_j/t_j - 1, \tilde{z}_k/u_k - 1)_{i,j,k}$$

is an isomorphism, the prism map $(A, I) \rightarrow (B, IB)$ is a flat cover by [BS22, Proposition 3.13 (1)] or [Ito24, Proposition 2.6.6 (1)].

We define a prelog ring map $(\mathfrak{S}_{R^0}, (E), \mathbb{N}^r) \rightarrow (B, IB, M_{C'})$ by $x_i \mapsto s_i$, $y_j \mapsto t_j$, $z_k \mapsto u_k$, and $t \mapsto v$. Since $E(v) \in J$ implies that $v^e \in (p, I)B$, this map is well-defined. Using the étaleness of $\mathfrak{S}_{R^0} \rightarrow \mathfrak{S}_R$, we obtain a unique lift $(\mathfrak{S}_R, (E), \mathbb{N}^r) \rightarrow (B, IB, M_{C'})$ inducing the natural map $(R, \mathbb{N}^r) \rightarrow (B/IB, M_A)$. Then $(B, IB, M_{C'}) \in (\mathrm{Spf}(R), \mathcal{M}_R)_{\Delta}^{\mathrm{str}}$ is the coproduct of $(\mathfrak{S}_R, (E), \mathbb{N}^r)^a$ and $(A, I, \mathcal{M}_A)$ in $(\mathrm{Spf}(R), \mathcal{M}_R)_{\Delta}$. $\square$

For a small affine log formal scheme $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) = (\mathrm{Spf}(R), \mathcal{M}_R)$ over $\mathcal{O}_K$ with a fixed framing, the $n + 1$ -fold self coproduct of $(\mathfrak{S}_R, (E), \mathcal{M}_{\mathfrak{S}_R})$ in $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}$ is denoted by $(\mathfrak{S}_R^{(n)}, (E), \mathcal{M}_{\mathfrak{S}_R^{(n)}}) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}$ for each $n \geq 0$.

Lemma 5.5.17. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a semi-stable log formal scheme over $\mathcal{O}_K$. Then the restriction functor

$$\mathrm{Vect}^*((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \mathrm{Vect}^{\mathrm{an}, *}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$$

is fully faithful for $\star \in \{\emptyset, \varphi\}$.

Proof. By working étale locally on $\mathfrak{X}$, we may assume that $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) = (\mathrm{Spf}(R), \mathcal{M}_R)$ is small affine with a fixed framing. The natural functors

$$\begin{aligned} \mathrm{Vect}^*((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) &\rightarrow \varprojlim_{\bullet \in \Delta} \mathrm{Vect}^*(\mathfrak{S}_R^{(\bullet)}, (E)), \\ \mathrm{Vect}^{\mathrm{an}, *}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) &\rightarrow \varprojlim_{\bullet \in \Delta} \mathrm{Vect}^{\mathrm{an}, *}(\mathfrak{S}_R^{(\bullet)}, (E)) \end{aligned}$$

give equivalences by Lemma 5.5.16. For each $n \geq 0$, the natural functor

$$\mathrm{Vect}^*(\mathfrak{S}_R^{(n)}, (E)) \rightarrow \mathrm{Vect}^{\mathrm{an}, *}(\mathfrak{S}_R^{(n)}, (E))$$

is fully faithful by [SP25, 0G7P]. Therefore, the functor in the statement is fully faithful. $\square$

Definition 5.5.18. ([DLMS24, Definition 3.7]) Let $\mathrm{DD}_{\mathfrak{S}_R}$ be the category of pairs $(\mathfrak{M}, f)$ where

- $\mathfrak{M}$ is a finitely generated $\mathfrak{S}_R$ -module such that, $\mathfrak{M}[1/p]$ (resp. $\mathfrak{M}[1/E]$) is finite projective over $\mathfrak{S}_R[1/p]$ (resp. $\mathfrak{S}_R[1/E]$), and $\mathfrak{M} = \mathfrak{M}[1/p] \cap \mathfrak{M}[1/E]$;
- $f : \mathfrak{M} \otimes_{\mathfrak{S}_R, p_1} \mathfrak{S}_R^{(1)} \xrightarrow{\sim} \mathfrak{M} \otimes_{\mathfrak{S}_R, p_2} \mathfrak{S}_R^{(1)}$ is an isomorphism of $\mathfrak{S}_R^{(1)}$ -modules satisfying the cocycle condition over $\mathfrak{S}_R^{(2)}$.

Let $\mathrm{DD}_{\mathfrak{S}_R}^{\mathrm{vect}}$ denote the full subcategory of $\mathrm{DD}_{\mathfrak{S}_R}$ consisting of objects $(\mathfrak{M}, f)$ with $\mathfrak{M}$ being finite projective over $\mathfrak{S}_R$.

Let $\mathrm{DD}_{\mathfrak{S}_R}^{\varphi}$ (resp. $\mathrm{DD}_{\mathfrak{S}_R}^{\varphi, \mathrm{vect}}$) be the category of pairs $(\mathfrak{M}, \varphi_{\mathfrak{M}}, f)$ where

- $(\mathfrak{M}, f)$ belongs to $\mathrm{DD}_{\mathfrak{S}_R}$ (resp. $\mathrm{DD}_{\mathfrak{S}_R}^{\mathrm{vect}}$);
- $\varphi_{\mathfrak{M}} : (\phi^* \mathfrak{M})[1/E] \xrightarrow{\sim} \mathfrak{M}[1/E]$ is an isomorphism of $\mathfrak{S}_R[1/E]$ -modules such that f is compatible with $\varphi_{\mathfrak{M}}$.

Lemma 5.5.19. For $\star \in \{\emptyset, \varphi\}$, we have natural bi-exact equivalences

$$\begin{aligned} \mathrm{Vect}^{\mathrm{an}, \star}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) &\xrightarrow{\sim} \mathrm{DD}_{\mathfrak{S}_R}^{\star}, \\ \mathrm{Vect}^{\star}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) &\xrightarrow{\sim} \mathrm{DD}_{\mathfrak{S}_R}^{\star, \mathrm{vect}}, \end{aligned}$$

sending $\mathcal{E}$ to $\Gamma(U(\mathfrak{S}_R, (E)), \mathcal{E}_{(\mathfrak{S}_R, (E), \mathcal{M}_{\mathfrak{S}_R})})$.

Proof. This follows from the same argument as [DLMS24, Lemma 3.8]. $\square$

5.6. Realization functors for log prismatic crystals

In this section, we define étale, crystalline, and de Rham realization functors for analytic log prismatic F -crystals on a semi-stable log formal scheme over $\mathcal{O}_K$ and study their properties.

5.6.1. The étale realization functor for log prismatic F -crystals.

Definition 5.6.1 (Laurent F -crystals, [KY25, Definition 7.34]). Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a bounded p -adic log formal scheme. Let $\mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]_p^{\wedge}$ denote the sheaf on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}$ given by $(A, I, \mathcal{M}_A) \mapsto A[1/I]_p^{\wedge}$ and $\phi: \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]_p^{\wedge} \rightarrow \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]_p^{\wedge}$ denote the ring map induced by a ring map ϕ_A for each log prism $(A, I, \mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}$. A vector bundle $\mathcal{E}$ on the ringed site $((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]_p^{\wedge})$ equipped with an isomorphism $\varphi_{\mathcal{E}}: \phi^* \mathcal{E} \xrightarrow{\sim} \mathcal{E}$ is called a *Laurent F -crystal* on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Let $\mathrm{Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]_p^{\wedge})$ denote the category of Laurent F -crystals on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$.

Remark 5.6.2. For a prism (A, I) , we let $\mathrm{Vect}^{\varphi}(A[1/I]_p^{\wedge})$ denote the category of a finite projective $A[1/I]_p^{\wedge}$ -module M with an isomorphism $\phi_A^* M \xrightarrow{\sim} M$, where the ring map $\phi_A: A[1/I]_p^{\wedge} \rightarrow A[1/I]_p^{\wedge}$ is induced by the ring map $\phi_A: A \rightarrow A$. By the same argument as in [BS23, Proposition 2.7], there exists a natural equivalence

$$\mathrm{Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]_p^{\wedge}) \simeq \varprojlim_{(A, I, \mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}} \mathrm{Vect}^{\varphi}(A[1/I]_p^{\wedge}).$$

For a Laurent F -crystal $\mathcal{E}$ on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ and a log prism $(A, I, \mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}$, the projection of $\mathcal{E}$ to $\mathrm{Vect}^{\varphi}(A[1/I]_p^{\wedge})$ via the above functor is denoted by $\mathcal{E}_{(A, I, \mathcal{M}_A)}$. When no confusion occurs, we write $\mathcal{E}_A$ for this.

Theorem 5.6.3 (cf. [KY25, Theorem 7.35]). For a bounded p -adic fs log formal scheme $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$, there exists a natural fully faithful functor

$$T_{\mathrm{ét}}: \mathrm{Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]_p^{\wedge}) \rightarrow \mathrm{Loc}_{\mathbb{Z}_p}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \mathrm{qprokét}}^{\diamond}),$$

where $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta}^{\diamond}$ is the log diamond of the generic fiber of $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ ([KY25, Example 7.6]).

Proof. For the proof, see [KY25, Theorem 7.35]. Although the definition of absolute prismatic sites in [KY25, Definition 7.33] is slightly different from Definition 5.5.6, the category of Laurent F -crystals is unchanged under this modification by Lemma 5.5.12 (see also Remark 5.5.7). $\square$

Remark 5.6.4. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a bounded p -adic fs log formal scheme and $\mathcal{E}$ be a Laurent F -crystal on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. By the proof of [KY25, Theorem 7.35], there exists a functorial isomorphism of $A[1/I]_p^{\wedge}$ -modules

$$\mathcal{E}_A \cong T_{\mathrm{ét}}(\mathcal{E})|_{(\mathrm{Spd}(S, S^+), \mathcal{M}_S)} \otimes_{\mathbb{Z}_p} A[1/I]_p^{\wedge}$$

for a perfect log prism $(A, I, \mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}$ associated with a perfect indiscrete log prism in the sense of [KY25, Definition 2.36 and Definition 2.39] satisfying the following conditions:

- the pullback of $T_{\text{ét}}(\mathcal{E})$ to $(\text{Spd}(S, S^+), \mathcal{M}_S) := (\text{Spf}(A/I), \mathcal{M}_{A/I})_{\eta}^{\diamond}$ is constant (we simply write $T_{\text{ét}}(\mathcal{E})|_{(\text{Spd}(S, S^+), \mathcal{M}_S)}$ for the corresponding $\mathbb{Z}_p$ -module);
- $\mathcal{M}_S/\mathcal{M}_S^{\times}$ is divisible.

Definition 5.6.5. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a bounded p -adic fs log formal scheme. By Remark 5.5.10 and Remark 5.6.2, we have a sequence of exact functors

$$\text{Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \text{Vect}^{\text{an}, \varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \text{Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]_p^{\wedge}).$$

Taking the composition with the functor in Proposition 5.6.3, we obtain exact functors

$$\begin{aligned} \text{Vect}^{\text{an}, \varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) &\rightarrow \text{Loc}_{\mathbb{Z}_p}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \text{qprokét}}^{\diamond}), \\ \text{Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) &\rightarrow \text{Loc}_{\mathbb{Z}_p}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \text{qprokét}}^{\diamond}). \end{aligned}$$

Both of them are also denoted by $T_{\text{ét}}$ and called *étale realization functors*.

Remark 5.6.6. When $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta}$ is a locally noetherian fs log adic space, the target category of the étale realization functor can be identified with the category of $\mathbb{Z}_p$ -local systems on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \text{két}}$ (or $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \text{prokét}}$) by Proposition 5.B.4.

5.6.2. The crystalline realization functor for log prismatic F -crystals.

Construction 5.6.7 (Breuil log prism). Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) = (\text{Spf}(R), \mathcal{M}_R)$ be a small affine p -adic fs log formal scheme and fix a framing t . We have a strict closed immersion

$$(\text{Spf}(R), \mathcal{M}_R) \hookrightarrow (\text{Spf}(\mathfrak{S}_{R,t}), \mathcal{M}_{\mathfrak{S}_{R,t}}).$$

Let $(\text{Spf}(S_{R,t}), \mathcal{M}_{S_{R,t}})$ denote a p -completed log PD-envelope of it. More generally, for an integer $i \geq 0$, let $(\text{Spf}(S_{R,t}^{(i)}), \mathcal{M}_{S_{R,t}^{(i)}})$ denote a p -completed log PD-envelope of a closed immersion

$$(\text{Spf}(R), \mathcal{M}_R) \hookrightarrow (\text{Spf}(\mathfrak{S}_{R,t}), \mathcal{M}_{\mathfrak{S}_{R,t}})^{(i)/W},$$

where $(\text{Spf}(\mathfrak{S}_{R,t}), \mathcal{M}_{\mathfrak{S}_{R,t}})^{(i)/W}$ is the $i+1$ -fold self product of $(\text{Spf}(\mathfrak{S}_{R,t}), \mathcal{M}_{\mathfrak{S}_{R,t}})$ over W . The p -adic log PD-thickening $(S_{R,t}^{(i)} \twoheadrightarrow R/p, \mathcal{M}_{S_{R,t}^{(i)}})$ belongs to $(X_{p=0}, \mathcal{M}_{X_{p=0}})_{\text{crys}}^{\text{str}}$. By construction, $(S_{R,t} \twoheadrightarrow R/p, \mathcal{M}_{S_{R,t}})$ covers the final object of the topos associated with $(X_{p=0}, \mathcal{M}_{X_{p=0}})_{\text{crys}}$, and $(S_{R,t}^{(i)}, \mathcal{M}_{S_{R,t}^{(i)}})$ is the i -th fold self product of $(S_{R,t}, \mathcal{M}_{S_{R,t}})$ in $(X_{p=0}, \mathcal{M}_{X_{p=0}})$. When no confusion occurs, we omit subscripts t .

Since $\text{Ker}(\mathfrak{S}_R \twoheadrightarrow R)$ is generated by a single regular element $E(t)$, the ring $S_R^{(i)}$ is also p -torsion free for each $i \geq 0$. Hence, the Frobenius lift $\phi_{S_R^{(i)}}$ on $(\text{Spf}(S_R^{(i)}), \mathcal{M}_{S_R^{(i)}})$ induced from $\phi_{\mathfrak{S}_R}$ uniquely gives a structure of a δ -ring on $S_R^{(i)}$. By [Kos22, Proposition 2.16], this δ -structure uniquely refines to a $\delta_{\log}$ -structure on $(\text{Spf}(S_R^{(i)}), \mathcal{M}_{S_R^{(i)}})$ such that the natural map

$$(\text{Spf}(S_R^{(i)}), \mathcal{M}_{S_R^{(i)}}) \rightarrow (\text{Spf}(\mathfrak{S}_R), \mathcal{M}_{\mathfrak{S}_R})^{(i)/W}$$

is compatible with $\delta_{\log}$ -structures. As a result, we obtain a log prism $(S_R^{(i)}, (p), \phi_{S_R^{(i)}}^* \mathcal{M}_{S_R^{(i)}}) \in (X_{p=0}, \mathcal{M}_{X_{p=0}})_{\Delta}^{\text{str}}$, where the $\delta_{\log}$ -structure on $(S_R^{(i)}, \phi_{S_R^{(i)}}^* \mathcal{M}_{S_R^{(i)}})$ is induced from that on

$(S_R^{(i)}, \mathcal{M}_{S_R^{(i)}})$, and the ring map $\mathfrak{S}_R \xrightarrow{\phi_{\mathfrak{S}_R}} \mathfrak{S}_R \rightarrow S_R$ induces a log prism map in $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\text{str}}$

$$\phi: (\mathfrak{S}_R^{(i)}, (E), \mathcal{M}_{\mathfrak{S}_R^{(i)}}) \rightarrow (S_R^{(i)}, (p), \phi_{S_R^{(i)}}^* \mathcal{M}_{S_R^{(i)}})$$

for each $i \geq 0$.

Remark 5.6.8. Let $(\text{Spf}(R), \mathcal{M}_R)$ be a small affine log formal scheme over $\mathcal{O}_K$ admitting a fixed framing with $n = 0$. Then $(\text{Spf}(R), \mathcal{M}_R \oplus \pi^{\mathbb{N}})^a$ is also small affine and admits an induced framing with $n = 1$. Let $(\text{Spf}(R_0), \mathcal{M}_{R_0})$ be the unramified model of $(\text{Spf}(R), \mathcal{M}_R)$ defined by the fixed framing and φ_{R_0} be the Frobenius lift on $(\text{Spf}(R_0), \mathcal{M}_{R_0})$ defined in Definition 5.2.5.

Let $(\text{Spf}(R_0[[t]]), \mathcal{M}_{R_0[[t]]})$ as in Remark 5.5.15. Consider a strict closed immersion

$$(\text{Spf}(R), \mathcal{M}_R) \hookrightarrow (\text{Spf}(R_0[[t]]), \mathcal{M}_{R_0[[t]]})$$

induced from $(\text{Spf}(R), \mathcal{M}_R) \rightarrow (\text{Spf}(R_0), \mathcal{M}_{R_0})$ and $t \mapsto \pi$. By Remark 5.5.15, the p -completed log PD-envelope of it is nothing but the Breuil ring $(S_R, \mathcal{M}_{S_R})$ associated with $(\text{Spf}(R), \mathcal{M}_R \oplus \pi^{\mathbb{N}})^a$. Explicitly, we have an isomorphism $S_R \cong R_0[t][\{\frac{t^{en}}{(en)!}\}_{n \geq 1}]^{\wedge}$, where $e := [K : K_0]$. We can also obtain the Breuil ring associated with $(\text{Spf}(R), \mathcal{M}_R)$ in a similar way as in Remark 5.5.15.

By using Breuil ring, we can prove the following.

Proposition 5.6.9. Let $(\text{Spf}(R), \mathcal{M}_R)$ be a small affine log formal scheme over $\mathcal{O}_K$ admitting a fixed framing with $n = 0$. Let $(\text{Spf}(R_0), \mathcal{M}_{R_0})$ be the unramified model defined by the fixed framing. Consider the Breuil ring $(\text{Spf}(S_R), \mathcal{M}_{S_R})$ associated with $(R, \mathcal{M}_R \oplus \pi^{\mathbb{N}})$ equipped with the induced framing. Then, for a locally free F -isocrystal $\mathcal{E}$ on $(\text{Spec}(R/p), \mathcal{M}_{R/p} \oplus \pi^{\mathbb{N}})$, there is a natural isomorphism

$$\mathcal{E}_{(S_R \rightarrow R/p, \mathcal{M}_{S_R})} \cong \mathcal{E}_{(R_0 \rightarrow R_0/p, \mathcal{M}_{R_0} \oplus 0^{\mathbb{N}})^a} \otimes_{R_0[1/p]} S_R[1/p]$$

that is compatible with Frobenius isomorphisms on both sides. In particular, there is a natural isomorphism

$$\mathcal{E}_{(R \rightarrow R/p, \mathcal{M}_R \oplus \pi^{\mathbb{N}})^a} \cong \mathcal{E}_{(R_0 \rightarrow R_0/p, \mathcal{M}_{R_0} \oplus 0^{\mathbb{N}})^a} \otimes_{R_0[1/p]} R[1/p].$$

Proof. The strict closed immersion

$$(\text{Spf}(R_0), \mathcal{M}_{R_0} \oplus 0^{\mathbb{N}})^a \hookrightarrow (\text{Spf}(R_0[[t]]), \mathcal{M}_{R_0} \oplus t^{\mathbb{N}})$$

given by $t \mapsto 0$ induces a strict closed immersion

$$(\text{Spf}(R_0), \mathcal{M}_{R_0} \oplus 0^{\mathbb{N}})^a \hookrightarrow (\text{Spf}(S_R), \mathcal{M}_{S_R})$$

by the universal property of PD-envelopes and Remark 5.6.8. The crystal property of $\mathcal{E}$ along the above map gives an isomorphism

$$\mathcal{E}_{(S_R \rightarrow R/p, \mathcal{M}_{S_R})} \otimes_{S_R[1/p]} R_0[1/p] \cong \mathcal{E}_{(R_0 \rightarrow R_0/p, \mathcal{M}_{R_0} \oplus 0^{\mathbb{N}})^a}.$$

Therefore, by applying Lemma 5.3.34 to ring maps $R_0 \rightarrow S_R \twoheadrightarrow R_0$, we obtain the former assertion. The latter one is proved by taking the former isomorphism along $S_R \twoheadrightarrow R$. $\square$

Definition 5.6.10. Let DD_{S_R} denote the category of pairs (M, ϵ) where

- M is a (not necessarily p -complete) finitely generated S_R -module such that $M[1/p]$ is finite projective over $S_R[1/p]$;
- $\epsilon: M \otimes_{S_R, p_1} S_R^{(1)} \xrightarrow{\sim} M \otimes_{S_R, p_2} S_R^{(1)}$ satisfying the cocycle condition over $S_R^{(2)}$.

Morphisms $(M, \epsilon) \rightarrow (M', \epsilon')$ in DD_{S_R} are S_R -linear morphisms $M \rightarrow M'$ compatible with ϵ and ϵ' . Let $\mathrm{DD}_{S_R}^{\mathrm{vect}}$ denote the full subcategory of DD_{S_R} consisting of objects (M, ϵ) with M being finite projective over S_R .

We let $\mathrm{DD}_{S_R}^{\varphi}$ (resp. $\mathrm{DD}_{S_R}^{\varphi, \mathrm{vect}}$) denote the category of triples (M, φ_M, ϵ) where

- (M, ϵ) belongs to DD_{S_R} (resp. $\mathrm{DD}_{S_R}^{\mathrm{vect}}$);
- $\varphi_M: (\phi_{S_R}^* M)[1/p] \xrightarrow{\sim} M[1/p]$ is an $S_R[1/p]$ -linear isomorphism that is compatible with ϵ .

Morphisms $(M, \varphi_M, \epsilon) \rightarrow (M', \varphi_{M'}, \epsilon')$ in $\mathrm{DD}_{S_R}^{\varphi}$ are S_R -linear morphisms $M \rightarrow M'$ compatible with $\epsilon, \epsilon', \varphi_M$, and $\varphi_{M'}$.

Lemma 5.6.11. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) = (\mathrm{Spf}(R), \mathcal{M}_R)$ be a small affine log formal scheme over $\mathcal{O}_K$ with a fixed framing. Then there exist natural functors

$$\begin{aligned} \mathrm{DD}_{S_R} &\rightarrow \mathrm{Crys}_{\mathrm{tf}}((X_{p=0}, \mathcal{M}_{X_{p=0}})_{\mathrm{crys}}), \\ \mathrm{DD}_{S_R}^{\mathrm{vect}} &\rightarrow \mathrm{Vect}((X_{p=0}, \mathcal{M}_{X_{p=0}})_{\mathrm{crys}}), \\ \mathrm{DD}_{S_R}^{\varphi} &\rightarrow \mathrm{Isoc}_{\mathrm{lftf}}^{\varphi}((X_{p=0}, \mathcal{M}_{X_{p=0}})_{\mathrm{crys}}), \\ \mathrm{DD}^{\varphi, \mathrm{vect}} &\rightarrow \mathrm{Vect}^{\varphi}((X_{p=0}, \mathcal{M}_{X_{p=0}})_{\mathrm{crys}}). \end{aligned}$$

Proof. It is enough to construct the first functor. Let $(M, \epsilon) \in \mathrm{DD}_{S_R}$. Let $\overline{M}$ denote the maximal p -torsion free quotient of M . Then $\overline{M}$ is p -complete by Lemma 5.3.18, and ϵ induces a $S_R^{(1)}$ -linear isomorphism

$$\bar{\epsilon}: \overline{M} \widehat{\otimes}_{S_R, p_1} S_R^{(1)} \xrightarrow{\sim} \overline{M} \widehat{\otimes}_{S_R, p_2} S_R^{(1)}$$

satisfying the cocycle condition over $S_R^{(2)}$ by Lemma 5.6.12 below. Since the p -adic log PD-thickening $(S_R \twoheadrightarrow R/p, \mathcal{M}_{S_R})$ covers the final object of the topos associated with $(X_{p=0}, \mathcal{M}_{X_{p=0}})_{\mathrm{crys}}$, the pair $(\overline{M}, \bar{\epsilon})$ gives an object of $\mathrm{Crys}_{\mathrm{tf}}((X_{p=0}, \mathcal{M}_{X_{p=0}})_{\mathrm{crys}})$. This construction gives the former functor in the statement. $\square$

Lemma 5.6.12. Let $A \rightarrow B$ be a p -adically flat ring map of p -complete and p -torsion free rings. Let M be a finitely generated A -module with $M[1/p]$ being finite projective over $A[1/p]$. Let $\overline{M}$ denote the maximal p -torsion free quotient of M (which is p -complete by Lemma 5.3.18). Then $\overline{M} \widehat{\otimes}_A B$ is isomorphic to the maximal p -torsion free quotient of $M \otimes_A B$.

Proof. Since $\overline{M} \otimes_A B$ is a finitely generated B -module, the natural map $\overline{M} \otimes_A B \rightarrow \overline{M} \widehat{\otimes}_A B$ is surjective. Since $A \rightarrow B$ is p -adically flat, $\overline{M} \widehat{\otimes}_A B$ is p -torsion free. By Lemma 5.3.17, we have a natural isomorphism $(\overline{M} \widehat{\otimes}_A B)[1/p] \cong \overline{M}[1/p] \otimes_A B$. As a summary, we obtain the following commutative diagram:

$$\begin{array}{ccccc} M \otimes_A B & \twoheadrightarrow & \overline{M} \otimes_A B & \twoheadrightarrow & \overline{M} \widehat{\otimes}_A B \\ \downarrow & & & & \downarrow \\ M[1/p] \otimes_A B & \xrightarrow{\sim} & \overline{M}[1/p] \otimes_A B & \xrightarrow{\sim} & (\overline{M} \widehat{\otimes}_A B)[1/p]. \end{array}$$

This diagram implies the assertion. $\square$

Construction 5.6.13 (Crystalline realization). Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) = (\mathrm{Spf}(R), \mathcal{M}_R)$ be a semi-stable log formal scheme over $\mathcal{O}_K$. First, we shall construct a functor

$$\mathrm{Vect}^{\mathrm{an}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \mathrm{Crys}_{\mathrm{tf}}((X_{p=0}, \mathcal{M}_{X_{p=0}})_{\mathrm{crys}}).$$

To do this, we may work étale locally on $\mathfrak{X}$ and assume that $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) = (\mathrm{Spf}(R), \mathcal{M}_R)$ is a small affine log formal scheme with a fixed framing. We use notations in Construction 5.5.14 and Construction 5.6.7. By Lemma 5.5.19, there is an equivalence

$$\mathrm{Vect}^{\mathrm{an}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \mathrm{DD}_{\mathfrak{S}_R}.$$

Taking the base change along log prism maps $\phi: (\mathfrak{S}_R^{(i)}, (E), \mathcal{M}_{\mathfrak{S}_R^{(i)}}) \rightarrow (S_R^{(i)}, (p), \mathcal{M}_{S_R^{(i)}})$ gives a functor $\mathrm{DD}_{\mathfrak{S}_R} \rightarrow \mathrm{DD}_{S_R}$. Then, by composing these functors with the functor in Lemma 5.6.11, we obtain a functor

$$\mathrm{Vect}^{\mathrm{an}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \mathrm{Crys}_{\mathrm{tf}}((X_{p=0}, \mathcal{M}_{X_{p=0}})_{\mathrm{crys}}),$$

denoted by T_{crys} . In the same way, we define a functor

$$\mathrm{Vect}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \mathrm{Vect}((X_{p=0}, \mathcal{M}_{X_{p=0}})_{\mathrm{crys}}),$$

which coincides with the restriction of T_{crys} . To globalize the construction of the functor T_{crys} , we need to prove the fact that T_{crys} is independent of the choice of a framing, which can be checked in the same way as [DLMS24, Lemma 3.31].

Next, we shall construct a functor

$$\mathrm{Vect}^{\mathrm{an}, \varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \mathrm{Isoc}_{\mathrm{lftf}}^{\varphi}((X_{p=0}, \mathcal{M}_{X_{p=0}})_{\mathrm{crys}}).$$

For $\mathcal{E} \in \mathrm{Vect}^{\mathrm{an}, \varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$, we have $T_{\mathrm{crys}}(\mathcal{E}) \in \mathrm{Crys}_{\mathrm{tf}}((X_{p=0}, \mathcal{M}_{X_{p=0}})_{\mathrm{crys}})$. By construction, $T_{\mathrm{crys}}(\mathcal{E})$ is locally free. Hence, it is enough to construct a Frobenius structure $F^*T_{\mathrm{crys}}(\mathcal{E})[1/p] \xrightarrow{\sim} T_{\mathrm{crys}}(\mathcal{E})[1/p]$. To do this, we may work étale locally on $\mathfrak{X}$, and so we may assume that $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) = (\mathrm{Spf}(R), \mathcal{M}_R)$ is a small affine log formal scheme with a fixed framing. Taking the base change along log prism maps $\phi: (\mathfrak{S}_R^{(i)}, (E), \mathcal{M}_{\mathfrak{S}_R^{(i)}}) \rightarrow (S_R^{(i)}, (p), \phi^*\mathcal{M}_{S_R^{(i)}})$ in Construction 5.6.7 gives a functor

$$\mathrm{DD}_{\mathfrak{S}_R}^{\varphi} \rightarrow \mathrm{DD}_{S_R}^{\varphi}.$$

Composing this functor with the functors in Lemma 5.5.19 and Lemma 5.6.11, we obtain a functor

$$\mathrm{Vect}^{\mathrm{an}, \varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \mathrm{Isoc}_{\mathrm{lftf}}^{\varphi}((X_{p=0}, \mathcal{M}_{X_{p=0}})_{\mathrm{crys}}),$$

denoted by T_{isoc} . Since the functor T_{isoc} is compatible with T_{crys} in the sense that there is an isomorphism $T_{\mathrm{isoc}}(\mathcal{E}) \cong T_{\mathrm{crys}}(\mathcal{E})[1/p]$, this gives a Frobenius structure on $T_{\mathrm{crys}}(\mathcal{E})[1/p]$. In the same way, we define a functor

$$\mathrm{Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \mathrm{Vect}^{\varphi}((X_{p=0}, \mathcal{M}_{X_{p=0}})_{\mathrm{crys}}),$$

which coincides with the restriction of T_{isoc} .

Definition 5.6.14 (Crystalline realization). For a semi-stable log formal scheme $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ over $\mathcal{O}_K$, we constructed functors

$$\begin{aligned} T_{\mathrm{crys}}: \mathrm{Vect}^{\mathrm{an}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) &\rightarrow \mathrm{Crys}_{\mathrm{tf}}((X_{p=0}, \mathcal{M}_{X_{p=0}})_{\mathrm{crys}}) \\ T_{\mathrm{crys}}: \mathrm{Vect}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) &\rightarrow \mathrm{Vect}((X_{p=0}, \mathcal{M}_{X_{p=0}})_{\mathrm{crys}}) \\ T_{\mathrm{isoc}}: \mathrm{Vect}^{\mathrm{an}, \varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) &\rightarrow \mathrm{Isoc}_{\mathrm{lftf}}^{\varphi}((X_{p=0}, \mathcal{M}_{X_{p=0}})_{\mathrm{crys}}), \\ T_{\mathrm{isoc}}: \mathrm{Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) &\rightarrow \mathrm{Vect}^{\varphi}((X_{p=0}, \mathcal{M}_{X_{p=0}})_{\mathrm{crys}}) \end{aligned}$$

in Construction 5.6.13. These are called *crystalline realization functors*.

5.6.3. The relationship between étale and crystalline realization. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a semi-stable log formal scheme over $\mathcal{O}_K$.

Setting 5.6.15. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a semi-stable log formal scheme over $\mathcal{O}_K$ and $\mathcal{E}$ be an analytic log prismatic F -crystal on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Consider a p -adic saturated log formal scheme $(\mathrm{Spf}(S), \mathcal{M}_S)$ over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ satisfying the following conditions:

- S is an integral perfectoid $\mathcal{O}_C$ -algebra;
- $T_{\text{ét}}(\mathcal{E})|_{(\mathrm{Spf}(S), \mathcal{M}_S)_{\eta}}$ is a constant $\mathbb{Z}_p$ -local system;
- the structure map $(\mathrm{Spf}(S), \mathcal{M}_S) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ factors as

$$(\mathrm{Spf}(S), \mathcal{M}_S) \rightarrow (\mathrm{Spf}(\tilde{R}_{\infty, \alpha}), \mathcal{M}_{\tilde{R}_{\infty, \alpha}}) \rightarrow (\mathrm{Spf}(R), \mathcal{M}_R) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}),$$

where $(\mathrm{Spf}(R), \mathcal{M}_R)$ is small affine with a fixed framing, $(\mathrm{Spf}(R), \mathcal{M}_R) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is strict étale, and $(\mathrm{Spf}(S), \mathcal{M}_S) \rightarrow (\mathrm{Spf}(\tilde{R}_{\infty, \alpha}), \mathcal{M}_{\tilde{R}_{\infty, \alpha}})$ is a strict map over $\mathcal{O}_C$.

In this situation, there exists a unique log structure $\mathcal{M}_{A_{\text{inf}}(S)}$ on $\mathrm{Spf}(A_{\text{inf}}(S))$ which restricts to $\mathcal{M}_S$ by Lemma 5.A.5. We have a log prism $(A_{\text{inf}}(S), (\xi), \mathcal{M}_{A_{\text{inf}}(S)})$.

Proposition 5.6.16. Let $\mathcal{E}$ be an analytic log prismatic F -crystal on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Let $(\mathrm{Spf}(S), \mathcal{M}_S)$ be a saturated p -adic log formal scheme over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ as in Setting 5.6.15. Then there exists a natural isomorphism

$$\mathcal{E}_{A_{\text{inf}}(S)}[1/\phi^{-1}(\mu)] \cong T_{\text{ét}}(\mathcal{E})|_{(\mathrm{Spf}(S), \mathcal{M}_S)_{\eta}} \otimes_{\mathbb{Z}_p} A_{\text{inf}}(S)[1/\phi^{-1}(\mu)].$$

Here, $T_{\text{ét}}(\mathcal{E})$ is regarded as a finite free $\mathbb{Z}_p$ -module.

Proof. By Remark 5.6.4, we have a natural isomorphism

$$\mathcal{E}_{A_{\text{inf}}(S)}[1/\phi^{-1}(\mu)] \cong T_{\text{ét}}(\mathcal{E})|_{(\mathrm{Spf}(S), \mathcal{M}_S)_{\eta}} \otimes_{\mathbb{Z}_p} A_{\text{inf}}(S)[1/\phi^{-1}(\mu)].$$

It suffices to prove that this restricts to the isomorphism in the statement. This follows from the argument in the proof of [GR24, Lemma 3.13], which reduces the problem to [BMS18, Lemma 4.26] via arc descent. Note that, in [GR24], the definition of Breuil-Kisin-Fargues modules over $\mathcal{O}_C$ in [BMS18, Definition 4.22] is adopted. This object naturally corresponds to analytic prismatic F -crystals over $(A_{\text{inf}}(\mathcal{O}_C), (\phi(\xi)))$ (not over $(A_{\text{inf}}(\mathcal{O}_C), \xi)$). $\square$

Theorem 5.6.17. Let $\mathcal{E}$ be an analytic log prismatic F -crystal $\mathcal{E}$ on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Then the $\mathbb{Q}_p$ -local system $T_{\text{ét}}(\mathcal{E})[1/p]$ on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta}$ and the F -isocrystal $T_{\text{isoc}}(\mathcal{E})$ on $(X, \mathcal{M}_X)$ are associated. In particular, $T_{\text{ét}}(\mathcal{E})[1/p]$ is a semi-stable local system.

Proof. It suffices to show that there exists a natural isomorphism

$$(T_{\text{ét}}(\mathcal{E}) \otimes_{\mathbb{Z}_p} \mathbb{B}_{\text{crys}})(U) \cong \mathbb{B}_{\text{crys}}(T_{\text{crys}}(\mathcal{E}))(U)$$

that is compatible with Frobenius for a big log affinoid perfectoid U in $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \text{prokét}}$ with $T_{\text{ét}}(\mathcal{E})|_U$ being constant. If we set $\widehat{U} := \mathrm{Spa}(S, S^+)$ to be the associated affinoid perfectoid space with U , the natural map $(\mathrm{Spf}(S^+), \mathcal{M}_{S^+}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ satisfies the conditions in Setting 5.6.15. Taking the base change of the isomorphism in Proposition 5.6.16 along a ring map

$$A_{\text{inf}}(S^+)[1/\phi^{-1}(\mu)] \xrightarrow{\phi} A_{\text{inf}}(S^+)[1/\mu] \rightarrow B_{\text{crys}}(S^+),$$

we obtain isomorphisms

$$\begin{aligned} (T_{\text{ét}}(\mathcal{E}) \otimes_{\mathbb{Z}_p} \mathbb{B}_{\text{crys}})(U) &\cong T_{\text{ét}}(\mathcal{E})|_U \otimes_{\mathbb{Z}_p} B_{\text{crys}}(S^+) \\ &\cong \mathcal{E}_{A_{\text{inf}}(S^+)}[1/\phi^{-1}(\mu)] \otimes_{A_{\text{inf}}(S^+)[1/\phi^{-1}(\mu)], \phi} B_{\text{crys}}(S^+) \end{aligned}$$

which are compatible with Frobenius.

For the right hand side, we have isomorphisms which are compatible with Frobenius isomorphisms

$$\begin{aligned} & \mathcal{E}_{A_{\text{inf}}(S^+)}[1/\phi^{-1}(\mu)] \otimes_{A_{\text{inf}}(S^+)[1/\phi^{-1}(\mu)], \phi} B_{\text{crys}}(S^+) \\ & \cong \mathcal{E}_{A_{\text{crys}}(S^+)} \otimes_{B_{\text{crys}}^+(S^+)} B_{\text{crys}}(S^+) \\ & \cong T_{\text{crys}}(\mathcal{E})_{(A_{\text{crys}}(S^+) \twoheadrightarrow S^+/p, \mathcal{M}_{A_{\text{crys}}(S^+)})} \otimes_{B_{\text{crys}}^+(S^+)} B_{\text{crys}}(S^+) \\ & \cong \mathbb{B}_{\text{crys}}(T_{\text{crys}}(\mathcal{E}))(U), \end{aligned}$$

where the first isomorphism is defined by the crystal property of $\mathcal{E}$ for a log prism map $\phi: (A_{\text{inf}}(S^+), (\xi), \mathcal{M}_{A_{\text{inf}}(S^+)}) \rightarrow (A_{\text{crys}}(S^+), (p), \mathcal{M}_{A_{\text{crys}}(S^+)})$, and the second isomorphism comes from the definition of crystalline realization functor. By composing this with the isomorphism in the previous paragraph, we obtain the desired isomorphism. $\square$

Theorem 5.6.18. The crystalline realization functor T_{isoc} is uniquely enhanced to a functor

$$T_{\text{fisoc}}: \text{Vect}^{\text{an}, \varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\underline{\Delta}}) \rightarrow \text{FilIsoc}^{\varphi}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$$

such that, for every $\mathcal{E} \in \text{Vect}^{\text{an}, \varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\underline{\Delta}})$, $T_{\text{ét}}(\mathcal{E})$ and $T_{\text{fisoc}}(\mathcal{E})$ are associated in the sense of Definition 5.4.18 (2).

Proof. By Proposition 5.4.20 and Theorem 5.6.17, there exists a unique filtration $\text{Fil}_{\text{dR}}^{\bullet}$ on $T_{\text{isoc}}(\mathcal{E})_{(X_{p=0}, \mathfrak{X}, \mathcal{M}_{\mathfrak{X}})}$ such that $T_{\text{ét}}(\mathcal{E})$ and $T_{\text{fisoc}}(\mathcal{E}) := (T_{\text{isoc}}(\mathcal{E})_{(X_{p=0}, \mathfrak{X}, \mathcal{M}_{\mathfrak{X}})}, \text{Fil}_{\text{dR}}^{\bullet})$ are associated in the sense of Definition 5.4.18 (2). Then the functor T_{fisoc} defined in this way is the desired one. $\square$

Definition 5.6.19 (De Rham realization). Let $\text{FilVect}(\mathfrak{X}_{\eta})$ be the category of filtered vector bundles on $\mathfrak{X}_{\eta}$. The functor

$$\text{Vect}^{\text{an}, \varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\underline{\Delta}}) \rightarrow \text{FilVect}(\mathfrak{X}_{\eta})$$

sending $\mathcal{E}$ to $(T_{\text{isoc}}(\mathcal{E})_{X_{p=0}, \mathfrak{X}, \mathcal{M}_{\mathfrak{X}}}, \text{Fil}_{\text{dR}}^{\bullet})$ is called a *de Rham realization functor*, denoted by T_{dR} .

In the next subsection, we give a description of T_{dR} using the Nygaard filtration. To do this, we make some preparations.

Let $(\text{Spf}(S), \mathcal{M}_S)$ be a p -adic saturated log formal scheme over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ as in Setting 5.6.15. We put a filtration $\text{Fil}_{\xi}^{\bullet}$ on $T_{\text{ét}}(\mathcal{E})|_{(\text{Spf}(S), \mathcal{M}_S)_{\eta}} \otimes_{\mathbb{Z}_p} B_{\text{dR}}(S)$ (resp. $\mathcal{E}_{A_{\text{inf}}(S)} \otimes_{A_{\text{inf}}(S)} B_{\text{dR}}(S)$) by

$$\begin{aligned} \text{Fil}_{\xi}^{\bullet} &:= T_{\text{ét}}(\mathcal{E})|_{(\text{Spf}(S), \mathcal{M}_S)_{\eta}} \otimes_{\mathbb{Z}_p} \text{Fil}^{\bullet} B_{\text{dR}}(S) \\ (\text{resp. } \text{Fil}_{\xi}^{\bullet} &:= \mathcal{E}_{A_{\text{inf}}(S)} \otimes_{A_{\text{inf}}(S)} \text{Fil}^{\bullet} B_{\text{dR}}(S)). \end{aligned}$$

Moreover we define a filtration $\text{Fil}_{\text{dR}, B_{\text{dR}}(S)}^{\bullet}$ on

$$\mathcal{E}_{A_{\text{inf}}(S)} \otimes_{A_{\text{inf}}(S), \phi} B_{\text{dR}}(S) \cong T_{\text{crys}}(\mathcal{E})_{(A_{\text{crys}}(S) \twoheadrightarrow S/p, \mathcal{M}_{A_{\text{crys}}(S)})} \otimes_{A_{\text{crys}}(S)} B_{\text{dR}}(S)$$

in the same way as Construction 5.4.17.

Lemma 5.6.20. We have the following commutative diagram of isomorphisms of filtered $B_{\mathrm{dR}}(S)$ -modules:

$$\begin{array}{ccc} (T_{\mathrm{\acute{e}t}}(\mathcal{E})|_{(\mathrm{Spf}(S), \mathcal{M}_S)_\eta} \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}(S), \mathrm{Fil}_\xi^\bullet) & \cong & (\mathcal{E}_{A_{\mathrm{inf}}(S)} \otimes_{A_{\mathrm{inf}}(S), \phi} B_{\mathrm{dR}}(S), \mathrm{Fil}_{\mathrm{dR}, B_{\mathrm{dR}}(S)}^\bullet) \\ \parallel & & \sim \downarrow \varphi_{\mathcal{E}} \otimes \mathrm{id} \\ (T_{\mathrm{\acute{e}t}}(\mathcal{E})|_{(\mathrm{Spf}(S), \mathcal{M}_S)_\eta} \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}(S), \mathrm{Fil}_\xi^\bullet) & \cong & (\mathcal{E}_{A_{\mathrm{inf}}(S)} \otimes_{A_{\mathrm{inf}}(S)} B_{\mathrm{dR}}(S), \mathrm{Fil}_\xi^\bullet). \end{array}$$

Proof. Since the isomorphism in Proposition 5.6.16 is obtained as the restriction of the isomorphism in Remark 5.6.4 which is compatible with Frobenius, we have a commutative diagram

$$\begin{array}{ccc} T_{\mathrm{\acute{e}t}}(\mathcal{E})|_{(\mathrm{Spf}(S), \mathcal{M}_S)_\eta} \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}(S)[1/\phi^{-1}(\mu)] & \xrightarrow{\sim} & \mathcal{E}_{A_{\mathrm{inf}}(S)}[1/\phi^{-1}(\mu)] \\ \sim \downarrow \mathrm{id} \otimes \phi & & \sim \downarrow \varphi_{\mathcal{E}} \\ T_{\mathrm{\acute{e}t}}(\mathcal{E})|_{(\mathrm{Spf}(S), \mathcal{M}_S)_\eta} \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}(S)[1/\mu] & \xrightarrow{\sim} & \mathcal{E}_{A_{\mathrm{inf}}(S)}[1/\mu], \end{array}$$

where the upper horizontal map is the isomorphism in Proposition 5.6.16, the bottom horizontal map is the localization of the upper horizontal isomorphism. Taking the base change of this diagram along ring maps

$$A_{\mathrm{inf}}(S)[1/\phi^{-1}(\mu)] \xrightarrow{\phi} A_{\mathrm{inf}}(S)[1/\mu] \rightarrow B_{\mathrm{dR}}(S),$$

we obtain the commutative diagram in the statement.

We shall prove that every isomorphism in the square in the statement is a filtered isomorphism. Since $\phi^{-1}(\mu)$ is invertible in $B_{\mathrm{dR}}^+(S)$, we have a ring map $A_{\mathrm{inf}}(S)[1/\phi^{-1}(\mu)] \rightarrow B_{\mathrm{dR}}^+(S)$, and the bottom horizontal isomorphism is obtained as the base change of the isomorphism in Proposition 5.6.16 along

$$A_{\mathrm{inf}}(S)[1/\phi^{-1}(\mu)] \rightarrow B_{\mathrm{dR}}^+(S) \rightarrow B_{\mathrm{dR}}(S).$$

This implies that the bottom horizontal isomorphism in the diagram in the statement is a filtered isomorphism.

For the upper horizontal isomorphism, note that $(\mathrm{Spf}(S), \mathcal{M}_S)_\eta$ does not necessarily live in $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \mathrm{prok\acute{e}t}}$. However, when we take a factorization of $(\mathrm{Spf}(S), \mathcal{M}_S) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ as in Setting 5.6.15, by taking the filtered base change of the filtered isomorphism

$$\begin{aligned} & (T_{\mathrm{\acute{e}t}}(\mathcal{E})|_{(\mathrm{Spf}(\tilde{R}_{\infty, \alpha}), \mathcal{M}_{\tilde{R}_{\infty, \alpha}})_\eta} \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}(\tilde{R}_{\infty, \alpha}), \mathrm{Fil}_\xi^\bullet) \\ & \cong (\mathcal{E}_{A_{\mathrm{inf}}(\tilde{R}_{\infty, \alpha})} \otimes_{A_{\mathrm{inf}}(\tilde{R}_{\infty, \alpha}), \phi} B_{\mathrm{dR}}(\tilde{R}_{\infty, \alpha}), \mathrm{Fil}_{\mathrm{dR}, B_{\mathrm{dR}}(\tilde{R}_{\infty, \alpha})}^\bullet) \end{aligned}$$

given by Proposition 5.4.20 along $B_{\mathrm{dR}}(\tilde{R}_{\infty, \alpha}) \rightarrow B_{\mathrm{dR}}(S)$, we see that the upper horizontal isomorphism is a filtered isomorphism. Therefore, the right vertical morphism is also a filtered isomorphism. $\square$

5.6.4. The de Rham realization functor for log prismatic F -crystals.

Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) = (\mathrm{Spf}(R), \mathcal{M}_R)$ be a small affine log formal scheme over $\mathcal{O}_K$ with a fixed framing t . Let $\mathcal{E}$ be an analytic log prismatic F -crystal on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. The goal of this subsection is to give the description of the filtered vector bundle $(T_{\mathrm{isoc}}(\mathcal{E})_{(X_{p=0}, \mathfrak{X}, \mathcal{M}_{\mathfrak{X}})}, \mathrm{Fil}_{\mathrm{dR}}^\bullet)$ by using Breuil-Kisin log prisms and Nygaard filtrations (Proposition 5.6.23).

For a log prism $(A, I, \mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}$, we set

$$\begin{aligned}\mathcal{E}[1/p]_A &:= \Gamma(\mathrm{Spec}(A[1/p]), \mathcal{E}_A), \\ (\phi^* \mathcal{E})[1/p]_A &:= \Gamma(\mathrm{Spec}(A[1/p]), \phi_A^* \mathcal{E}_A), \\ \overline{\mathcal{E}[1/p]}_A &:= \mathcal{E}_A[1/p] \otimes_{A[1/p]} A/I[1/p], \\ \overline{(\phi^* \mathcal{E})[1/p]}_A &:= (\phi^* \mathcal{E})[1/p]_A \otimes_{A[1/p]} A/I[1/p].\end{aligned}$$

The map $\varphi_{\mathcal{E}} : (\phi^* \mathcal{E})[1/\mathcal{I}_{\Delta}] \xrightarrow{\sim} \mathcal{E}[1/\mathcal{I}_{\Delta}]$ induces a map

$$\varphi_{\mathcal{E}_A} : (\phi^* \mathcal{E})_A[1/pI] \xrightarrow{\sim} \mathcal{E}_A[1/pI]$$

of $A[1/pI]$ -modules. We define a Nygaard filtration $\mathrm{Fil}_{\mathcal{N}, A}^{\bullet}$ on $\overline{(\phi^* \mathcal{E})[1/p]}_A$ by

$$\mathrm{Fil}_{\mathcal{N}, A}^n := \mathrm{Im}(\varphi_{\mathcal{E}_A}^{-1}(I^n \mathcal{E}_A[1/p]) \cap (\phi^* \mathcal{E})[1/p]_A \rightarrow \overline{(\phi^* \mathcal{E})[1/p]}_A).$$

Lemma 5.6.21. There exists a natural isomorphism of $R[1/p]$ -modules

$$T_{\mathrm{isoc}}(\mathcal{E})_{(X_{p=0}, \mathfrak{X}, \mathcal{M}_{\mathfrak{X}})} \cong \overline{(\phi^* \mathcal{E})[1/p]}_{\mathfrak{S}_{R,t}}.$$

Proof. The crystal property of $T_{\mathrm{isoc}}(\mathcal{E})$ for the map

$$(S_{R,t} \twoheadrightarrow R/p, \mathcal{M}_{S_{R,t}}) \rightarrow (R \twoheadrightarrow R/p, \mathcal{M}_R)$$

in $(X_{p=0}, \mathcal{M}_{X_{p=0}})_{\mathrm{cryst}}^{\mathrm{str}}$ gives an isomorphism of $R[1/p]$ -modules

$$T_{\mathrm{isoc}}(\mathcal{E})_{(X_{p=0}, \mathfrak{X}, \mathcal{M}_{\mathfrak{X}})} \cong T_{\mathrm{isoc}}(\mathcal{E})_{(S_{R,t} \twoheadrightarrow R/p, \mathcal{M}_{S_{R,t}})} \otimes_{S_{R,t}[1/p]} R[1/p].$$

By the definition of $T_{\mathrm{isoc}}(\mathcal{E})$ and the crystal property of $\mathcal{E}$ of the map

$$\phi : (\mathfrak{S}_{R,t}, (E), \mathcal{M}_{\mathfrak{S}_{R,t}}) \rightarrow (S_{R,t}, (p), \phi_{S_{R,t}}^* \mathcal{M}_{S_{R,t}})$$

in $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}$, we have isomorphisms of $R[1/p]$ -modules

$$\begin{aligned}& T_{\mathrm{isoc}}(\mathcal{E})_{(S_{R,t} \twoheadrightarrow R/p, \mathcal{M}_{S_{R,t}})} \otimes_{S_{R,t}[1/p]} R[1/p] \\ & \cong (\Gamma(\mathrm{Spec}(\mathfrak{S}_{R,t}[1/p]), \mathcal{E}_{\mathfrak{S}_{R,t}}) \otimes_{\mathfrak{S}_{R,t}[1/p], \phi} S_{R,t}[1/p]) \otimes_{S_{R,t}[1/p]} R[1/p] \\ & \cong \Gamma(\mathrm{Spec}(\mathfrak{S}_{R,t}[1/p]), \phi_{\mathfrak{S}_{R,t}}^* \mathcal{E}_{\mathfrak{S}_{R,t}}) \otimes_{\mathfrak{S}_{R,t}[1/p]} R[1/p] \\ & \cong \overline{(\phi^* \mathcal{E})[1/p]}_{\mathfrak{S}_{R,t}}.\end{aligned}$$

The combination of these isomorphisms give the desired isomorphism. $\square$

Lemma 5.6.22. There exists a p -adic saturated log formal scheme $(\mathrm{Spf}(S), \mathcal{M}_S)$ over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ satisfying the following conditions:

- S is an integral perfectoid $\mathcal{O}_C$ -algebra, and $R \rightarrow S$ is faithfully flat;
- $\mathbb{L}|_{(\mathrm{Spf}(S), \mathcal{M}_S)_{\eta}}$ is constant for every $\mathbb{Z}_p$ -local system $\mathbb{L}$ on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \mathrm{k\acute{e}t}}$;
- the structure map $(\mathrm{Spf}(S), \mathcal{M}_S) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ factors as

$$(\mathrm{Spf}(S), \mathcal{M}_S) \rightarrow (\mathrm{Spf}(\tilde{R}_{\infty, \alpha}), \mathcal{M}_{\tilde{R}_{\infty, \alpha}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}),$$

where $(\mathrm{Spf}(S), \mathcal{M}_S) \rightarrow (\mathrm{Spf}(\tilde{R}_{\infty, \alpha}), \mathcal{M}_{\tilde{R}_{\infty, \alpha}})$ is a strict map over $\mathcal{O}_C$.

Proof. We may assume that $\mathfrak{X}$ is connected. Let S denote the p -adic completion of the integral closure of R in an algebraic closure of $\mathrm{Frac}(R)$. The ring S is an integral perfectoid $\mathcal{O}_C$ -algebra by [Bha20, Lemma 4.20 (2)], and $R \rightarrow S$ is faithfully flat by [Bha20, Theorem 5.16]. Since the ring map $R \rightarrow S$ admits a factorization

$$R \rightarrow \tilde{R}_{\infty, \alpha} \rightarrow S,$$

we fix such a factorization and let $\mathcal{M}_S$ denote the pullback log structure of $\mathcal{M}_{\tilde{R}_{\infty,\alpha}}$.

It suffices to prove that $(\mathrm{Spf}(S), \mathcal{M}_S)$ satisfies the second condition in the statement. Let $\mathbb{L}$ be a $\mathbb{Z}_p$ -local system on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \text{két}}$ and let $\mathbb{L}_n$ denote the $\mathbb{Z}/p^n$ -local system $\mathbb{L} \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n$ for $n \geq 1$. Since $\mathrm{GL}_s(\mathbb{Z}/p^{n+1}) \rightarrow \mathrm{GL}_s(\mathbb{Z}/p^n)$ is surjective for $s := \mathrm{rk}(\mathbb{L})$ and each $n \geq 1$, it is enough to show that $\mathbb{L}_n|_{(\mathrm{Spf}(S), \mathcal{M}_S)_\eta}$ is constant for each $n \geq 1$.

Take a finite Kummer étale cover $(Y_n, \mathcal{M}_{Y_n}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_\eta$ such that $\mathbb{L}_n|_{(Y_n, \mathcal{M}_{Y_n})}$ is constant. By [DLLZ23b, Lemma 4.2.5], we may refine $(Y_n, \mathcal{M}_{Y_n})$ and assume that $(Y_n, \mathcal{M}_{Y_n}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_\eta$ factors as

$$(Y_n, \mathcal{M}_{Y_n}) \rightarrow (\mathfrak{X}_{m,\alpha}, \mathcal{M}_{\mathfrak{X}_{m,\alpha}})_\eta \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_\eta$$

for some $m \geq 1$, where $(Y_n, \mathcal{M}_{Y_n}) \rightarrow (\mathfrak{X}_{m,\alpha}, \mathcal{M}_{\mathfrak{X}_{m,\alpha}})_\eta$ is strict finite étale. Here, $(\mathfrak{X}_{m,\alpha}, \mathcal{M}_{\mathfrak{X}_{m,\alpha}})$ is defined by

$$(\mathfrak{X}_{m,\alpha}, \mathcal{M}_{\mathfrak{X}_{m,\alpha}}) := (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \otimes_{(\mathbb{Z}[\mathbb{N}^r], \mathbb{N}^r)^a} (\mathbb{Z}[\frac{1}{m}\mathbb{N}^r], \frac{1}{m}\mathbb{N}^r)^a.$$

Then $(\mathrm{Spf}(S), \mathcal{M}_S) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ factors as

$$(\mathrm{Spf}(S), \mathcal{M}_S) \rightarrow (\mathrm{Spf}(\tilde{R}_{\infty,\alpha}), \mathcal{M}_{\tilde{R}_{\infty,\alpha}}) \rightarrow (\mathfrak{X}_{m,\alpha}, \mathcal{M}_{\mathfrak{X}_{m,\alpha}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}),$$

and, when we fix such a factorization, $(\mathrm{Spf}(S), \mathcal{M}_S)_\eta \rightarrow (\mathfrak{X}_{m,\alpha}, \mathcal{M}_{\mathfrak{X}_{m,\alpha}})_\eta$ factors as

$$(\mathrm{Spf}(S), \mathcal{M}_S)_\eta \rightarrow (Y_n, \mathcal{M}_{Y_n}) \rightarrow (\mathfrak{X}_{m,\alpha}, \mathcal{M}_{\mathfrak{X}_{m,\alpha}})_\eta$$

by the same argument as the forth paragraph of the proof of [GR24, Lemma 4.10]. Therefore, $\mathbb{L}_n|_{(\mathrm{Spf}(S), \mathcal{M}_S)_\eta}$ is constant. $\square$

Proposition 5.6.23. The isomorphism in Lemma 5.6.21 is refined to an isomorphism of filtered vector bundles on $\mathfrak{X}_\eta$

$$T_{\mathrm{dR}}(\mathcal{E}) \cong (\overline{(\phi^*\mathcal{E})[1/p]}_{\mathfrak{S}_{R,t}}, \mathrm{Fil}_{\mathcal{N}, \mathfrak{S}_{R,t}}^\bullet).$$

In particular, the filtration $\mathrm{Fil}_{\mathcal{N}, \mathfrak{S}_{R,t}}^\bullet$ on $\overline{(\phi^*\mathcal{E})[1/p]}_{\mathfrak{S}_{R,t}}$ is a filtration with locally free graded pieces.

Proof. Take a sequence of p -adic saturated log formal schemes

$$(\mathrm{Spf}(S), \mathcal{M}_S) \rightarrow (\mathrm{Spf}(\tilde{R}_{\infty,\alpha}), \mathcal{M}_{\tilde{R}_{\infty,\alpha}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$$

as in Lemma 5.6.22. Since $R \rightarrow S$ is faithfully flat, it is enough to show that the isomorphism in Lemma 5.6.21 is a filtered isomorphism after taking the base change along $R[1/p] \rightarrow S[1/p]$.

We prepare some categories and functors as follows:

- $\mathrm{Mod}^\varphi(\mathfrak{S}_R[1/p])$ denote the category of a pair (M, φ_M) consisting of a finite projective $\mathfrak{S}_R[1/p]$ -module M with an isomorphism $\varphi_M: (\phi_{\mathfrak{S}_R}^* M)[1/E] \xrightarrow{\sim} M[1/E]$;
- $\mathrm{Mor}(B_{\mathrm{dR}}^+(S))$ denote the category of a triple (M_1, M_2, f) consisting of finite projective $B_{\mathrm{dR}}^+(S)$ -modules M_1 and M_2 with a $B_{\mathrm{dR}}(S)$ -linear isomorphism

$$M_1 \otimes_{B_{\mathrm{dR}}^+(S)} B_{\mathrm{dR}}(S) \xrightarrow{\sim} M_2 \otimes_{B_{\mathrm{dR}}^+(S)} B_{\mathrm{dR}}(S);$$

- the functor $\mathrm{Mod}^\varphi(\mathfrak{S}_R[1/p]) \rightarrow \mathrm{Mor}(B_{\mathrm{dR}}^+(S))$ given by

$$(M, \varphi_M) \mapsto (M \otimes_{\mathfrak{S}_{R,\phi}} B_{\mathrm{dR}}^+(S), M \otimes_{\mathfrak{S}_R} B_{\mathrm{dR}}^+(S), \varphi_M \otimes \mathrm{id}),$$

where $M \otimes_{\mathfrak{S}_R, \phi} B_{\mathrm{dR}}^+(S)$ (resp. $M \otimes_{\mathfrak{S}_R} B_{\mathrm{dR}}^+(S)$) is the base change of M along the ring map

$$\mathfrak{S}_R \xrightarrow{\phi} \mathfrak{S}_R \rightarrow A_{\mathrm{inf}}(S) \rightarrow B_{\mathrm{dR}}(S) \quad (\text{resp. } \mathfrak{S}_R \rightarrow A_{\mathrm{inf}}(S) \rightarrow B_{\mathrm{dR}}(S))$$

(for the definition of $\mathfrak{S}_R \rightarrow A_{\mathrm{inf}}(S)$, see the last paragraph of Construction 5.5.16);

- for a ring A , $\mathrm{FilMod}(A)$ denote the category of a finite projective A -module with a descending filtration;
- the functor $\mathrm{Mod}^\varphi(\mathfrak{S}_R[1/p]) \rightarrow \mathrm{FilMod}(R[1/p])$ sending (M, φ_M) to the $R[1/p]$ -module $M \otimes_{\mathfrak{S}_R[1/p]} R[1/p]$ equipped with the filtration defined as

$$\mathrm{Im}(\phi_{\mathfrak{S}_R}^* M \cap \varphi_M^{-1}(E^\bullet M) \rightarrow M \otimes_{\mathfrak{S}_R[1/p]} R[1/p]);$$

- the functor $\mathrm{Mor}(B_{\mathrm{dR}}^+(S)) \rightarrow \mathrm{FilMod}(S[1/p])$ sending (M_1, M_2, f) to the $B_{\mathrm{dR}}^+(S)$ -module $M_1 \otimes_{B_{\mathrm{dR}}^+(S)} S[1/p]$ equipped with the filtration defined as

$$\mathrm{Im}(M_1 \cap f^{-1}(\xi^\bullet M_2) \rightarrow M_1 \otimes_{B_{\mathrm{dR}}^+(S)} S[1/p]).$$

Then, by flatness of the ring map $\mathfrak{S}_R \rightarrow B_{\mathrm{dR}}^+(S)$, the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Mod}^\varphi(\mathfrak{S}_R[1/p]) & \longrightarrow & \mathrm{FilMod}(R[1/p]) \\ \downarrow & & \downarrow -\otimes_{R[1/p]} S[1/p] \\ \mathrm{Mor}(B_{\mathrm{dR}}^+(S)) & \longrightarrow & \mathrm{FilMod}(S[1/p]). \end{array}$$

Let

$$\varphi_{B_{\mathrm{dR}}(S)}: \mathcal{E}_{\mathfrak{S}_R} \otimes_{\mathfrak{S}_R, \phi} B_{\mathrm{dR}}(S) \xrightarrow{\sim} \mathcal{E}_{\mathfrak{S}_R} \otimes_{\mathfrak{S}_R} B_{\mathrm{dR}}(S)$$

be the image of $\mathcal{E}[1/p]_{\mathfrak{S}_R} \in \mathrm{Mod}^\varphi(\mathfrak{S}_R[1/p])$ via the left vertical functor. We define a filtration $\mathrm{Fil}_{\mathcal{N}, B_{\mathrm{dR}}(S)}^\bullet$ on $\mathcal{E}_{\mathfrak{S}_R} \otimes_{\mathfrak{S}_R, \phi} B_{\mathrm{dR}}^+(S)$ by

$$\mathrm{Fil}_{\mathcal{N}, B_{\mathrm{dR}}(S)}^\bullet := (\mathcal{E}_{\mathfrak{S}_R} \otimes_{\mathfrak{S}_R, \phi} B_{\mathrm{dR}}^+(S)) \cap \varphi_{B_{\mathrm{dR}}(S)}^{-1}(\xi^\bullet \cdot (\mathcal{E}_{\mathfrak{S}_R} \otimes_{\mathfrak{S}_R, \phi} B_{\mathrm{dR}}^+(S))).$$

Let $\overline{\mathrm{Fil}}_{\mathcal{N}, B_{\mathrm{dR}}(S)}^\bullet$ denote the filtration on $\mathcal{E}_{\mathfrak{S}_R} \otimes_{\mathfrak{S}_R, \phi} B_{\mathrm{dR}}^+(S) \otimes_{B_{\mathrm{dR}}^+(S)} S[1/p]$ defined as the image of $\mathrm{Fil}_{\mathcal{N}, B_{\mathrm{dR}}(S)}^\bullet$ via the surjection

$$\mathcal{E}_{\mathfrak{S}_R} \otimes_{\mathfrak{S}_R, \phi} B_{\mathrm{dR}}^+(S) \twoheadrightarrow \mathcal{E}_{\mathfrak{S}_R} \otimes_{\mathfrak{S}_R, \phi} B_{\mathrm{dR}}^+(S) \otimes_{B_{\mathrm{dR}}^+(S)} S[1/p].$$

Applying the commutativity of the above diagram to $\mathcal{E}[1/p]_{\mathfrak{S}_R} \in \mathrm{Mod}^\varphi(\mathfrak{S}_R[1/p])$ gives an isomorphism

$$(5.6.1) \quad (\overline{(\phi^* \mathcal{E})[1/p]_{\mathfrak{S}_R}}, \mathrm{Fil}_{\mathcal{N}, \mathfrak{S}_R}^\bullet) \otimes_R S \cong (\mathcal{E}_{A_{\mathrm{inf}}(S)} \otimes_{A_{\mathrm{inf}}(S), \phi} B_{\mathrm{dR}}^+(S) \otimes_{B_{\mathrm{dR}}^+(S)} S, \overline{\mathrm{Fil}}_{\mathcal{N}, B_{\mathrm{dR}}(S)}^\bullet).$$

By Lemma 5.6.20, we have the identity of filtrations on $\mathcal{E}_{A_{\mathrm{inf}}(S)} \otimes_{A_{\mathrm{inf}}(S), \phi} B_{\mathrm{dR}}^+(S)$

$$\mathrm{Fil}_{\mathcal{N}, B_{\mathrm{dR}}(S)}^\bullet = (\mathcal{E}_{A_{\mathrm{inf}}(S)} \otimes_{A_{\mathrm{inf}}(S), \phi} B_{\mathrm{dR}}^+(S)) \cap \mathrm{Fil}_{\mathrm{dR}, B_{\mathrm{dR}}(S)}^\bullet.$$

By the definition of $\mathrm{Fil}_{\mathrm{dR}}^\bullet$ and the argument in Construction 5.4.17, when we choose $s: (R, \mathcal{M}_R) \rightarrow (A_{\mathrm{crys}, K}(S), \mathcal{M}_{A_{\mathrm{crys}, K}(S)})$ as in *loc. cit.*, there is an isomorphism

$$\mathcal{E}_{A_{\mathrm{inf}}(S)} \otimes_{A_{\mathrm{inf}}(S), \phi} B_{\mathrm{dR}}^+(S) \cong T_{\mathrm{isoc}}(\mathcal{E})_{(X_{p=0}, \mathfrak{X}, \mathcal{M}_{\mathfrak{X}})} \otimes_{R[1/p], s} B_{\mathrm{dR}}^+(S)$$

such that the induced isomorphism

$$\mathcal{E}_{A_{\mathrm{inf}}(S)} \otimes_{A_{\mathrm{inf}}(S), \phi} B_{\mathrm{dR}}(S) \cong T_{\mathrm{isoc}}(\mathcal{E})_{(X_{p=0}, \mathfrak{X}, \mathcal{M}_{\mathfrak{X}})} \otimes_{R[1/p], s} B_{\mathrm{dR}}(S)$$

sends the filtration on the left hand side $\mathrm{Fil}_{\mathrm{dR}, B_{\mathrm{dR}}(S)}^\bullet$ to the product filtration $\mathrm{Fil}_{\mathrm{dR}}^\bullet \otimes_{R[1/p], s} \mathrm{Fil}^\bullet B_{\mathrm{dR}}(S)$ on the right hand side. Therefore, we have filtered isomorphisms

$$\begin{aligned} & (\mathcal{E}_{\mathfrak{S}_R} \otimes_{\mathfrak{S}_{R, \phi}} B_{\mathrm{dR}}^+(S), \mathrm{Fil}_{\mathcal{N}, B_{\mathrm{dR}}(S)}^\bullet) \\ &= (\mathcal{E}_{\mathfrak{S}_R} \otimes_{\mathfrak{S}_{R, \phi}} B_{\mathrm{dR}}^+(S), (\mathcal{E}_{\mathfrak{S}_R} \otimes_{\mathfrak{S}_{R, \phi}} B_{\mathrm{dR}}^+(S)) \cap \mathrm{Fil}_{\mathrm{dR}, B_{\mathrm{dR}}(S)}^\bullet) \\ &\cong (T_{\mathrm{isoc}}(\mathcal{E})_{(R \rightarrow R/p, \mathcal{M}_R)} \otimes_{R[1/p]} B_{\mathrm{dR}}^+(S), \bigoplus_{n \geq 0} (\mathrm{Fil}_{\mathrm{dR}}^{\bullet-n} \otimes_{R[1/p]} \mathrm{Fil}^n B_{\mathrm{dR}}(S))). \end{aligned}$$

Taking the mod- ξ reductions with induced filtrations, we get a filtered isomorphism

$$(5.6.2) \quad (\mathcal{E}_{\mathfrak{S}_R} \otimes_{\mathfrak{S}_{R, \phi}} B_{\mathrm{dR}}^+(S) \otimes_{B_{\mathrm{dR}}^+(S)} S[1/p], \overline{\mathrm{Fil}}_{\mathcal{N}, B_{\mathrm{dR}}^+(S)}^\bullet) \cong T_{\mathrm{dR}}(\mathcal{E}) \otimes_{R[1/p]} S[1/p].$$

Then the combination of (5.6.1) and (5.6.2) proves the assertion. $\square$

5.6.5. The proof of fully faithfulness part of Theorem 5.1.4.

Lemma 5.6.24. Let $(\mathrm{Spf}(S), \mathcal{M}_S)$ be a p -adic saturated log formal scheme. Suppose that S is a p -torsion free qrsp ring, and we are given a chart $P \rightarrow \mathcal{M}_S$ with P being and uniquely n -divisible for every $n \geq 1$ and saturated. Then the étale realization functor

$$T_{\mathrm{ét}} : \mathrm{Vect}^{\mathrm{an}, \varphi}((\mathrm{Spf}(S), \mathcal{M}_S)_{\Delta}) \rightarrow \mathrm{Loc}_{\mathbb{Z}_p}((\mathrm{Spf}(S), \mathcal{M}_S)_{\eta, \mathrm{qprokét}}^\diamond)$$

is faithful.

Proof. We see that the étale realization functor

$$T_{\mathrm{ét}} : \mathrm{Vect}^{\mathrm{an}, \varphi}(S_{\Delta}) \rightarrow \mathrm{Loc}_{\mathbb{Z}_p}(\mathrm{Spf}(S)_{\eta, \mathrm{proét}})$$

is faithful by the same argument as the proof of [GR24, Lemma 4.1]. By Lemma 5.5.13, the source category is equivalent to $\mathrm{Vect}^{\mathrm{an}, \varphi}((\mathrm{Spf}(S), \mathcal{M}_S)_{\Delta})$. On the other hand, Lemma 5.B.1 gives an equivalence

$$\mathrm{Loc}_{\mathbb{Z}_p}((\mathrm{Spf}(S), \mathcal{M}_S)_{\eta, \mathrm{qprokét}}^\diamond) \simeq \mathrm{Loc}_{\mathbb{Z}_p}(\mathrm{Spf}(S)_{\eta, \mathrm{qproét}}^\diamond) \simeq \mathrm{Loc}_{\mathbb{Z}_p}(\mathrm{Spf}(S)_{\eta, \mathrm{proét}}).$$

This proves the assertion. $\square$

Construction 5.6.25 ([GR24, Construction 4.12]). Let S be a p -torsion free integral perfectoid $\mathcal{O}_C$ -algebra and set $\mathfrak{X} := \mathrm{Spf}(S)$. The *lattice realization functor* is a functor

$$\begin{aligned} \mathrm{Vect}^{\mathrm{an}, \varphi}(A_{\mathrm{inf}}(S), (\xi)) &\rightarrow \mathbb{B}_{\mathrm{dR}}^+ \text{-Latt} \\ &:= \{(\mathbb{L}, M) \mid \mathbb{L} \in \mathrm{Loc}_{\mathbb{Z}_p}(\mathfrak{X}_{\eta, \mathrm{proét}}), M(\subset \mathbb{L} \otimes_{\mathbb{Z}_p} \mathbb{B}_{\mathrm{dR}}) : \mathbb{B}_{\mathrm{dR}}^+ \text{-lattice}\} \end{aligned}$$

given on an object $(\mathcal{E}, \varphi_{\mathcal{E}}) \in \mathrm{Vect}^{\mathrm{an}, \varphi}(A_{\mathrm{inf}}(S), (\xi))$ as follows:

- $\mathbb{L}$ is the étale realization of $(\mathcal{E}, \varphi_{\mathcal{E}})$.
- Taking the base change of the isomorphism in Proposition 5.6.16 along

$$\mathbb{A}_{\mathrm{inf}}[1/\phi^{-1}(\mu)] \xrightarrow{\phi} \mathbb{A}_{\mathrm{inf}}[1/\mu] \rightarrow \mathbb{B}_{\mathrm{dR}},$$

we get an isomorphism

$$\phi_{\mathbb{B}_{\mathrm{inf}}}^* \mathbb{B}_{\mathrm{inf}}(\mathcal{E}) \otimes_{\mathbb{B}_{\mathrm{inf}}} \mathbb{B}_{\mathrm{dR}} \cong \mathbb{L} \otimes_{\mathbb{Z}_p} \mathbb{B}_{\mathrm{dR}}.$$

and a $\mathbb{B}_{\mathrm{dR}}^+$ -lattice $M := \phi_{\mathbb{B}_{\mathrm{inf}}}^* \mathbb{B}_{\mathrm{inf}}(\mathcal{E}) \otimes_{\mathbb{B}_{\mathrm{inf}}} \mathbb{B}_{\mathrm{dR}}^+$ in $\mathbb{L} \otimes_{\mathbb{Z}_p} \mathbb{B}_{\mathrm{dR}}$.

Lemma 5.6.26 ([GR24, Lemma 4.3 and Proposition 4.4]). Let S be a p -torsion free perfectoid $\mathcal{O}_C$ -algebra. Suppose that S/π is a free $\mathcal{O}_C/\pi$ -module. Then the following assertions hold.

(1) We have the following formulas.

$$\begin{aligned} \bigcap_{r \geq 1} \frac{\mu}{\phi^{-r}(\mu)} \Delta_S &= \mu \Delta_S \\ \bigcap_{r \geq 1} \left(\frac{\mu}{\phi^{-r}(\mu)} \Delta_S[1/p] \right) &= \mu \Delta_S[1/p] \\ \bigcap_{r \geq 1} \left(\frac{\mu}{\phi^{-r}(\mu)} \Delta_S[1/\xi] \right) &= \mu \Delta_S[1/\xi] \end{aligned}$$

(2) The lattice realization functor in Construction 5.6.25 is fully faithful.

Proof. (1) In [GR24, Lemma 4.3], this assertion is proved for a specified integral perfectoid ring S . The argument in *loc. cit.* works in this setting as well.

(2) This follows from the same argument as [GR24, Proposition 4.4]. We can use (1) as a key input. $\square$

Theorem 5.6.27. For a semi-stable log formal scheme $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ over $\mathcal{O}_K$, the étale realization functor

$$T_{\text{ét}}: \text{Vect}^{\text{an}, \varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \text{Loc}_{\mathbb{Z}_p}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta})$$

is fully faithful.

Proof. By working étale locally on $\mathfrak{X}$, we may assume that $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) = (\text{Spf}(R), \mathcal{M}_R)$ is small affine. Fix a framing $(\text{Spf}(R), \mathcal{M}_R) \rightarrow (\text{Spf}(R^0), \mathbb{N}^r)^a$. By Proposition 6.4.17 below and Lemma 5.6.24, it is enough to prove that $T_{\text{ét}}$ is full. Let $\mathcal{E}_1, \mathcal{E}_2 \in \text{Vect}^{\text{an}, \varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ and $f: T_{\text{ét}}(\mathcal{E}_1) \rightarrow T_{\text{ét}}(\mathcal{E}_2)$ be a morphism of $\mathbb{Z}_p$ -local systems.

First, suppose that $n \leq 1$ (in other words, the special fiber of $\mathfrak{X}$ is smooth). In this case, by isomorphisms

$$\begin{aligned} \tilde{R}/\pi &\cong (R \otimes_{\mathcal{O}_K \langle \{x_i^{\pm 1}\}, \{y_j\} \rangle} \mathcal{O}_K \langle \{x_i^{\pm 1/p^\infty}\}, \{y_j^{\mathbb{Q}_{\geq 0}}\} \rangle \otimes_{\mathcal{O}_K} \mathcal{O}_C) / \pi \\ &\cong (R/\pi \otimes_{k[\{x_i^{\pm 1}\}, \{y_j\}]} k[\{x_i^{\pm 1/p^\infty}\}, \{y_j^{\mathbb{Q}_{\geq 0}}\}]) \otimes_k \mathcal{O}_C / \pi, \end{aligned}$$

$\tilde{R}/\pi$ is a free $\mathcal{O}_C/\pi$ -module. Hence we can use Lemma 5.6.26 and obtain the following diagram.

$$\begin{array}{ccc} \text{Vect}^{\text{an}, \varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) & \xrightarrow{\mathcal{E} \mapsto (T_{\text{ét}}(\mathcal{E}), \mathbb{B}_{\text{dR}}^+(T_{\text{crys}}(\mathcal{E})))} & \mathbb{B}_{\text{dR}}(\tilde{R})\text{-Latt} \\ \downarrow & & \parallel \\ \text{Vect}^{\text{an}, \varphi}(\tilde{R}_{\Delta}) & \xrightarrow{\sim} \text{Vect}^{\text{an}, \varphi}(\Delta_{\tilde{R}}, I) \xrightarrow{\sim} & \mathbb{B}_{\text{dR}}(\tilde{R})\text{-Latt} \end{array}$$

By Proposition 5.4.22, there exists a unique map $T_{\text{crys}}(\mathcal{E}_1) \rightarrow T_{\text{crys}}(\mathcal{E}_2)$ of isocrystals which induces the map

$$\mathbb{B}_{\text{crys}}(T_{\text{crys}}(\mathcal{E}_1)) \cong T_{\text{ét}}(\mathcal{E}_1) \otimes_{\mathbb{Z}_p} \mathbb{B}_{\text{crys}} \xrightarrow{f \otimes \text{id}} T_{\text{ét}}(\mathcal{E}_2) \otimes_{\mathbb{Z}_p} \mathbb{B}_{\text{crys}} \cong \mathbb{B}_{\text{crys}}(T_{\text{crys}}(\mathcal{E}_2)),$$

and so $f \otimes \text{id}_{\mathbb{B}_{\text{dR}}}: T_{\text{ét}}(\mathcal{E}_1) \otimes_{\mathbb{Z}_p} \mathbb{B}_{\text{dR}} \rightarrow T_{\text{ét}}(\mathcal{E}_2) \otimes_{\mathbb{Z}_p} \mathbb{B}_{\text{dR}}$ maps the $\mathbb{B}_{\text{dR}}^+(T_{\text{crys}}(\mathcal{E}_1))$ into $\mathbb{B}_{\text{dR}}^+(T_{\text{crys}}(\mathcal{E}_2))$. Hence, the above diagram implies that there exists a unique morphism $g: \mathcal{E}_1|_{\text{Spf}(\tilde{R})} \rightarrow \mathcal{E}_2|_{\text{Spf}(\tilde{R})}$ which induces $f|_{\text{Spf}(\tilde{R})_{\eta}}$ via étale realization. Lemma 5.6.24 implies that $\text{pr}_1^*(g) = \text{pr}_2^*(g)$, and so g descends to a morphism $\mathcal{E}_1 \rightarrow \mathcal{E}_2$ by Proposition 6.4.17 below.

We shall prove the assertion in general cases by induction on n . What we proved in the previous paragraph allows us to assume that $n \geq 2$. Let $\mathfrak{U}_i$ be the vanishing locus of $x_i \in R$ for $i = 1, 2$ and $\mathfrak{U}_3 := \mathfrak{U}_1 \cap \mathfrak{U}_2$. Let $\mathcal{M}_{\mathfrak{U}_i}$ be log structures obtained as restrictions of $\mathcal{M}_{\mathfrak{X}}$ for $i = 1, 2, 3$. By the induction hypothesis, we have morphisms $\mathcal{E}_1|_{(\mathfrak{U}_i, \mathcal{M}_{\mathfrak{U}_i})} \rightarrow \mathcal{E}_2|_{(\mathfrak{U}_i, \mathcal{M}_{\mathfrak{U}_i})}$ which induces f via the étale realization for $i = 1, 2$, and these morphisms coincide on $(\mathfrak{U}_3, \mathcal{M}_{\mathfrak{U}_3})$ by the faithfulness of the étale realization. Hence, it suffices to prove that the restriction functor

$$\begin{aligned} & \text{Vect}^{\text{an}, \varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \\ & \rightarrow \text{Vect}^{\text{an}, \varphi}((\mathfrak{U}_1, \mathcal{M}_{\mathfrak{U}_1})_{\Delta}) \times_{\text{Vect}^{\text{an}, \varphi}((\mathfrak{U}_3, \mathcal{M}_{\mathfrak{U}_3})_{\Delta})} \text{Vect}^{\text{an}, \varphi}((\mathfrak{U}_2, \mathcal{M}_{\mathfrak{U}_2})_{\Delta}) \end{aligned}$$

is fully faithful.

Let $x_3 := x_1 x_2$. A log prism $(\mathfrak{S}_R[1/x_i]_{(p, E)}^{\wedge}, (E), \mathcal{M}_{\mathfrak{S}_R[1/x_i]_{(p, E)}^{\wedge}}) \in (\mathfrak{U}_i, \mathcal{M}_{\mathfrak{U}_i})_{\Delta}$ covers the final object of the associated topos, and the $n + 1$ -fold product of this log prism in $(\mathfrak{U}_i, \mathcal{M}_{\mathfrak{U}_i})_{\Delta}^{\text{str}}$ coincides with

$$(\mathfrak{S}_R^{(n)}[1/x_i]_{(p, E)}^{\wedge}, (E), \mathcal{M}_{\mathfrak{S}_R^{(n)}[1/x_i]_{(p, E)}^{\wedge}}).$$

It suffices to show that

$$\begin{aligned} & \text{Vect}^{\text{an}, \varphi}(\mathfrak{S}_R^{(n)}, (E)) \\ & \rightarrow \text{Vect}^{\text{an}, \varphi}(\mathfrak{S}_R^{(n)}[1/x_1]^{\wedge}, (E)) \times_{\text{Vect}^{\text{an}, \varphi}(\mathfrak{S}_R^{(n)}[1/x_3]^{\wedge}, (E))} \text{Vect}^{\text{an}, \varphi}(\mathfrak{S}_R^{(n)}[1/x_2]^{\wedge}, (E)) \end{aligned}$$

is fully faithful for $n \geq 0$. Since $\text{Spf}(A) \setminus V(p, d)$ has a covering $\text{Spec}(A[1/p]) \cap \text{Spec}(A[1/d])$ with an intersection $\text{Spec}(A[1/pd])$ for a prism $(A, (d))$, it is enough to prove that

$$\begin{aligned} & \text{Vect}(\mathfrak{S}_R^{(n)}[1/a]) \\ & \rightarrow \text{Vect}(\mathfrak{S}_R^{(n)}[1/x_1]^{\wedge}[1/a]) \times_{\text{Vect}(\mathfrak{S}_R^{(n)}[1/x_3]^{\wedge}[1/a])} \text{Vect}(\mathfrak{S}_R^{(n)}[1/x_2]^{\wedge}[1/a]) \end{aligned}$$

is fully faithful for $a \in \{p, E, pE\}$. Taking internal homomorphisms, we are reduced to show that

$$0 \rightarrow \mathfrak{S}_R^{(n)} \rightarrow \mathfrak{S}_R^{(n)}[1/x_1]^{\wedge} \times \mathfrak{S}_R^{(n)}[1/x_2]^{\wedge} \rightarrow \mathfrak{S}_R^{(n)}[1/x_3]^{\wedge} \quad (1)$$

is exact. We have an exact sequence

$$0 \rightarrow R \rightarrow R[1/x_1]_p^{\wedge} \times R[1/x_2]_p^{\wedge} \rightarrow R[1/x_3]_p^{\wedge} \quad (2)$$

because the mod- π reductions of this is obtained by the base change of an exact sequence

$$0 \rightarrow R_0/\pi \rightarrow (R_0/\pi)[1/x_1] \times (R_0/\pi)[1/x_2] \rightarrow (R_0/\pi)[1/x_3]$$

(the exactness of which can be checked by a direct computation) along the étale map $R_0/\pi \rightarrow R/\pi$. Therefore, the sequence (1) is exact because the mod- E reduction of the sequence (2) is obtained by the base change of the sequence (1) along the p -completely flat map $R \rightarrow \mathfrak{S}_R^{(n)}/(E)$. $\square$

Corollary 5.6.28. For a semi-stable log formal scheme $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ over $\mathcal{O}_K$, the étale realization functor

$$T_{\text{ét}} : \text{Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \text{Loc}_{\mathbb{Z}_p}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta})$$

is fully faithful.

Proof. This follows from Lemma 5.5.17 and Theorem 5.6.27. $\square$

5.6.6. The proof of bi-exactness part of Theorem 5.1.4.

Setting 5.6.29. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) = (\mathrm{Spf}(R), \mathcal{M}_R)$ be a small affine log formal scheme over $\mathcal{O}_K$ with a fixed framing. Let $\mathcal{E}$ be an analytic log prismatic F -crystal on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Let $(\mathrm{Spf}(S), \mathcal{M}_S)$ be a p -adic saturated log formal scheme over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ as in Lemma 5.6.22.

We simply write $T_{\mathrm{\acute{e}t}}(\mathcal{E})|_{(\mathrm{Spf}(S), \mathcal{M}_S)_\eta}$ for the $\mathbb{Z}_p$ -module corresponding to the constant $\mathbb{Z}_p$ -local system $T_{\mathrm{\acute{e}t}}(\mathcal{E})|_{(\mathrm{Spf}(S), \mathcal{M}_S)_\eta}$.

We consider the following rings:

$$\begin{aligned} B_{[0,1]} &:= A_{\mathrm{inf}}(S)[p/[p^b]]_{[p^b]}^\wedge [1/[p^b]] \\ B_{[1,\infty]} &:= A_{\mathrm{inf}}(S)[[p^b]/p]_p^\wedge [1/p] \\ B_{[1,1]} &:= A_{\mathrm{inf}}(S)[[p^b]/p, p/[p^b]]_p^\wedge [1/p[p^b]]. \end{aligned}$$

Lemma 5.6.30. ([Ked20, Theorem 3.8]) The functor

$$\mathrm{Vect}(U(A_{\mathrm{inf}}(S)), (\xi)) \rightarrow \mathrm{Vect}(B_{[0,1]}) \times_{\mathrm{Vect}(B_{[1,1]})} \mathrm{Vect}(B_{[1,\infty]})$$

is a bi-exact equivalence.

Lemma 5.6.31. We have canonical isomorphisms

$$\begin{aligned} \mathcal{E}_{A_{\mathrm{inf}}(S)} \otimes_{A_{\mathrm{inf}}(S)} B_{[0,1]} &\cong T_{\mathrm{\acute{e}t}}(\mathcal{E}) \otimes_{\mathbb{Z}_p} B_{[0,1]} \\ (\mathcal{E}_{A_{\mathrm{inf}}(S)} \otimes_{A_{\mathrm{inf}}(S)} B_{[1,\infty]})[1/\xi] &\cong (\mathcal{E}_{A_{\mathrm{crys}}(S)} \otimes_{A_{\mathrm{crys}}(S)} B_{[1,\infty]})[1/\xi]. \end{aligned}$$

Proof. First, we construct the former isomorphism. The intersection formula in Lemma 5.6.26 (1) implies that $\phi^{-1}(\mu)$ is invertible in $B_{[0,1]}$ (cf. [GR24, footnote in p.44]), and so we have a ring map $A_{\mathrm{inf}}[1/\phi^{-1}(\mu)] \rightarrow B_{[0,1]}$. Taking the base change of the isomorphism in Proposition 5.6.16 along this ring map, we obtain an isomorphism

$$\mathcal{E}_{A_{\mathrm{inf}}(S)} \otimes_{A_{\mathrm{inf}}(S)} B_{[0,1]} \cong T_{\mathrm{\acute{e}t}}(\mathcal{E}) \otimes_{\mathbb{Z}_p} B_{[0,1]}.$$

The remaining is to construct the latter isomorphism. The crystal property of $\mathcal{E}$ for a log prism map $\phi: (A_{\mathrm{inf}}(S), (\xi), \mathcal{M}_{A_{\mathrm{inf}}(S)}) \rightarrow (A_{\mathrm{crys}}(S), (p), \mathcal{M}_{A_{\mathrm{crys}}(S)})$ induces an isomorphism of $B_{\mathrm{crys}}^+(S)$ -modules

$$\mathcal{E}_{A_{\mathrm{crys}}(S)} \cong \mathcal{E}_{A_{\mathrm{inf}}(S)} \otimes_{A_{\mathrm{inf}}(S), \phi} B_{\mathrm{crys}}^+(S).$$

Therefore, we have isomorphisms

$$\begin{aligned} (\mathcal{E}_{A_{\mathrm{crys}}(S)} \otimes_{A_{\mathrm{crys}}(S)} B_{[1,\infty]})[1/\xi] &\cong (\mathcal{E}_{A_{\mathrm{inf}}(S)} \otimes_{A_{\mathrm{inf}}(S), \phi} B_{[1,\infty]})[1/\xi] \\ &\cong (\phi^* \mathcal{E}_{A_{\mathrm{inf}}(S)})[1/\xi] \otimes_{A_{\mathrm{inf}}(S)} B_{[1,\infty]} \\ &\stackrel{\varphi_{\mathcal{E}}}{\cong} \mathcal{E}_{A_{\mathrm{inf}}(S)}[1/\xi] \otimes_{A_{\mathrm{inf}}(S)} B_{[1,\infty]} \\ &\cong (\mathcal{E}_{A_{\mathrm{inf}}(S)} \otimes_{A_{\mathrm{inf}}(S)} B_{[1,\infty]})[1/\xi]. \end{aligned}$$

□

Theorem 5.6.32. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a semi-stable log formal scheme over $\mathcal{O}_K$ which admits framings with $n \leq 1$ étale locally. Then the étale realization functor

$$T_{\mathrm{\acute{e}t}} : \mathrm{Vect}^{\mathrm{an}, \varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \mathrm{Loc}_{\mathbb{Z}_p}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta})$$

gives a bi-exact equivalence to the essential image.

Proof. By working étale locally on $\mathfrak{X}$, we may assume that $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is a small affine log formal scheme with a fixed framing such that $n \leq 1$. We treat just the $n = 1$ case because the same argument works in the $n = 0$ case.

Let $(\mathrm{Spf}(R), \mathcal{M}_R)$ be a small affine log formal scheme over $\mathcal{O}_K$ admitting a fixed framing with $n = 0$ such that $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \cong (\mathrm{Spf}(R), \mathcal{M}_R \oplus \pi^{\mathbb{N}})^a$. Let $\mathcal{E}_1 \rightarrow \mathcal{E}_2 \rightarrow \mathcal{E}_3$ be a sequence in $\mathrm{Vect}^{\mathrm{an}, \varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ and suppose that

$$0 \rightarrow T_{\mathrm{ét}}(\mathcal{E}_1) \rightarrow T_{\mathrm{ét}}(\mathcal{E}_2) \rightarrow T_{\mathrm{ét}}(\mathcal{E}_3) \rightarrow 0$$

is exact. The goal is to prove that

$$0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_2 \rightarrow \mathcal{E}_3 \rightarrow 0$$

is exact. To do this, it is enough to show that the sequence

$$0 \rightarrow \mathcal{E}_{1, A_{\mathrm{inf}}(S)} \rightarrow \mathcal{E}_{2, A_{\mathrm{inf}}(S)} \rightarrow \mathcal{E}_{3, A_{\mathrm{inf}}(S)} \rightarrow 0$$

is exact by Proposition 6.4.17 below.

First, we shall show that the sequence of $B_{\mathrm{crys}}^+(S)$ -modules

$$0 \rightarrow \mathcal{E}_{1, A_{\mathrm{crys}}(S)} \rightarrow \mathcal{E}_{2, A_{\mathrm{crys}}(S)} \rightarrow \mathcal{E}_{3, A_{\mathrm{crys}}(S)} \rightarrow 0$$

is exact. In [IKY24], this is proved by using Brinon's formulation. Since this argument does not work in this setting, we discuss in a different way. Taking the grading pieces of an exact sequence of filtered $B_{\mathrm{dR}}(S)$ -modules

$$0 \rightarrow T_{\mathrm{ét}}(\mathcal{E}_1) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}(S) \rightarrow T_{\mathrm{ét}}(\mathcal{E}_2) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}(S) \rightarrow T_{\mathrm{ét}}(\mathcal{E}_3) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}(S) \rightarrow 0,$$

we obtain an exact sequence

$$\begin{aligned} 0 \rightarrow \mathrm{gr}^{\bullet} T_{\mathrm{dR}}(\mathcal{E}_1) \otimes_{R[1/p]} \mathrm{gr}^{\bullet} B_{\mathrm{dR}}(S) &\rightarrow \mathrm{gr}^{\bullet} T_{\mathrm{dR}}(\mathcal{E}_2) \otimes_{R[1/p]} \mathrm{gr}^{\bullet} B_{\mathrm{dR}}(S) \\ &\rightarrow \mathrm{gr}^{\bullet} T_{\mathrm{dR}}(\mathcal{E}_3) \otimes_{R[1/p]} \mathrm{gr}^{\bullet} B_{\mathrm{dR}}(S) \rightarrow 0. \end{aligned}$$

Since $\mathrm{gr}^n B_{\mathrm{dR}}(S) \cong S[1/p]$ is faithfully flat over $R[1/p]$ for each $n \geq 1$, we see that the sequence

$$0 \rightarrow T_{\mathrm{dR}}(\mathcal{E}_1) \rightarrow T_{\mathrm{dR}}(\mathcal{E}_2) \rightarrow T_{\mathrm{dR}}(\mathcal{E}_3) \rightarrow 0$$

is exact. Let $(\mathrm{Spf}(R_0), \mathcal{M}_{R_0})$ be the unramified model of $(\mathrm{Spf}(R), \mathcal{M}_R)$ defined by the framing. Let $(\mathrm{Spf}(S_R), \mathcal{M}_{S_R})$ be the Breuil ring associated with $(\mathrm{Spf}(R), \mathcal{M}_R \oplus \pi^{\mathbb{N}})^a$. By Proposition 5.6.9 and Proposition 5.6.21, we have isomorphisms

$$(5.6.3) \quad T_{\mathrm{dR}}(\mathcal{E}_i) \cong T_{\mathrm{isoc}}(\mathcal{E}_i)_{(R_0, \mathcal{M}_{R_0} \oplus 0^{\mathbb{N}})^a} \otimes_{K_0} K,$$

$$(5.6.4) \quad T_{\mathrm{isoc}}(\mathcal{E}_i)_{(S_R, \mathcal{M}_{S_R})} \cong T_{\mathrm{isoc}}(\mathcal{E}_i)_{(R_0, \mathcal{M}_{R_0} \oplus 0^{\mathbb{N}})^a} \otimes_{R_0[1/p]} S_R[1/p],$$

for $i = 1, 2, 3$. By (5.6.3), the sequence

$$0 \rightarrow T_{\mathrm{isoc}}(\mathcal{E}_1)_{(R_0, \mathcal{M}_{R_0} \oplus 0^{\mathbb{N}})^a} \rightarrow T_{\mathrm{isoc}}(\mathcal{E}_2)_{(R_0, \mathcal{M}_{R_0} \oplus 0^{\mathbb{N}})^a} \rightarrow T_{\mathrm{isoc}}(\mathcal{E}_3)_{(R_0, \mathcal{M}_{R_0} \oplus 0^{\mathbb{N}})^a} \rightarrow 0$$

is exact. By (5.6.4), the sequence

$$0 \rightarrow T_{\mathrm{isoc}}(\mathcal{E}_1)_{(S_R, \mathcal{M}_{S_R})} \rightarrow T_{\mathrm{isoc}}(\mathcal{E}_2)_{(S_R, \mathcal{M}_{S_R})} \rightarrow T_{\mathrm{isoc}}(\mathcal{E}_3)_{(S_R, \mathcal{M}_{S_R})} \rightarrow 0$$

is exact. Since $(\mathrm{Spf}(S_R), \mathcal{M}_{S_R})$ covers the final object of the topos associated with $(X_{p=0}, \mathcal{M}_{X_{p=0}})_{\mathrm{crys}}$, we have an exact sequence

$$0 \rightarrow T_{\mathrm{isoc}}(\mathcal{E}_1)_{(A_{\mathrm{crys}}(S), \mathcal{M}_{A_{\mathrm{crys}}(S)})} \rightarrow T_{\mathrm{isoc}}(\mathcal{E}_1)_{(A_{\mathrm{crys}}(S), \mathcal{M}_{A_{\mathrm{crys}}(S)})} \rightarrow T_{\mathrm{isoc}}(\mathcal{E}_1)_{(A_{\mathrm{crys}}(S), \mathcal{M}_{A_{\mathrm{crys}}(S)})} \rightarrow 0.$$

Since the crystal property of $\mathcal{E}_i$ for a log prism map $(\mathfrak{S}_R, (E), \mathcal{M}_{\mathfrak{S}_R}) \rightarrow (A_{\text{crys}}(S), (p), \mathcal{M}_{A_{\text{crys}}})$ gives isomorphisms

$$\begin{aligned} T_{\text{isoc}}(\mathcal{E}_i)_{(A_{\text{crys}}(S), \mathcal{M}_{A_{\text{crys}}})} &\cong T_{\text{isoc}}(\mathcal{E}_i)_{S_R, \mathcal{M}_{S_R}} \otimes_{S_R[1/p]} A_{\text{crys}}(S)[1/p] \\ &\cong (\mathcal{E}_{i, \mathfrak{S}_R} \otimes_{\mathfrak{S}_R} S_R[1/p]) \otimes_{S_R[1/p]} A_{\text{crys}}(S)[1/p] \\ &\cong \mathcal{E}_{i, A_{\text{crys}}(S)} \end{aligned}$$

for $i = 1, 2, 3$, we conclude that the sequence

$$0 \rightarrow \mathcal{E}_{1, A_{\text{crys}}(S)} \rightarrow \mathcal{E}_{2, A_{\text{crys}}(S)} \rightarrow \mathcal{E}_{3, A_{\text{crys}}(S)} \rightarrow 0$$

is exact.

By Lemma 5.6.30, it suffices to prove that the following sequences are exact:

$$(5.6.5) \quad 0 \rightarrow \mathcal{E}_{1, A_{\text{inf}}(S)} \otimes B_{[0,1]} \rightarrow \mathcal{E}_{2, A_{\text{inf}}(S)} \otimes B_{[0,1]} \rightarrow \mathcal{E}_{3, A_{\text{inf}}(S)} \otimes B_{[0,1]} \rightarrow 0$$

$$(5.6.6) \quad 0 \rightarrow \mathcal{E}_{1, A_{\text{inf}}(S)} \otimes B_{[1,\infty]} \rightarrow \mathcal{E}_{2, A_{\text{inf}}(S)} \otimes B_{[1,\infty]} \rightarrow \mathcal{E}_{3, A_{\text{inf}}(S)} \otimes B_{[1,\infty]} \rightarrow 0$$

The exactness of the sequence (5.6.5) follows from Lemma 5.6.31. For the exactness of the sequence (5.6.6), Beauville-Laszlo glueing reduces the problem to prove that the following sequences are exact:

$$(5.6.7) \quad 0 \rightarrow \mathcal{E}_{1, A_{\text{inf}}(S)} \otimes B_{[1,\infty]}[1/\xi] \rightarrow \mathcal{E}_{2, A_{\text{inf}}(S)} \otimes B_{[1,\infty]}[1/\xi] \rightarrow \mathcal{E}_{3, A_{\text{inf}}(S)} \otimes B_{[1,\infty]}[1/\xi] \rightarrow 0$$

$$(5.6.8) \quad 0 \rightarrow (\mathcal{E}_{1, A_{\text{inf}}(S)} \otimes B_{[1,\infty]})_{\xi}^{\wedge} \rightarrow (\mathcal{E}_{2, A_{\text{inf}}(S)} \otimes B_{[1,\infty]})_{\xi}^{\wedge} \rightarrow (\mathcal{E}_{3, A_{\text{inf}}(S)} \otimes B_{[1,\infty]})_{\xi}^{\wedge} \rightarrow 0.$$

The exactness of (5.6.7) follows from Lemma 5.6.31 and what we proved in the previous paragraph, and the exactness of (5.6.8) follows from the exactness of (5.6.5) and the identification $(B_{[1,\infty]})_{\xi}^{\wedge} \cong (B_{[0,1]})_{\xi}^{\wedge} (\cong B_{\text{dR}}^+(S))$. This completes the proof. $\square$

5.A. Some lemmas on log schemes

Lemma 5.A.1. Let $i : S \hookrightarrow T$ be a nilpotent closed immersion of affine schemes with an ideal sheaf $\mathcal{I}$. Let $\mathcal{M}_T$ be an integral log structure on T . We let $\mathcal{M}_S$ denote the pullback log structure $i^* \mathcal{M}_T$. Then a natural map $\Gamma(T, \mathcal{M}_T) \rightarrow \Gamma(S, \mathcal{M}_S)$ is a $\Gamma(T, 1 + \mathcal{I})$ -torsor.

Proof. It is obvious that each fiber of the map $\Gamma(T, \mathcal{M}_T) \rightarrow \Gamma(S, \mathcal{M}_S)$ is a $\Gamma(T, 1 + \mathcal{I})$ -orbit. Since $\mathcal{M}_T$ is integral, the action of $\Gamma(T, 1 + \mathcal{I})$ on $\Gamma(T, \mathcal{M}_T)$ is faithful. Hence it suffices to prove that the map $\Gamma(T, \mathcal{M}_T) \rightarrow \Gamma(S, \mathcal{M}_S)$ is surjective. We may assume that $\mathcal{I}^2 = 0$. We have an exact sequence

$$1 \rightarrow 1 + \mathcal{I} \rightarrow \mathcal{M}_T^{\text{gp}} \rightarrow \mathcal{M}_S^{\text{gp}} \rightarrow 1$$

Since we have $H^1(T, 1 + \mathcal{I}) \cong H^1(T, \mathcal{I}) = 0$, the natural map $\Gamma(T, \mathcal{M}_T^{\text{gp}}) \rightarrow \Gamma(S, \mathcal{M}_S^{\text{gp}})$ is surjective, which implies that the natural map $\Gamma(T, \mathcal{M}_T) \rightarrow \Gamma(S, \mathcal{M}_S)$ is also surjective because $\mathcal{M}_T$ is isomorphic to $\mathcal{M}_S \times_{\mathcal{M}_S^{\text{gp}}} \mathcal{M}_T^{\text{gp}}$. $\square$

Definition 5.A.2. Let P be a monoid. For an integer $n \geq 1$, a monoid P is called *uniquely n -divisible* (resp. *n -divisible*) if a monoid map $P \rightarrow P$ mapping p to p^n is an isomorphism (resp. a surjection).

Lemma 5.A.3. Let $n \geq 1$ be an integer. Let P be a uniquely n -divisible monoid and $\pi : Q \rightarrow R$ is a surjection of monoids. Assume that there exists an integer $k \geq 1$ such that, for any $p_1, p_2 \in P$ with $\pi(p_1) = \pi(p_2)$, we have $p_1^{n^k} = p_2^{n^k}$. Then the natural map

$$\text{Hom}(P, Q) \rightarrow \text{Hom}(P, R)$$

is bijective.

Proof. Let $f : P \rightarrow R$ be a monoid map. We define a monoid map $\tilde{f} : P \rightarrow Q$ lifting f as follows. For $p \in P$, we take a unique $p' \in P$ such that $(p')^{n^k} = p$ and choose a lift $q' \in Q$ of $\pi(p')$. We put $\tilde{f}(p) := (q')^{n^k}$. For another lift $q'' \in Q$ of $\pi(p')$, the assumption implies $(q')^{n^k} = (q'')^{n^k}$. Therefore $\tilde{f}$ is a well-defined monoid map. By construction, $\tilde{f}$ is a unique lift of f . $\square$

Remark 5.A.4. when $\pi : Q \rightarrow R$ is either of the following types, the assumption in Lemma 5.A.3 is satisfied:

- a natural surjection $\pi : A \rightarrow A/I$ where A is a ring, I is an ideal of A with $I^k = 0$ and $n \in I$
- a natural surjection $\pi : M \rightarrow M/G$ where M is a monoid and G is a subgroup of $M^\times$ with $G^{n^k} = \{1\}$.

Lemma 5.A.5. Let $i : S \hookrightarrow T$ be a nilpotent closed immersion of schemes with an ideal sheaf $\mathcal{I}$. Suppose that an integer $n \geq 1$ is nilpotent on S .

- (1) Let $\mathcal{M}_T, \mathcal{N}_T$ be integral log structures on T . We let $\mathcal{M}_S, \mathcal{N}_S$ denote the pullback log structures $i^*\mathcal{M}_T, i^*\mathcal{N}_T$ respectively. Suppose that, étale locally, $\mathcal{M}_T$ admits a chart $P \rightarrow \mathcal{M}_T$ for an n -divisible monoid P . Then any map of log structures $\mathcal{M}_S \rightarrow \mathcal{N}_S$ extends uniquely to a map of log structures $\mathcal{M}_T \rightarrow \mathcal{N}_T$.
- (2) Let $\mathcal{M}_S$ be an integral log structure on S . Suppose that, étale locally, $\mathcal{M}_S$ admits a chart $P \rightarrow \mathcal{M}_S$ for a uniquely n -divisible monoid P . Then there exists a unique log structure $\mathcal{M}_T$ on T with $i^*\mathcal{M}_T \cong \mathcal{M}_S$.

Proof. (1) By working étale locally on T , we may assume that T is affine and we have an n -divisible monoid P and a chart $P \rightarrow \mathcal{M}_T$. It is enough to prove that a natural map

$$\mathrm{Hom}(P, \Gamma(T, \mathcal{M}_T)) \rightarrow \mathrm{Hom}(P, \Gamma(S, \mathcal{M}_S))$$

is bijective. We may assume that $\mathcal{I}^2 = 0$. Since we have $\mathcal{M}_T \cong \mathcal{M}_S \times_{\mathcal{M}_S^{\mathrm{gp}}} \mathcal{M}_T^{\mathrm{gp}}$, we are reduced to show that a natural map

$$\mathrm{Hom}(P^{\mathrm{gp}}, \Gamma(T, \mathcal{M}_T^{\mathrm{gp}})) \rightarrow \mathrm{Hom}(P^{\mathrm{gp}}, \Gamma(S, \mathcal{M}_S^{\mathrm{gp}}))$$

is bijective. Lemma 5.A.1 gives an exact sequence

$$\mathrm{Hom}(P^{\mathrm{gp}}, \Gamma(T, \mathcal{I})) \rightarrow \mathrm{Hom}(P^{\mathrm{gp}}, \Gamma(T, \mathcal{M}_T^{\mathrm{gp}})) \rightarrow \mathrm{Hom}(P^{\mathrm{gp}}, \Gamma(S, \mathcal{M}_S^{\mathrm{gp}})) \rightarrow \rightarrow \mathrm{Ext}^1(P^{\mathrm{gp}}, H^1(T, \mathcal{I})),$$

by $1 + \mathcal{I} \cong \mathcal{I}$. Since P^{gp} is n -divisible and $n^k \mathcal{I} = 0$ for a sufficiently large k , we have

$$\mathrm{Hom}(P^{\mathrm{gp}}, \Gamma(T, \mathcal{I})) = \mathrm{Ext}^1(P^{\mathrm{gp}}, H^1(T, \mathcal{I})) = 0.$$

This proves the assertion.

- (2) By working étale locally on T , we may assume that T is affine and we have a uniquely n -divisible monoid P and a chart $P \rightarrow \mathcal{M}_S$. Lemma 5.A.3 gives a unique lift $P \rightarrow \mathcal{O}_T$ of the map $P \rightarrow \mathcal{M}_S \rightarrow \mathcal{O}_S$. Let $\mathcal{M}_T$ denote the associated log structure with the map $P \rightarrow \mathcal{O}_T$. By construction, we have $i^*\mathcal{M}_T \cong \mathcal{M}_S$. To prove the uniqueness, we take another log structure $\mathcal{N}_T$ on T with $i^*\mathcal{N}_T \cong \mathcal{M}_S$. By (1), there exist a unique lift $\mathcal{M}_T \rightarrow \mathcal{N}_T$ of the identity map on $\mathcal{M}_S$. This map $\mathcal{M}_T \rightarrow \mathcal{N}_T$ is an isomorphism because so is the pullback to S . This proves the uniqueness. $\square$

5.B. Log adic spaces and log diamonds

In this section, we study the relationship between log adic spaces (introduced in [DLLZ23b]) and associated log diamond (introduced in [KY25]).

Lemma 5.B.1. Let $f: (X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y)$ be a map of log diamonds. Suppose that we are given a divisible saturated monoid P and a chart $P \rightarrow \mathcal{M}_Y$. Then f is quasi-pro-Kummer-étale if and only if it is strict quasi-pro-étale.

Proof. Since the "if" part is trivial, we shall prove the "only if" part. Take a strictly totally disconnected perfectoid space Z and a quasi-pro-étale cover $Z \rightarrow Y$. Let $\mathcal{M}_Z$ denote the log structure on Z obtained as the pullback of $\mathcal{M}_Y$. Consider the following Cartesian diagram:

$$\begin{array}{ccc} (X, \mathcal{M}_X) \times_{(Y, \mathcal{M}_Y)} (Z, \mathcal{M}_Z) & \longrightarrow & (Z, \mathcal{M}_Z) \\ \downarrow & & \downarrow \\ (X, \mathcal{M}_X) & \xrightarrow{f} & (Y, \mathcal{M}_Y). \end{array}$$

In this diagram, the upper horizontal map is strict quasi-pro-étale by the definition of quasi-pro-Kummer-étale maps, and both of vertical maps are strict quasi-pro-étale covers. Therefore, f is strict quasi-pro-étale. $\square$

Definition 5.B.2. Let $(X, \mathcal{M}_X)$ be an fs log adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. We have the associated locally spatial diamond $X^\diamond$. there exists a natural morphism of ringed sites

$$\nu: (X_{\mathrm{qpro\acute{e}t}}^\diamond, \widehat{\mathcal{O}}_{X^\diamond}) \rightarrow (X_{\acute{e}t}, \mathcal{O}_X).$$

Let $\mathcal{M}_{X^\diamond} \rightarrow \widehat{\mathcal{O}}_{X^\diamond}$ be the associated log structure with the prelog structure

$$\nu^{-1}\mathcal{M}_X \rightarrow \nu^{-1}\mathcal{O}_X \rightarrow \widehat{\mathcal{O}}_{X^\diamond}.$$

The locally spatial log diamond $(X^\diamond, \mathcal{M}_{X^\diamond})$ is called *the associated log diamond* with $(X, \mathcal{M}_X)$ and denoted by $(X, \mathcal{M}_X)^\diamond$. By definition, a chart $P \rightarrow \mathcal{M}_X$ induces a chart $P \rightarrow \mathcal{M}_{X^\diamond}$.

Lemma 5.B.3. Let $(X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y)$ be a Kummer log étale morphism of locally noetherian fs log adic spaces over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. Then the induced morphism of log diamonds $(X, \mathcal{M}_X)^\diamond \rightarrow (Y, \mathcal{M}_Y)^\diamond$ is Kummer log étale.

Proof. By working étale locally on Y , we may assume that Y is quasi-compact and that $(Y, \mathcal{M}_Y)$ admits an fs chart $P \rightarrow \mathcal{M}_Y$ with P being a torsion free. We set

$$(Y_n, \mathcal{M}_{Y_n}) := (Y, \mathcal{M}_Y) \times_{(\mathrm{Spa}(\mathbb{Q}_p \langle P \rangle, \mathbb{Z}_p \langle P \rangle), P)^a} (\mathrm{Spa}(\mathbb{Q}_p \langle P^{1/n} \rangle, \mathbb{Z}_p \langle P^{1/n} \rangle), P)^a$$

and let $(X_n, \mathcal{M}_{X_n})$ denote the saturated fiber product of $(Y_n, \mathcal{M}_{Y_n})$ and $(X, \mathcal{M}_X)$ over $(Y, \mathcal{M}_Y)$ for an integer. By [DLLZ23b, Lemma 4.2.5], there exists an integer $n \geq 1$ such that the projection map $(X_n, \mathcal{M}_{X_n}) \rightarrow (Y_n, \mathcal{M}_{Y_n})$ is strict étale.

Let $f: (Z, \mathcal{M}_Z) \rightarrow (Y, \mathcal{M}_Y)^\diamond$ be a map from strictly totally disconnected log perfectoid space $(Z, \mathcal{M}_Z)$. Since $\Gamma(Z, \mathcal{M}_Z)$ is divisible by [KY25, Definition 7.12], the map $P \rightarrow \mathcal{M}_Z$ extends to a map $P_{\mathbb{Q}_{\geq 0}} \rightarrow \mathcal{M}_Z$. Hence, the map f factors through $(Y_n, \mathcal{M}_{Y_n})$ and the base-changed map

$$g: (Z, \mathcal{M}_Z) \times_{(Y, \mathcal{M}_Y)^\diamond} (X, \mathcal{M}_X)^\diamond \rightarrow (Z, \mathcal{M}_Z)$$

can be written as the base change of the strict étale map $(X_n, \mathcal{M}_{X_n})^\diamond \rightarrow (Y_n, \mathcal{M}_{Y_n})^\diamond$. Therefore, g is also strict étale, and so the map $(X, \mathcal{M}_X)^\diamond \rightarrow (Y, \mathcal{M}_Y)^\diamond$ is Kummer étale. $\square$

Proposition 5.B.4. Let $(X, \mathcal{M}_X)$ be a locally noetherian fs log adic space. For a site $\mathcal{C}$, we let $\mathrm{Loc}_{\mathbb{Z}/p^n}(\mathcal{C})$ denote the category of $\mathbb{Z}/p^n$ -local systems on $\mathcal{C}$. Then the natural functor

$$\mathrm{Loc}_{\mathbb{Z}/p^n}((X, \mathcal{M}_X)_{\mathrm{k\acute{e}t}}) \rightarrow \mathrm{Loc}_{\mathbb{Z}/p^n}((X, \mathcal{M}_X)_{\mathrm{qprok\acute{e}t}}^{\diamond})$$

is an equivalence.

Proof. By working étale locally on X , we may assume that $(X, \mathcal{M}_X)$ admits a chart $P \rightarrow \mathcal{M}_X$ and that X is qcqs. We define log adic spaces

$$(X_m, \mathcal{M}_{X_m}) := (X, \mathcal{M}_X) \times_{(\mathrm{Spa}(\mathbb{Q}_p\langle P \rangle, \mathbb{Z}_p\langle P \rangle), P)^a} (\mathrm{Spa}(\mathbb{Q}_p\langle P^{1/m} \rangle, \mathbb{Z}_p\langle P^{1/m} \rangle), P^{1/m})^a$$

and a log diamond

$$(X_{\infty}, \mathcal{M}_{X_{\infty}}) := (X, \mathcal{M}_X)^{\diamond} \times_{(\mathrm{Spd}(\mathbb{Q}_p\langle P \rangle, \mathbb{Z}_p\langle P \rangle), P)^a} (\mathrm{Spd}(\mathbb{Q}_p\langle P_{\mathbb{Q}_{\geq 0}} \rangle, \mathbb{Z}_p\langle P_{\mathbb{Q}_{\geq 0}} \rangle), P_{\mathbb{Q}_{\geq 0}})^a.$$

Then the natural map $(X_{\infty}, \mathcal{M}_{X_{\infty}}) \rightarrow (X, \mathcal{M}_X)^{\diamond}$ is a quasi-pro-Kummer étale cover by [KY25, Proposition 7.20 (2)], and we have the following isomorphisms:

$$X_{\infty} \cong \varprojlim_m X_m^{\diamond}, \quad (X_{\infty}, \mathcal{M}_{X_{\infty}}) \cong \varprojlim_m (X_m, \mathcal{M}_{X_m})^{\diamond}.$$

Let $(X_m^{(\bullet)}, \mathcal{M}_{X_m^{(\bullet)}})$ (resp. $(X_{\infty}^{(\bullet)}, \mathcal{M}_{X_{\infty}^{(\bullet)}})$) is the Čech nerve of $(X_m, \mathcal{M}_{X_m}) \rightarrow (X, \mathcal{M}_X)$ (resp. $(X_{\infty}, \mathcal{M}_{X_{\infty}}) \rightarrow (X, \mathcal{M}_X)^{\diamond}$) in the category of fs log adic spaces (resp. saturated log diamonds).

By [DLLZ23b, Lemma 4.2.5 and Theorem 4.4.15], every object of $\mathrm{Loc}_{\mathbb{Z}/p^n}((X, \mathcal{M}_X)_{\mathrm{k\acute{e}t}})$ comes from an object of $\mathrm{Loc}_{\mathbb{Z}/p^n}((X_m)_{\mathrm{\acute{e}t}})$ after being pulled back to $(X_m, \mathcal{M}_{X_m})$ for some $m \geq 1$. In other words, we have a natural equivalence

$$\mathrm{Loc}_{\mathbb{Z}/p^n}((X, \mathcal{M}_X)_{\mathrm{k\acute{e}t}}) \simeq \varinjlim_m \varprojlim_{\bullet \in \Delta} \mathrm{Loc}_{\mathbb{Z}/p^n}((X_m^{(\bullet)})_{\mathrm{\acute{e}t}}).$$

On the other hand, we have natural equivalences

$$\begin{aligned} \mathrm{Loc}_{\mathbb{Z}/p^n}((X, \mathcal{M}_X)_{\mathrm{qprok\acute{e}t}}^{\diamond}) &\simeq \varprojlim_{\bullet \in \Delta} \mathrm{Loc}_{\mathbb{Z}/p^n}((X_{\infty}^{(\bullet)}, \mathcal{M}_{X_{\infty}^{(\bullet)}})_{\mathrm{qprok\acute{e}t}}) \\ &\simeq \varprojlim_{\bullet \in \Delta} \mathrm{Loc}_{\mathbb{Z}/p^n}((X_{\infty}^{(\bullet)})_{\mathrm{qpro\acute{e}t}}) \\ &\simeq \varinjlim_m \varprojlim_{\bullet \in \Delta} \mathrm{Loc}_{\mathbb{Z}/p^n}((X_m^{(\bullet)})_{\mathrm{qpro\acute{e}t}}^{\diamond}), \end{aligned}$$

where the first equivalence is quasi-pro-Kummer étale descent, the second one comes from Lemma 5.B.1, and the third one is the limit argument [Sch22, Proposition 11.23 (i)]. Moreover, [Sch22, Lemma 15.6] and [MW23, Proposition 3.7] imply that there exists an equivalence

$$\mathrm{Loc}_{\mathbb{Z}/p^n}((X_m^{(k)})_{\mathrm{\acute{e}t}}) \simeq \mathrm{Loc}_{\mathbb{Z}/p^n}((X_m^{(k)})_{\mathrm{\acute{e}t}}^{\diamond}) \simeq \mathrm{Loc}_{\mathbb{Z}/p^n}((X_m^{(k)})_{\mathrm{qpro\acute{e}t}}^{\diamond})$$

for each $m \geq 1$ and $k \geq 0$. This proves the assertion. $\square$

CHAPTER 6

Log prismatic Dieudonné theory and its application to Shimura varieties

6.1. Introduction

Shimura varieties form a class of algebraic varieties that play an important role in number theory and algebraic geometry. A guiding principle for Shimura varieties is that a Shimura variety associated with a Shimura datum (G, X) should be the moduli space parameterizing “motives with G -structure”. In particular, there ought to be “a universal motive with G -structure” (which is just a conjectural object). Several cohomological realizations of “a universal motive with G -structure” are constructed so far. As a recent development in this direction, a realization in the framework of prismatic cohomology has been constructed in [IKY23].

On the other hand, there is a satisfactory theory of compactifications of Shimura varieties. For canonical models of Shimura varieties, see [Pin90]. For integral models of Shimura varieties with hyperspecial levels, a theory of compactifications in Siegel cases, PEL-type cases, and Hodge type cases is established in [FC90], [Lan13], and [Mad19], respectively. In some cases, “the universal motive with G -structure” on an interior Shimura variety is known to extend to “the universal degenerating motive with G -structure” on a toroidal compactification (for example, the realization as log Hodge structures in [KU09] or G -zip realization in [GK19]). In this chapter, we construct a realization of the hypothetical degenerating motive within the framework of log prismatic cohomology established by [Kos22, KY25] over a toroidal compactification of the integral canonical models of Shimura varieties of Hodge type with a hyperspecial level constructed in [Mad19]. The resulting object is called *log prismatic realization*. The log prismatic realization restricts to the prismatic realization on the interior constructed in [IKY23].

To give a precise statement, let (G, X) be a Shimura datum of Hodge type with reflex field E . Let $p > 2$ be a prime number and fix a place v of E above p . Let E_v be the completion of E at v and $\mathcal{O}_{E_v}$ be a ring of integers of E_v . Let $K_p \subset G(\mathbb{Q}_p)$ be an open compact subgroup and $K^p \subset G(\mathbb{A}^{\infty, p})$ be a neat open compact subgroup. Set $K := K_p K^p$. Then the Shimura variety $\mathrm{Sh}_K(G, X)$ is an algebraic variety over E . Suppose that K_p is hyperspecial and let $\mathcal{G}$ be a reductive model of $G_{\mathbb{Q}_p}$ over $\mathbb{Z}_p$ with $\mathcal{G}(\mathbb{Z}_p) = K_p$. In this situation, for a sufficiently small K^p , an integral canonical model $\mathcal{S}_K(G, X)$ of $\mathrm{Sh}_K(G, X)_{E_v}$ over $\mathcal{O}_{E_v}$ is constructed in [Kis10], and the toroidal compactification $\mathcal{S}_K^\Sigma(G, X)$ of $\mathcal{S}_K(G, X)$ for a suitable (smooth projective) cone decomposition Σ is constructed in [Mad19]. Let $\mathrm{Sh}_K^\Sigma(G, X) := \mathcal{S}_K^\Sigma(G, X) \otimes_{\mathcal{O}_{E_v}} E_v$. Let $\widehat{\mathcal{S}}_K(G, X)$ and $\widehat{\mathcal{S}}_K^\Sigma(G, X)$ denote the p -adic completion of $\mathcal{S}_K(G, X)$ and $\mathcal{S}_K^\Sigma(G, X)$ respectively. Let $(\widehat{\mathcal{S}}_K^\Sigma(G, X), \mathcal{M}_{\widehat{\mathcal{S}}_K^\Sigma(G, X)})$ denote the log scheme equipped with the log structure defined by the boundary divisor. Similarly, we define $(\mathrm{Sh}_K^\Sigma(G, X), \mathcal{M}_{\mathrm{Sh}_K^\Sigma(G, X)})$ and $(\widehat{\mathcal{S}}_K^\Sigma(G, X), \mathcal{M}_{\widehat{\mathcal{S}}_K^\Sigma(G, X)})$. Note that $(\widehat{\mathcal{S}}_K^\Sigma(G, X), \mathcal{M}_{\widehat{\mathcal{S}}_K^\Sigma(G, X)})$ is a horizontally semi-stable log formal scheme over $\mathcal{O}_{E_v}$ in the sense of Definition 5.2.4.

The tower

$$\varprojlim_{K_p' \subset K_p} \mathrm{Sh}_{K_p' K_p}(G, X) \rightarrow \mathrm{Sh}_K(G, X)$$

gives a $\mathbb{Z}_p$ -linear exact tensor functor

$$\omega_{\text{ét}}: \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \rightarrow \mathrm{Loc}_{\mathbb{Z}_p}(\mathrm{Sh}_K(G, X)),$$

which is regarded as the étale realization of “the universal motive with G -structure” on $\mathrm{Sh}_K(G, X)$. Here, $\mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G})$ denotes the category of representations of $\mathcal{G}$ over $\mathbb{Z}_p$. By purity, $\omega_{\text{ét}}$ induces a $\mathbb{Z}_p$ -linear exact tensor functor

$$\omega_{\text{két}}: \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \rightarrow \mathrm{Loc}_{\mathbb{Z}_p}(\mathrm{Sh}_K^{\Sigma}(G, X), \mathcal{M}_{\mathrm{Sh}_K^{\Sigma}(G, X)}),$$

where the target category is the category of Kummer étale $\mathbb{Z}_p$ -local systems. The object $\omega_{\text{két}}$ can be regarded as the Kummer étale realization of “the universal degenerating motive with G -structure” on $\mathrm{Sh}_K^{\Sigma}(G, X)$. In [IKY23], the prismatic realization of “the universal motive with G -structure” is constructed as a $\mathbb{Z}_p$ -linear exact tensor functor

$$\omega_{\Delta}: \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \rightarrow \mathrm{Vect}^{\varphi}(\widehat{\mathcal{S}}_K(G, X)_{\Delta})$$

with $T_{\text{ét}} \circ \omega_{\Delta} \cong \omega_{\text{ét}}$, where $T_{\text{ét}}$ is the étale realization functor

$$T_{\text{ét}}: \mathrm{Vect}^{\varphi}(\widehat{\mathcal{S}}_K(G, X)_{\Delta}) \rightarrow \mathrm{Loc}_{\mathbb{Z}_p}(\widehat{\mathcal{S}}_K(G, X)_{\eta}).$$

Then we can state our main theorem as follows.

Theorem 6.1.1 (see Theorem 6.7.2). There exists a unique $\mathbb{Z}_p$ -linear exact tensor functor

$$\omega_{\Delta, \log}: \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \rightarrow \mathrm{Vect}^{\varphi}((\widehat{\mathcal{S}}_K^{\Sigma}(G, X), \mathcal{M}_{\widehat{\mathcal{S}}_K^{\Sigma}(G, X)})_{\Delta})$$

with $T_{\text{ét}} \circ \omega_{\Delta, \log} \cong \omega_{\text{két}}$ and $\omega_{\Delta, \log}|_{\widehat{\mathcal{S}}_K(G, X)} \cong \omega_{\Delta}$, where $T_{\text{ét}}$ is the étale realization functor

$$T_{\text{ét}}: \mathrm{Vect}^{\varphi}((\widehat{\mathcal{S}}_K^{\Sigma}(G, X), \mathcal{M}_{\widehat{\mathcal{S}}_K^{\Sigma}(G, X)})_{\Delta}) \rightarrow \mathrm{Loc}_{\mathbb{Z}_p}((\widehat{\mathcal{S}}_K^{\Sigma}(G, X), \mathcal{M}_{\widehat{\mathcal{S}}_K^{\Sigma}(G, X)})_{\eta}).$$

As an application of our construction, we give the p -adic comparison isomorphism on (not necessarily compact) Shimura varieties, which is conjectured in [Lov17, 3.6.2].

Theorem 6.1.2. (see Theorem 6.7.5) Let $(\overline{\mathcal{S}}_K^{\Sigma}(G, X), \mathcal{M}_{\overline{\mathcal{S}}_K^{\Sigma}(G, X)})$ be the special fiber of $(\mathcal{S}_K^{\Sigma}(G, X), \mathcal{M}_{\mathcal{S}_K^{\Sigma}(G, X)})$. Consider a $\mathbb{Z}_p$ -linear exact tensor functor

$$\omega_{\text{fisoc}}: \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \rightarrow \mathrm{FillSoc}^{\varphi}(\widehat{\mathcal{S}}_K^{\Sigma}(G, X), \mathcal{M}_{\widehat{\mathcal{S}}_K^{\Sigma}(G, X)})$$

defined by $\omega_{\text{fisoc}} := T_{\text{fisoc}} \circ \omega_{\Delta, \log}$, where T_{fisoc} is the realization functor as filtered F -isocrystals defined in Theorem 5.6.18. Then the $\mathrm{Gal}(\overline{E}_v/E_v)$ -representation

$$H_{\text{ét}}^i(\mathrm{Sh}_K(G, X)_{\overline{E}_v}, \omega_{\text{ét}}(\xi)[1/p])$$

is crystalline, and we have a comparison isomorphism

$$\begin{aligned} & H_{\text{ét}}^i(\mathrm{Sh}_K(G, X)_{\overline{E}_v}, \omega_{\text{ét}}(\xi)[1/p]) \otimes_{\mathbb{Q}_p} B_{\text{crys}} \\ & \cong H_{\text{logcrys}}^i((\overline{\mathcal{S}}_K^{\Sigma}(G, X), \mathcal{M}_{\overline{\mathcal{S}}_K^{\Sigma}(G, X)}), \omega_{\text{fisoc}}(\xi)) \otimes_W B_{\text{crys}} \end{aligned}$$

that is compatible with Galois actions and Frobenius isomorphisms and a filtered isomorphism after taking the base change along $B_{\text{crys}} \rightarrow B_{\text{dR}}$ for any $\xi \in \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G})$.

Let us explain our construction. One of the key steps is the construction of a log prismatic F -crystal on $\widehat{\mathcal{S}}_K^\Sigma(G, X)$ which is the extension of the prismatic F -crystal $R^1 f_* \mathcal{O}_\Delta$ on $\mathcal{S}_K(G, X)$, where $f: \mathcal{A}_{\text{univ}} \rightarrow \mathcal{S}_K(G, X)$ is the universal abelian scheme (i.e. the pullback of the universal abelian scheme on the Siegel modular variety along the Hodge embedding).

For simplicity, we assume that $(G, X) = (\text{GSp}, S^\pm)$. The universal abelian variety $\mathcal{A}_{\text{univ}}$ on $\mathcal{S}_K(G, X)$ degenerates to a semi-abelian scheme $\mathcal{A}_{\text{univ}}^{\text{sab}}$ on $\mathcal{S}_K^\Sigma(G, X)$. However, the relative prismatic cohomology of $\mathcal{A}_{\text{univ}}^{\text{sab}} \rightarrow \mathcal{S}_K^\Sigma(G, X)$ does not work well because this morphism is not proper. As a substitute, we have two candidates: the log abelian scheme on $\mathcal{S}_K^\Sigma(G, X)$ extending $\mathcal{A}_{\text{univ}}$ and a compactification of $\mathcal{A}_{\text{univ}}^{\text{sab}}$ constructed in [Lan12, Theorem 2.15]. In this chapter, we adopt the former one.

The compactification $\mathcal{S}_K^\Sigma(G, X)$ is reinterpreted as the moduli space of principally polarized log abelian schemes with level K -structure and local monodromies in Σ (see [KKN21] or [KKN22, Theorem 2.2.2, Proposition 4.3.4]). In particular, $\mathcal{A}_{\text{univ}}$ on $\mathcal{S}_K(G, X)$ extends to the universal log abelian scheme $\mathcal{A}_{\text{univ}}^{\text{log}}$ on $\mathcal{S}_K^\Sigma(G, X)$. Then “the relative log prismatic cohomology of $\mathcal{A}_{\text{univ}}^{\text{log}}$ over $\widehat{\mathcal{S}}_K^\Sigma(G, X)$ ” should be the desired object. However, it seems technically difficult to consider “the log prismatic cohomology of log abelian schemes” because log abelian varieties are just sheaves (not even log algebraic spaces). Circumventing this problem, we pass to the associated log p -divisible groups which are easier to treat.

The key ingredient is the following theorem.

Theorem 6.1.3 (see Theorem 6.5.12). Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a semi-stable log formal scheme over $\mathcal{O}_K$. Then there exists an exact anti-equivalence

$$\mathcal{M}_\Delta: \text{BT}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow \text{DM}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_\Delta)$$

(which is called a *log prismatic Dieudonné functor*), where $\text{BT}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is the category of log p -divisible groups over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ and $\text{DM}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_\Delta)$ is the category of log prismatic Dieudonné crystals on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ (for the definition, see Definition 5.5.9).

In [ALB23], it is proved that the relative prismatic cohomology of an abelian scheme is isomorphic to the prismatic Dieudonné crystal of the associated p -divisible group. As considered by analogy, the log prismatic Dieudonné crystal of $\mathcal{A}_{\text{univ}}^{\text{log}}[p^\infty]$ is the desired object.

The idea of the proof of Theorem 6.1.3 is to reduce the problem to non-log cases ([ALB23]) by applying saturated descent. Since the category of prismatic crystals does not satisfy good saturated descent properties, we define a larger category whose objects are called *kfl prismatic crystals* (Definition 6.4.1) and prove this category satisfies certain saturated descent properties which can be regarded as a log version of quasi-syntomic descent (Proposition 6.4.12). In a similar way, while the category of log p -divisible groups does not satisfy such descent properties, we have a larger category whose objects are called *weak log p -divisible groups* (which is introduced in [Kat23]) and prove certain saturated descent properties for weak log p -divisible groups which are applicable to non-fs log formal schemes (Proposition 6.3.27). These descent properties and (non-log) prismatic Dieudonné theory established in [ALB23] give an equivalence between larger categories (Proposition 6.5.6). To prove that this equivalence restricts to the desired equivalence, we use crystalline realization functors. In this step, we need the assumption that $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is semi-stable log formal scheme over $\mathcal{O}_K$.

Remark 6.1.4. Theorem 6.1.3 is proved in [WZ23] when $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is $\mathrm{Spf}(\mathcal{O}_K)$ with the standard log structure, and they use a similar approach. They define the category of kfl prismatic crystals as the category of vector bundles on the log prismatic site equipped with Kummer log flat topology ([WZ23, Definition 2.1]). However, it is doubtful that the log prismatic site equipped with Kummer log flat topology is indeed a site because coverings in this site may not be stable under base change due to the fact that the underlying ring map of a Kummer log flat map is not necessarily flat.

We briefly explain the structure of this chapter. In Section 6.2, we generalize the theory of kfl topology and kfl vector bundles to the setting of log formal schemes. The main goal of this section is to formulate and prove certain saturated descent properties for kfl vector bundles which are applicable to a class of non-fs log formal schemes (Proposition 6.2.26). In Section 6.3, we generalize the theory of (weak) log finite group schemes and (weak) log p -divisible groups to the setting of log formal schemes and show similar saturated descent properties for weak log finite group schemes and weak log p -divisible groups (Proposition 6.3.15 and Proposition 6.3.27) in an analogous way. In Section 6.4, we introduce the notion of kfl prismatic crystals and prove certain saturated descent properties for kfl prismatic crystals (Proposition 6.4.12), which is the most technical part in this chapter. We also obtain analogous results for crystalline crystals. In Section 6.5, we prove the main theorem of log prismatic Dieudonné theory (Theorem 6.5.12). In Section 6.6, we prepare several results generalizing those in [IKY24] to give an application to Shimura varieties. A primary result in this section is Theorem 6.6.16. In Section 6.7, we construct log prismatic realization functors (Theorem 6.7.2) and apply them to the proof of Lovering’s conjecture (Theorem 6.7.5).

Notation and conventions.

- Limits of categories mean 2-limits.
- Formal schemes are assumed to admit a finitely generated ideal of definition Zariski locally. For a property $\mathcal{P}$ of morphisms of schemes, an adic morphism of formal schemes is called $\mathcal{P}$ if it is adically $\mathcal{P}$.
- The i -th standard basis of $\mathbb{N}^r$ is denote by e_i for $1 \leq i \leq r$.
- For I -adically complete ring A and a monoid M , a ring $A\langle M \rangle$ denotes the I -adic completion of the monoid algebra $A[M]$.
- Let $\mathfrak{X}$ be a formal scheme and $n \geq 1$ be an integer. Assume that p is topologically nilpotent on $\mathfrak{X}$ when $n = 1$. The category of finite locally free group schemes (resp. truncated Barsotti-Tate groups of level n) (resp. p -divisible groups) on $\mathfrak{X}$ is denoted by $\mathrm{Fin}(\mathfrak{X})$ (resp. $\mathrm{BT}_n(\mathfrak{X})$) (resp. $\mathrm{BT}(\mathfrak{X})$).

6.2. Kfl topology on fs log formal schemes

Throughout this section, let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ denote an fs log formal scheme.

6.2.1. Kummer morphisms of log formal schemes. We recall fundamental properties of Kummer morphisms.

Definition 6.2.1 (Kummer morphisms of log formal schemes). Let $f: (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ be a morphism of saturated log formal schemes. The morphism f is called *Kummer* if, for every $x \in \mathfrak{X}$, the monoid map $\overline{\mathcal{M}_{\mathfrak{Y}, \overline{y}}} \rightarrow \overline{\mathcal{M}_{\mathfrak{X}, \overline{x}}}$ is Kummer. Here, we set $y := f(x)$.

Lemma 6.2.2 (cf. [Kat21, Proposition 2.7 (2)]). Let $f: (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ be a Kummer morphism of fs log formal schemes. Suppose that $\mathfrak{X}$ is quasi-compact and that we are given an fs chart $P \rightarrow \mathcal{M}_{\mathfrak{Y}}$. Then there exists an integer $n \geq 1$ such that, if we put $(\mathfrak{Y}', \mathcal{M}_{\mathfrak{Y}'}):= (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \otimes_{(\mathbb{Z}[P], P^a)} (\mathbb{Z}[P^{1/n}], P^{1/n})^a$, the natural morphism

$$(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \times_{(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})}^{\text{sat}} (\mathfrak{Y}', \mathcal{M}_{\mathfrak{Y}'}) \rightarrow (\mathfrak{Y}', \mathcal{M}_{\mathfrak{Y}'})$$

is strict.

Proof. This follows from the same argument as [Kat21, Proposition 2.7 (2)]. $\square$

The following lemma is a generalization of [Kat21, Lemma 2.4] to non-fs cases.

Lemma 6.2.3 (cf. [Kat21, Lemma 2.4 (2)]). Consider the following Cartesian diagram in the category of saturated log formal schemes:

$$\begin{array}{ccc} (\mathfrak{X}', \mathcal{M}_{\mathfrak{X}'}) & \xrightarrow{g'} & (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \\ \downarrow f' & & \downarrow f \\ (\mathfrak{Y}', \mathcal{M}_{\mathfrak{Y}'}) & \xrightarrow{g} & (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}). \end{array}$$

Suppose that $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ is fs and that f is Kummer. Then, for $x \in \mathfrak{X}$ and $y' \in \mathfrak{Y}'$ such that $f(x) = g(y')$, there exists $x' \in \mathfrak{X}'$ satisfying $g'(x') = x$ and $f'(x') = y'$.

Proof. We may assume that $\mathfrak{X} = \mathfrak{Y} = \mathfrak{Y}' = \text{Spec}(k)$ for an algebraically closed field k . It suffices to show that $\mathfrak{X}'$ is nonempty. We can write $\mathcal{M}_{\mathfrak{X}} = k^{\times} \times Q$, $\mathcal{M}_{\mathfrak{Y}} = k^{\times} \times P$, and $\mathcal{M}_{\mathfrak{Y}'} = k^{\times} \times P'$, where P, P', Q are saturated sharp monoids. The monoid map $\mathcal{M}_{\mathfrak{Y}} \rightarrow \mathcal{M}_{\mathfrak{X}}$ induced by f has the form of $(a, p) \mapsto (a\alpha(p), \beta(p))$ for a monoid map $\alpha: P \rightarrow k^{\times}$ and a Kummer monoid map $\beta: P \rightarrow Q$. Writing Q as the union of fs submonoids of Q containing $\beta(P)$, one can write $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ as an inverse limit of fs log schemes which are Kummer over $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$. In a similar way, we can write $(\mathfrak{Y}', \mathcal{M}_{\mathfrak{Y}'})$ as an inverse limit of fs log schemes over $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$. Then the assertion follows from the result in fs cases ([Kat21, Lemma 2.4 (2)]) and the limit argument. $\square$

6.2.2. Kfl sites of fs log formal schemes. First, we recall the definition of log flat morphisms of fs log schemes introduced by Kato in [Kat21].

Definition 6.2.4 (Kummer log flat morphisms of log formal schemes). Let $f: (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ be an adic morphism of fs log formal schemes.

- (1) The morphism f is called *log flat* if, for any open subset $\mathfrak{U}$ of $\mathfrak{Y}$ and any finitely generated ideal of definition $\mathcal{I}$, the morphism of log schemes induced from f

$$(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \times_{(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})} (U, \mathcal{M}_U) \rightarrow (U, \mathcal{M}_U)$$

is log flat, where $(U, \mathcal{M}_U)$ is the strict closed subscheme of $(\mathfrak{U}, \mathcal{M}_{\mathfrak{U}})$ defined by $\mathcal{I}$.

- (2) The morphism f is called *Kummer log flat* (or *kfl* for short) if f is Kummer and log flat.

Lemma 6.2.5 (cf. [Ogu18, Kat21]). The following statements are true.

- (1) A strict and log flat morphism of fs log formal schemes is strict flat.
- (2) The composite of log flat morphisms of fs log formal schemes is also log flat.

- (3) Let $f: (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ be a log flat morphism of fs log formal schemes, and let $(\mathfrak{Y}', \mathcal{M}_{\mathfrak{Y}'}) \rightarrow (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ be an adic morphism of fs log formal schemes. Then the morphism

$$(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \times_{(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})}^{\text{sat}} (\mathfrak{Y}', \mathcal{M}_{\mathfrak{Y}'}) \rightarrow (\mathfrak{Y}', \mathcal{M}_{\mathfrak{Y}'})$$

induced from f is log flat.

- (4) For a kfl morphism of fs log formal schemes $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ that is locally of finite presentation, the morphism $\mathfrak{X} \rightarrow \mathfrak{Y}$ is open.

Proof. Taking the reduction by an ideal of definition allows us to assume that every formal scheme is a scheme. For the proof in this case, see [Ogu18, Chapter IV, Proposition 4.1.2] and [Kat21, Lemma 2.4, Proposition 2.5]. $\square$

Definition 6.2.6. A family of morphisms of fs log formal schemes $\{f_i: (\mathfrak{U}_i, \mathcal{M}_{\mathfrak{U}_i}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})\}_{i \in I}$ is called a *Kummer log flat covering* (or a *kfl covering* for short) if the following conditions are satisfied.

- (1) Each f_i is kfl and locally of finite presentation.
- (2) The family is set-theoretically surjective (i.e. $\mathfrak{X} = \bigcup_{i \in I} f_i(\mathfrak{U}_i)$).

Definition 6.2.7 (Kfl sites, (cf. [Kat21, Definition 2.3])). We let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{kfl}}$ denote the category of fs log (not formal) schemes that are adic over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ equipped with the Grothendieck topology given by kfl coverings. This topology is called *kfl topology*, and the resulting site $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{kfl}}$ is called a *kfl site*.

Proposition 6.2.8 (cf. [Kat21, Theorem 3.1]). Let $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be an adic morphism of fs log formal schemes. Then the presheaf on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{kfl}}$ given by

$$(Z, \mathcal{M}_Z) \mapsto \text{Mor}_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})}((Z, \mathcal{M}_Z), (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}))$$

for $(Z, \mathcal{M}_Z) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{kfl}}$ is a sheaf. Here, $\text{Mor}_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})}((Z, \mathcal{M}_Z), (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}))$ denotes the set of morphisms from $(Z, \mathcal{M}_Z)$ to $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$.

Proof. This immediately follows from [Kat21, Theorem 3.1]. $\square$

Remark 6.2.9. A formal scheme $\mathfrak{Y}$ over $\mathfrak{X}$ is regarded as a log formal scheme over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ by equipping $\mathfrak{Y}$ with the pullback log structure of $\mathcal{M}_{\mathfrak{X}}$. Hence, Proposition 6.2.8 gives natural fully faithful functors

$$\text{FSch}_{/\mathfrak{X}} \hookrightarrow \text{LFSch}_{/(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})} \hookrightarrow \text{Shv}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{kfl}}).$$

Here, $\text{FSch}_{/\mathfrak{X}}$ denotes the category of formal schemes adic over $\mathfrak{X}$, and $\text{LFSch}_{/(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})}$ denotes the category of fs log formal schemes adic over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$.

Definition 6.2.10. We define a presheaf $\mathcal{O}_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})}$ on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{kfl}}$ by

$$\Gamma((Y, \mathcal{M}_Y), \mathcal{O}_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})}) := \Gamma(Y, \mathcal{O}_Y)$$

for $(Y, \mathcal{M}_Y) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{kfl}}$. By Proposition 6.2.8, this is a sheaf on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{kfl}}$.

6.2.3. Kfl vector bundles on fs log formal schemes. In this subsection, we generalize the notion and properties of kfl vector bundles on lg schemes to the case of log formal schemes.

Definition 6.2.11 (Kfl vector bundles). Locally free sheaves of finite rank on the ringed site $((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{kfl}}, \mathcal{O}_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})})$ are called *kfl vector bundles*, and the category of kfl vector bundles on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is denoted by $\text{Vect}_{\text{kfl}}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. We equip $\text{Vect}_{\text{kfl}}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ with the structure of exact category derived from the exactness as sheaves.

For a morphism of fs log formal schemes $f: (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$, the exact pullback functor

$$f^*: \text{Vect}_{\text{kfl}}(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow \text{Vect}_{\text{kfl}}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$$

is naturally defined. For a kfl vector bundle $\mathcal{E}$ on $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$, the pullback $f^*\mathcal{E}$ is also denoted by $\mathcal{E}|_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})}$.

Lemma 6.2.12. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be an fs log formal scheme with a finitely generated ideal of definition $\mathcal{I}$. Let $(X_n, \mathcal{M}_{X_n})$ be the strict closed subscheme of $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ defined by $\mathcal{I}^{n+1}$ for $n \geq 0$. Then the natural functor

$$\text{Vect}_{\text{kfl}}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow \varprojlim_{n \geq 0} \text{Vect}_{\text{kfl}}(X_n, \mathcal{M}_{X_n})$$

gives a bi-exact equivalence.

Proof. For an integer $n \geq 0$, $(X_n, \mathcal{M}_{X_n})_{\text{kfl}}$ is naturally regarded as the full subcategory of $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{kfl}}$, and the full subcategory $\bigcup_{n \geq 0} (X_n, \mathcal{M}_{X_n})_{\text{kfl}}$ forms a basis of $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{kfl}}$, which implies the assertion. $\square$

Definition 6.2.13. For a vector bundle $\mathcal{E}$ on $\mathfrak{X}$, we define a presheaf $\iota(\mathcal{E})$ on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{kfl}}$ by

$$\Gamma((Y, \mathcal{M}_Y), \iota(\mathcal{E})) := \Gamma(Y, f^*\mathcal{E}),$$

where $(Y, \mathcal{M}_Y)$ is an object of $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{kfl}}$ with structure morphism $f: (Y, \mathcal{M}_Y) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. By [Niz08, Proposition 2.19], $\iota(\mathcal{E})$ is a sheaf. Furthermore, $\iota(\mathcal{E})$ is a kfl vector bundle on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$.

Lemma 6.2.14. The functor

$$\iota: \text{Vect}(\mathfrak{X}) \rightarrow \text{Vect}_{\text{kfl}}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$$

gives a bi-exact equivalence to the essential image.

Proof. Lemma 6.2.12 allows us to assume that $\mathfrak{X}$ is a scheme, in which case the assertion follows from Lemma 3.2.6. $\square$

We regard $\text{Vect}(\mathfrak{X})$ as a full subcategory of $\text{Vect}_{\text{kfl}}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ via ι , and we simply write $\mathcal{E}$ for $\iota(\mathcal{E})$ when there is no possibility of confusion.

Definition 6.2.15. A kfl vector bundle $\mathcal{E}$ on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is said *classical* if $\mathcal{E}$ belongs to $\text{Vect}(\mathfrak{X})$.

Lemma 6.2.16 (cf. [Kat21, Theorem 6.2]). Let $\mathcal{E}$ be a kfl vector bundle on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. For a point $x \in \mathfrak{X}$ with residue field $k(x)$, we let $\mathcal{M}_{k(x)}$ denote the log structure on $\text{Spec}(k(x))$ induced from $\mathcal{M}_{\mathfrak{X}}$. Suppose that the pullback $\mathcal{E}|_{(\text{Spec}(k(x)), \mathcal{M}_{k(x)})}$ is classical for a point $x \in \mathfrak{X}$. Then there exists an open neighborhood $\mathfrak{U}$ of x such that $\mathcal{E}|_{(\mathfrak{U}, \mathcal{M}_{\mathfrak{U}})}$ is classical. In particular, if $\mathcal{E}|_{(\text{Spec}(k(x)), \mathcal{M}_{k(x)})}$ is classical for every $x \in \mathfrak{X}$, then $\mathcal{E}$ itself is classical.

Proof. By shrinking $\mathfrak{X}$ to an open neighborhood of x , we may assume that $\mathfrak{X}$ admits a finitely generated ideal of definition $\mathcal{I}$. Let $(X_n, \mathcal{M}_{X_n})$ be the strict closed subscheme of $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ defined by $\mathcal{I}^{n+1}$ for an integer $n \geq 0$. Applying Proposition 3.2.9 to the kfl vector bundle $\mathcal{E}|_{(X_0, \mathcal{M}_{X_0})}$ on $(X_0, \mathcal{M}_{X_0})$, we see that there exists an open neighborhood $\mathfrak{U}$ of x such

that $\mathcal{E}|_{(U_0, \mathcal{M}_{U_0})}$ is classical, where we set $(U_n, \mathcal{M}_{U_n})$ to be the strict closed subscheme of $(\mathfrak{U}, \mathcal{M}_{\mathfrak{U}})$ defined by $\mathcal{I}^{n+1}$ for $n \geq 0$. Using Proposition 3.2.9 again, we see that $\mathcal{E}|_{(U_n, \mathcal{M}_{U_n})}$ is classical for every integer $n \geq 0$. Then the system $\{\mathcal{E}|_{(U_n, \mathcal{M}_{U_n})}\}_{n \geq 0}$ corresponds to a (usual) vector bundle $\mathcal{E}'$ on $\mathfrak{X}$, which is isomorphic to $\mathcal{E}|_{(\mathfrak{U}, \mathcal{M}_{\mathfrak{U}})}$ by Lemma 6.2.12. In particular, $\mathcal{E}|_{(\mathfrak{U}, \mathcal{M}_{\mathfrak{U}})}$ is classical. $\square$

Lemma 6.2.17. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a quasi-compact fs log formal scheme with an fs chart $P \rightarrow \mathcal{M}_{\mathfrak{X}}$, and let $\mathcal{E}$ be a kfl vector bundle on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Then the pullback of $\mathcal{E}$ by a kfl covering

$$(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \otimes_{(\mathbb{Z}[P], P)^a} (\mathbb{Z}[P^{1/n}], P^{1/n})^a \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$$

is classical for some $n \geq 1$.

Proof. By working Zariski locally on $\mathfrak{X}$, we may assume that $\mathfrak{X}$ admits a finitely generated ideal of definition $\mathcal{I}$. Let $(X_0, \mathcal{M}_{X_0})$ denote the strict closed subscheme defined by $\mathcal{I}$. By Lemma 3.2.7, the pullback of $\mathcal{E}$ to $(X_0, \mathcal{M}_{X_0}) \otimes_{(\mathbb{Z}[P], P)^a} (\mathbb{Z}[P^{1/n}], P^{1/n})^a$ is classical for some $n \geq 1$. Hence, the pullback of $\mathcal{E}$ to $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \otimes_{(\mathbb{Z}[P], P)^a} (\mathbb{Z}[P^{1/n}], P^{1/n})^a$ is also classical by Lemma 6.2.16. $\square$

Proposition 6.2.18 (cf. [Niz08, Proposition 3.29]). Consider an exact sequence of kfl vector bundles on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$

$$0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_2 \rightarrow \mathcal{E}_3 \rightarrow 0.$$

Then $\mathcal{E}_2$ is classical if and only if $\mathcal{E}_1$ and $\mathcal{E}_3$ are classical.

Proof. By Lemma 6.2.16, we are reduced to the case where $\mathfrak{X}$ is a scheme. In this case, the assertion is proved in [Niz08, Proposition 3.29]. $\square$

Our goal of the rest of this subsection is the proof of a certain saturated descent property of kfl vector bundles (Proposition 6.2.26). To formulate it, we generalize the formalism of pullback functors for kfl vector bundles to non-fs cases. In the rest of this section, let $f: (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be an adic morphism from a (not necessarily fs) saturated log formal scheme $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ to an fs log formal scheme $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$.

Definition 6.2.19. Take a geometric point $\bar{x}$ on $\mathfrak{X}$, and let $\mathcal{M}_{\bar{x}}$ denote the pullback log structure of $\mathcal{M}_{\mathfrak{X}}$. [Kat21, Theorem 6.2] implies that the isomorphism classes of kfl vector bundles on $(\bar{x}, \mathcal{M}_{\bar{x}})$ of rank r is naturally in bijection with

$$(\overline{\mathcal{M}_{\bar{x}}}^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z})^r / S_r$$

as pointed sets, where the permutation group S_r acts by permutations. For a kfl vector bundle $\mathcal{E}$ of rank r on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$, the element of $(\overline{\mathcal{M}_{\bar{x}}}^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z})^r / S_r$ corresponding to $\mathcal{E}|_{(\bar{x}, \mathcal{M}_{\bar{x}})}$ via the above bijection is denoted by $c(\mathcal{E}, \bar{x})$.

Lemma 6.2.20. Let $\mathcal{E}$ be a kfl vector bundle of rank r on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Let $\bar{y}$ be a geometric point on $\mathfrak{Y}$, and let $\bar{x} := f(\bar{y})$. The following statements are equivalent.

- (1) The image of $c(\mathcal{E}, \bar{x})$ by the map induced from f

$$(\overline{\mathcal{M}_{\bar{x}}}^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z})^r / S_r \rightarrow (\overline{\mathcal{M}_{\bar{y}}}^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z})^r / S_r$$

is trivial.

- (2) There exist a strict fppf morphism $p: (\mathfrak{Z}, \mathcal{M}_{\mathfrak{Z}}) \rightarrow (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ with $y \in p(\mathfrak{Z})$ and a factorization of $f \circ p$

$$(\mathfrak{Z}, \mathcal{M}_{\mathfrak{Z}}) \xrightarrow{g} (\mathfrak{Z}, \mathcal{M}_{\mathfrak{Z}}) \xrightarrow{h} (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$$

such that $(\mathfrak{Z}, \mathcal{M}_\mathfrak{Z})$ is fs and $h^*\mathcal{E}$ is classical.

Proof. The implication (2) $\Rightarrow$ (1) is obvious. We shall prove the implication (1) $\Rightarrow$ (2). Let $f^*: \Gamma(\mathfrak{X}, \mathcal{M}_\mathfrak{X}) \rightarrow \Gamma(\mathfrak{Y}, \mathcal{M}_\mathfrak{Y})$ denote the monoid map induced from f . We set $P := \overline{\mathcal{M}_{\mathfrak{X}, \bar{x}}}$, $Q := \overline{\mathcal{M}_{\mathfrak{Y}, \bar{y}}}$. Write $\alpha: P \rightarrow Q$ for the monoid map induced from f . Let $\beta: P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow Q^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ denote the map induced from α . The element $c(\mathcal{E}, \bar{x})$ is represented by an r -tuple $(c_1, \dots, c_r)$ of elements of $P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$. By Lemma 6.2.21 below, there exist a Kummer map $\gamma: P \rightarrow P'$ of fs sharp monoids and a factorization $P \rightarrow P' \rightarrow Q$ of α such that the image of c_i by $P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow P'^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is contained in P'^{gp} for $1 \leq i \leq r$.

Since P^{gp} and P'^{gp} are finite free, by shrinking $\mathfrak{X}$ and $\mathfrak{Y}$ to an étale neighborhood of $\bar{x}$ and $\bar{y}$ respectively, we may assume that there exist a section $s: P \rightarrow \Gamma(\mathfrak{X}, \mathcal{M}_\mathfrak{X})$ of the natural map $\pi_{\bar{x}}: \Gamma(\mathfrak{X}, \mathcal{M}_\mathfrak{X}) \rightarrow P$ and a monoid map $t: P' \rightarrow \Gamma(\mathfrak{Y}, \mathcal{M}_\mathfrak{Y})$ such that the composite of it with $\pi_{\bar{y}}: \Gamma(\mathfrak{Y}, \mathcal{M}_\mathfrak{Y}) \rightarrow Q$ is equal to the natural map. Although the monoid maps $t \circ \gamma$ and $f^* \circ s$ are not necessarily equal, they coincide after taking the composition with $\pi_{\bar{y}}$. Hence, shrinking $\mathfrak{Y}$ to an étale neighborhood of $\bar{y}$ further, we may assume that there exists a monoid map $u: P \rightarrow \Gamma(\mathfrak{Y}, \mathcal{O}_\mathfrak{Y}^\times)$ such that $f^* \circ s$ coincides with the multiplication of $t \circ \gamma$ and u . By replacing $(\mathfrak{Y}, \mathcal{M}_\mathfrak{Y})$ with its strict finite flat cover

$$(\mathfrak{X}, \mathcal{M}_\mathfrak{X}) \otimes_{u, (\mathbb{Z}[P^{\text{gp}}], P^{\text{gp}})^a} (\mathbb{Z}[P'^{\text{gp}}], P'^{\text{gp}})^a,$$

we are reduced to the case that u extends to a monoid map $u': P' \rightarrow \Gamma(\mathfrak{Y}, \mathcal{O}_\mathfrak{Y}^\times)$. Replacing t with its multiplication with u' , we may assume that $t \circ \gamma$ and $f^* \circ s$ are equal. Then, if we set

$$(\mathfrak{Z}, \mathcal{M}_\mathfrak{Z}) := (\mathfrak{X}, \mathcal{M}_\mathfrak{X}) \otimes_{(\mathbb{Z}[P], P)^a} (\mathbb{Z}[P'], P')^a,$$

the morphism f factors as

$$(\mathfrak{Y}, \mathcal{M}_\mathfrak{Y}) \xrightarrow{g} (\mathfrak{Z}, \mathcal{M}_\mathfrak{Z}) \xrightarrow{h} (\mathfrak{X}, \mathcal{M}_\mathfrak{X}).$$

By the construction of P' , the pullback of $h^*\mathcal{E}$ to $(\bar{z}, \mathcal{M}_{\bar{z}})$ is classical, where $\bar{z} := g(\bar{y})$. By Lemma 6.2.16, replacing $\mathfrak{Z}$ with an open neighborhood of z and $\mathfrak{Y}$ with its preimage by g allows us to assume that $h^*\mathcal{E}$ is classical. This proves the assertion. $\square$

Lemma 6.2.21. Let $\alpha: P \rightarrow Q$ be a monoid map from an fs sharp monoid P to a saturated sharp monoid Q . Write $\beta: P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow Q^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ for the induced map. Let $c_1, \dots, c_r$ be elements of $P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ with $\beta(c_i) \in Q^{\text{gp}}$ for every $1 \leq i \leq r$. Then there exists a Kummer map of fs sharp monoids $P \rightarrow P'$ such that α factors as $P \rightarrow P' \rightarrow Q$ and the image of c_i by $P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow P'^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is contained in P'^{gp} for $1 \leq i \leq r$.

Proof. First, we claim that $\beta^{-1}(Q^{\text{gp}}) = \beta^{-1}(Q)^{\text{gp}}$. Since $P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is written as $P_{\mathbb{Q}_{\geq 0}} - P$, we get

$$\beta^{-1}(Q^{\text{gp}}) = (P_{\mathbb{Q}_{\geq 0}} \cap \beta^{-1}(Q^{\text{gp}})) - P.$$

Since Q is saturated, $P_{\mathbb{Q}_{\geq 0}} \cap \beta^{-1}(Q^{\text{gp}})$ is equal to $\beta^{-1}(Q)$. Hence, we obtain $\beta^{-1}(Q^{\text{gp}}) \subset \beta^{-1}(Q)^{\text{gp}}$. The inverse inclusion is trivial.

Write the saturated monoid $\beta^{-1}(Q)$ as the union $\bigcup_{j \in J} P_j$ where P_j runs over all fs submonoids of $\beta^{-1}(Q)$ containing P . Since the claim we proved in the previous paragraph implies $\beta^{-1}(Q^{\text{gp}}) = \bigcup_{j \in J} P_j^{\text{gp}}$, there exists $j_0 \in J$ such that c_i is contained in $P_{j_0}^{\text{gp}}$ for every $1 \leq i \leq r$. Then the Kummer map $P \rightarrow P_{j_0}$ is the desired one. $\square$

Definition 6.2.22. Let $\mathcal{E}$, $\bar{x}$, and $\bar{y}$ be as in Lemma 6.2.20. We say that $\mathcal{E}$ is *f-classical at y* if the equivalent conditions in Lemma 6.2.20 are satisfied. This terminology is justified because this property is independent of a choice of $\bar{y}$ above $y \in \mathfrak{Y}$. Lemma 6.2.20 implies

that, if $\mathcal{E}$ is f -classical at y , there exists an open neighborhood $\mathfrak{V}$ of y such that $\mathcal{E}$ is f -classical at every $y' \in \mathfrak{V}$. When $\mathcal{E}$ is f -classical at y for every $y \in \mathfrak{Y}$, the kfl vector bundle $\mathcal{E}$ is called *f -classical*.

Let $\mathrm{Vect}_{\mathrm{kfl}}^{f\text{-cl}}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ denote the full subcategory of $\mathrm{Vect}_{\mathrm{kfl}}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ consisting of f -classical objects. We define a pullback functor

$$f^*: \mathrm{Vect}_{\mathrm{kfl}}^{f\text{-cl}}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow \mathrm{Vect}(\mathfrak{Y})$$

as follows. Let $\mathcal{E} \in \mathrm{Vect}_{\mathrm{kfl}}^{f\text{-cl}}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. By working fppf locally on $\mathfrak{Y}$, we may assume that f admits a factorization as in Lemma 6.2.20 (2). In this situation, we set

$$f^*\mathcal{E} := g^*(h^*\mathcal{E}),$$

where g^* is the pullback functor for (non-log) vector bundles along $g: \mathfrak{Y} \rightarrow \mathfrak{Z}$. This definition is independent of the choice of a factorization $f = hg$ because two such factorizations are dominated by another factorization (for example, by taking the fiber product). When $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ is fs, a kfl vector bundle $\mathcal{E}$ on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is f -classical if and only if the pullback of $\mathcal{E}$ along f in the usual sense is classical because the attachment of $c(\mathcal{E}, \bar{x})$ is functorial. In this case, $f^*\mathcal{E}$ coincides with the pullback of $\mathcal{E}$ along f in the usual sense.

Lemma 6.2.23. Consider a sequence of adic morphisms of log formal schemes

$$(\mathfrak{Z}, \mathcal{M}_{\mathfrak{Z}}) \xrightarrow{g} (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \xrightarrow{f} (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \xrightarrow{h} (\mathfrak{W}, \mathcal{M}_{\mathfrak{W}}),$$

where $(\mathfrak{Z}, \mathcal{M}_{\mathfrak{Z}})$ is saturated and $(\mathfrak{W}, \mathcal{M}_{\mathfrak{W}})$ is fs. Let $\mathcal{E}$ be a kfl vector bundle on $(\mathfrak{W}, \mathcal{M}_{\mathfrak{W}})$ with $h^*\mathcal{E}$ being f -classical. Then $\mathcal{E}$ is hfg -classical, and there is a natural isomorphism of vector bundles on $\mathfrak{Z}$

$$(hfg)^*\mathcal{E} \cong g^*(f^*(h^*\mathcal{E})).$$

Conversely, if $\mathcal{E}$ is hfg -classical and g is a strict surjection, then $h^*\mathcal{E}$ is f -classical.

Proof. All assertions follow directly from the definition. $\square$

Lemma 6.2.24. Suppose that f admits a chart of the form $P \rightarrow P_{\mathbb{Q}_{\geq 0}}$ for an fs monoid P . Then every kfl vector bundle $\mathcal{E}$ on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is f -classical.

Proof. Since the map f factors as

$$(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \otimes_{(\mathbb{Z}[P], P)^a} (\mathbb{Z}[P^{1/n}], P^{1/n})^a \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$$

for each integer $n \geq 1$, the assertion follows from Lemma 6.2.17. $\square$

Definition 6.2.25. We say that an adic morphism $f: (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ from saturated log formal scheme $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ to an fs log formal scheme $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is a *strongly pro-kfl morphism* if there exists a chart $P \rightarrow Q$ of f such that P is fs, Q is saturated, $P \rightarrow Q$ is Kummer, and the induced map

$$(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \otimes_{(\mathbb{Z}[P], P)^a} (\mathbb{Z}[Q], Q)^a$$

is strict flat. The morphism f is called a *strongly pro-kfl covering* if f is strongly pro-kfl, quasi-compact, and surjective.

Proposition 6.2.26 (Strongly pro-kfl descent for kfl vector bundles). Let $f: (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a strongly pro-kfl covering. Let $(\mathfrak{Y}^{(\bullet)}, \mathcal{M}_{\mathfrak{Y}^{(\bullet)}})$ denote the Čech nerve of f in the category of saturated log formal schemes. Then there exists a natural bi-exact equivalence

$$\mathrm{Vect}_{\mathrm{kfl}}^{f\text{-cl}}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \xrightarrow{\sim} \varprojlim_{\bullet \in \Delta} \mathrm{Vect}(\mathfrak{Y}^{(\bullet)}).$$

Proof. By Lemma 6.2.23, the pullback functor f^* induces an exact functor

$$\mathrm{Vect}_{\mathrm{kfl}}^{f\text{-cl}}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow \varprojlim_{\bullet \in \Delta} \mathrm{Vect}(\mathfrak{Y}^{(\bullet)}).$$

We shall prove that this functor is a bi-exact equivalence.

Fix a chart $P \rightarrow Q$ of f as in Definition 6.2.25. Set

$$(\mathfrak{Z}, \mathcal{M}_{\mathfrak{Z}}) := (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \otimes_{(\mathbb{Z}[P], P)^a} (\mathbb{Z}[Q], Q)^a,$$

and let $(\mathfrak{Z}^{(\bullet)}, \mathcal{M}_{\mathfrak{Z}^{(\bullet)}})$ denote its Čech nerve in the category of saturated log formal schemes over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Consider the following diagram in which every square is Cartesian:

$$\begin{array}{ccc} (\mathfrak{W}, \mathcal{M}_{\mathfrak{W}}) & \longrightarrow & (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \\ \downarrow & & \downarrow \\ (\mathfrak{Z}^{(1)}, \mathcal{M}_{\mathfrak{Z}^{(1)}}) & \longrightarrow & (\mathfrak{Z}, \mathcal{M}_{\mathfrak{Z}}) \\ \downarrow & & \downarrow g \\ (\mathfrak{Z}, \mathcal{M}_{\mathfrak{Z}}) & \xrightarrow{g} & (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}). \end{array}$$

Since there is an isomorphism

$$Q \oplus_P^{\mathrm{sat}} Q \cong Q \oplus Q^{\mathrm{gp}} / P^{\mathrm{gp}},$$

both projection maps $(\mathfrak{Z}^{(1)}, \mathcal{M}_{\mathfrak{Z}^{(1)}}) \rightarrow (\mathfrak{Z}, \mathcal{M}_{\mathfrak{Z}})$ are strict fpqc coverings. By Lemma 6.2.3, the composite of the left vertical maps is surjective. Hence, the upper horizontal map and the composite of left vertical maps are strict fpqc coverings. Therefore, due to fpqc descent for (usual) vector bundles, it is enough to show that the functor

$$\mathrm{Vect}_{\mathrm{kfl}}^{f\text{-cl}}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow \varprojlim_{\bullet \in \Delta} \mathrm{Vect}(\mathfrak{Z}^{(\bullet)})$$

induced from g^* is a bi-exact equivalence. This claim makes sense because we see that a kfl vector bundle on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is f -classical if and only if g -classical by considering the condition Lemma 6.2.20 (1).

Lemma 6.2.12 reduces the problem to the case that $\mathfrak{X}$ is a scheme. Write Q as a union $\bigcup_{i \in I} P_i$, where P_i runs over all fs submonoids of Q including P . Set

$$(\mathfrak{Z}_i, \mathcal{M}_{\mathfrak{Z}_i}) := (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \otimes_{(\mathbb{Z}[P], P)^a} (\mathbb{Z}[P_i], P_i)^a,$$

and let $(\mathfrak{Z}_i^{(\bullet)}, \mathcal{M}_{\mathfrak{Z}_i^{(\bullet)}})$ denote its Čech nerve in the category of saturated log schemes over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Let $g_i: (\mathfrak{Z}_i, \mathcal{M}_{\mathfrak{Z}_i}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ denote the structure map. Then we get bi-exact equivalences

$$\begin{aligned} \mathrm{Vect}_{\mathrm{kfl}}^{g\text{-cl}}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) &\simeq \varinjlim_{i \in I} \mathrm{Vect}_{\mathrm{kfl}}^{g_i\text{-cl}}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \\ &\simeq \varinjlim_{i \in I} \varprojlim_{\bullet \in \Delta} \mathrm{Vect}(\mathfrak{Z}_i^{(\bullet)}) \\ &\simeq \varprojlim_{\bullet \in \Delta} \varinjlim_{i \in I} \mathrm{Vect}(\mathfrak{Z}_i^{(\bullet)}) \\ &\simeq \varprojlim_{\bullet \in \Delta} \mathrm{Vect}(\mathfrak{Z}^{(\bullet)}), \end{aligned}$$

where the first equivalence follows from the condition Lemma 6.2.20 (1) because $(\mathfrak{Z}, \mathcal{M}_{\mathfrak{Z}})$ is isomorphic to the inverse limit of $(\mathfrak{Z}_i, \mathcal{M}_{\mathfrak{Z}_i})$ for $i \in I$, and the forth one follows from the limit argument for (usual) vector bundles on schemes. $\square$

6.3. Log p -divisible groups on fs formal log schemes

Throughout this section, let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be an fs log formal scheme.

6.3.1. Log finite group schemes. In this subsection, we introduce the notion of log finite group schemes over log formal schemes and study some properties.

Definition 6.3.1 (cf. [Kat23, Definition 1.3 and §1.6]). Let $n \geq 1$ be an integer. Assume that p is topologically nilpotent on $\mathfrak{X}$ when $n = 1$.

- (1) Let G be a sheaf of abelian groups on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{kfl}}$. We call G a *weak log finite group scheme* (resp. a *weak log truncated Barsotti-Tate group scheme of level n*) if there exists a kfl covering $\{(\mathfrak{U}_i, \mathcal{M}_{\mathfrak{U}_i}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})\}_{i \in I}$ such that the restriction of G to $(\mathfrak{U}_i, \mathcal{M}_{\mathfrak{U}_i})_{\text{kfl}}$ belongs to $\text{Fin}(\mathfrak{U}_i)$ (resp. $\text{BT}_n(\mathfrak{U}_i)$) via the inclusion functors in Remark 6.2.9 for each $i \in I$. We let $\text{wFin}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ (resp. $\text{wBT}_n(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$) denote the category of weak log finite group schemes (resp. weak log truncated Barsotti-Tate group schemes of level n) over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. The category $\text{Fin}(\mathfrak{X})$ (resp. $\text{BT}_n(\mathfrak{X})$) is regarded as the full subcategory of $\text{wFin}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ (resp. $\text{wBT}_n(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$), and an object $G \in \text{wFin}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is called *classical* if G belongs to $\text{Fin}(\mathfrak{X})$.
- (2) For a weak log finite group scheme G over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$, we define a sheaf G^* on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{kfl}}$ as the sheaf of homomorphisms $\mathcal{H}om(G, \mathbb{G}_m)$, called the *Cartier dual* of G . By working kfl locally on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$, we see that G^* is a weak log finite group scheme over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$.
- (3) We say that a weak log finite group scheme G over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is a *log finite group scheme* if G and G^* are representable by finite and kfl log formal schemes of finite presentation over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. We let $\text{Fin}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ denote the category of log finite group schemes over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$.
- (4) An object $G \in \text{wFin}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is called a *log truncated Barsotti-Tate group schemes of level n* if G belongs to both $\text{Fin}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ and $\text{wBT}_n(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. We let $\text{BT}_n(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ denote the category of log truncated Barsotti-Tate group schemes of level n over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$.

For a morphism of fs log formal schemes $f: (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$, the natural cocontinuous functor $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{kfl}} \rightarrow (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})_{\text{kfl}}$ induces an exact pullback functor

$$f^*: \text{wFin}(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow \text{wFin}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}).$$

For a weak log finite group scheme G over $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$, the pullback f^*G is also denoted by $G|_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})}$.

Lemma 6.3.2. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a quasi-compact fs log formal scheme with an fs chart $P \rightarrow \mathcal{M}_{\mathfrak{X}}$, and let G be a weak log finite group scheme over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Then the pullback of G by a kfl covering

$$(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \otimes_{(\mathbb{Z}[P], P)^a} (\mathbb{Z}[P^{1/n}], P^{1/n})^a \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$$

is classical for some $n \geq 1$.

Proof. This follows from Lemma 6.2.2 and flat descent for (non-log) finite locally free group schemes. $\square$

Lemma 6.3.3. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be an fs log formal scheme with a finitely generated ideal of definition $\mathcal{I}$. Let $(X_n, \mathcal{M}_{X_n})$ be the strict closed subscheme of $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ defined by $\mathcal{I}^{n+1}$ for $n \geq 0$. Let $m \geq 1$ be an integer, and assume that p is topologically nilpotent on $\mathfrak{X}$ when $m = 1$. Then the natural functors

$$(6.3.1) \quad \mathrm{wFin}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow \varprojlim_n \mathrm{wFin}(X_n, \mathcal{M}_{X_n})$$

$$(6.3.2) \quad \mathrm{Fin}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow \varprojlim_n \mathrm{Fin}(X_n, \mathcal{M}_{X_n})$$

$$(6.3.3) \quad \mathrm{wBT}_m(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow \varprojlim_n \mathrm{wBT}_m(X_n, \mathcal{M}_{X_n})$$

$$(6.3.4) \quad \mathrm{BT}_m(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow \varprojlim_n \mathrm{BT}_m(X_n, \mathcal{M}_{X_n})$$

give bi-exact equivalences.

Proof. Since the full subcategory $\bigcup_{n \geq 0} (X_n, \mathcal{M}_{X_n})_{\mathrm{kfl}}$ of $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\mathrm{kfl}}$ forms a basis, there is a natural bi-exact equivalence

$$(6.3.5) \quad \mathrm{Shv}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\mathrm{kfl}}) \xrightarrow{\sim} \varprojlim_n \mathrm{Shv}((X_n, \mathcal{M}_{X_n})_{\mathrm{kfl}}),$$

where $\mathrm{Shv}(\mathcal{C})$ denotes the category of sheaves of abelian groups on $\mathcal{C}$ for a site $\mathcal{C}$. Hence, the functor (6.3.1) gives a bi-exact equivalence to the essential image. We shall prove the essential surjectivity. Let $\{G_n\}_{n \geq 0}$ be an object of $\varprojlim_{n \geq 0} \mathrm{wFin}(X_n, \mathcal{M}_{X_n})$. In other words, $\{G_n\}_{n \geq 0}$ is a family of weak log finite group schemes G_n on $(X_n, \mathcal{M}_{X_n})$ with isomorphisms $G_{n+1}|_{(X_n, \mathcal{M}_{X_n})} \cong G_n$ for each $n \geq 0$. Due to the equivalence (6.3.5), there is a unique sheaf G on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\mathrm{kfl}}$ such that the system $\{G|_{(X_n, \mathcal{M}_{X_n})}\}_{n \geq 0}$ is isomorphic to $\{G_n\}_{n \geq 0}$. It is enough to prove that G is a weak log finite group scheme. By working étale locally on $\mathfrak{X}$, we may assume that $\mathfrak{X}$ is quasi-compact and there exists an fs chart $P \rightarrow \mathcal{M}_{\mathfrak{X}}$. Lemma 6.3.2 implies that the pullback of G_0 to $(X_0, \mathcal{M}_{X_0}) \otimes_{(\mathbb{Z}[P], P)^a} (\mathbb{Z}[P^{1/m}], P^{1/m})^a$ is classical for some $m \geq 1$. We set

$$\begin{aligned} (\mathfrak{X}', \mathcal{M}_{\mathfrak{X}'}) &:= (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \otimes_{(\mathbb{Z}[P], P)^a} (\mathbb{Z}[P^{1/m}], P^{1/m})^a \\ (X'_n, \mathcal{M}_{X'_n}) &:= (X_n, \mathcal{M}_{X_n}) \otimes_{(\mathbb{Z}[P], P)^a} (\mathbb{Z}[P^{1/m}], P^{1/m})^a \end{aligned}$$

for each integer $n \geq 0$. By Proposition 3.2.24, the pullback $G'_n := G_n|_{(X'_n, \mathcal{M}_{X'_n})}$ is classical for each $n \geq 0$. Let G' be a unique finite locally free group scheme over $\mathfrak{X}'$ such that the system $\{G'|_{X'_n}\}_{n \geq 0}$ is isomorphic to $\{G'_n\}_{n \geq 0}$. By the equivalence (6.3.5), there is an isomorphism $G' \cong G|_{(\mathfrak{X}', \mathcal{M}_{\mathfrak{X}'})}$, which is classical. This proves that G is a weak log finite group scheme over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. The equivalence of the functor (6.3.3) also follows in the same way.

To complete the proof, it is enough to prove the equivalence of the functor (6.3.2). Since the functor (6.3.1) is an equivalence, we are reduced to showing the following: a weak log finite group scheme H over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ belongs to $\mathrm{Fin}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ if $H_n := H|_{(X_n, \mathcal{M}_{X_n})}$ belongs to $\mathrm{Fin}(X_n, \mathcal{M}_{X_n})$ for each $n \geq 0$. Let $(Z_n, \mathcal{M}_{Z_n})$ be a finite and kfl fs log scheme of finite presentation over $(X_n, \mathcal{M}_{X_n})$ representing H_n . If we set $\mathfrak{Z} := \varinjlim_{n \geq 0} Z_n$, the formal scheme $\mathfrak{Z}$ is finite (in particular, adic) over $\mathfrak{X}$ by [SP25, Tag 09B8]. If we put $\mathcal{M}_{\mathfrak{Z}} := \varprojlim_{n \geq 0} \mathcal{M}_{Z_n}$ under the identification $\mathfrak{Z}_{\mathrm{ét}} \simeq (Z_n)_{\mathrm{ét}}$, the pair $(\mathfrak{Z}, \mathcal{M}_{\mathfrak{Z}})$ is a finite and kfl fs log formal scheme of finite presentation over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ with isomorphisms

$$(\mathfrak{Z}, \mathcal{M}_{\mathfrak{Z}}) \times_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})} (X_n, \mathcal{M}_{X_n}) \cong (Z_n, \mathcal{M}_{Z_n})$$

for every $n \geq 0$. Hence, the sheaf H is representable by $(\mathfrak{Z}, \mathcal{M}_3)$. By the same argument, we see that H^* is also represented by a finite and kfl fs log formal scheme of finite presentation over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Therefore, H belongs to $\text{Fin}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. $\square$

Lemma 6.3.4 (cf. [Kat23, Proposition 2.16]). Let G be a weak log finite group scheme over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. For a point $x \in \mathfrak{X}$ with residue field $k(x)$, we let $\mathcal{M}_{k(x)}$ denote the log structure on $\text{Spec}(k(x))$ induced from $\mathcal{M}_{\mathfrak{X}}$. Suppose that the pullback $G|_{(\text{Spec}(k(x)), \mathcal{M}_{k(x)})}$ is classical (resp. a log finite group scheme) for some point $x \in \mathfrak{X}$. Then there exists an open neighborhood $\mathfrak{U}$ of x such that $G|_{(\mathfrak{U}, \mathcal{M}_{\mathfrak{U}})}$ is classical (resp. a log finite group scheme). In particular, if $G|_{(\text{Spec}(k(x)), \mathcal{M}_{k(x)})}$ is classical (resp. a log finite group scheme) for every $x \in \mathfrak{X}$, then G itself is classical (resp. a log finite group scheme).

Proof. Applying the same argument as the proof of Lemma 6.2.16, we see that the assertion follows from Proposition 3.2.24 and Lemma 6.3.3. $\square$

One can generalize the construction of coordinate rings associated with weak log finite group schemes as follows.

Proposition 6.3.5 (cf. [Kat23, Proposition 2.15]). There is a natural contravariant equivalence between the category of weak log finite group schemes over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ and the category of commutative Hopf objects in the unitary symmetric monoidal category $\text{Vect}_{\text{kfl}}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Moreover, this equivalence restricts to the equivalence between the category of finite locally free group schemes over $\mathfrak{X}$ and the category of Hopf objects of $\text{Vect}(\mathfrak{X})$ given by taking coordinate rings.

Proof. The equivalence in Proposition 3.2.18 gives the equivalence in the assertion by Lemma 6.2.12. The latter assertion follows from the latter assertion of Proposition 3.2.18 and Lemma 6.2.16. $\square$

Lemma 6.3.6. Let G be a log finite group scheme over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ killed by some power of p . Suppose that $\mathfrak{X} = \text{Spf} R$ and, for some (equivalently any) ideal of definition I of R , the ring R/I is a henselian local ring whose residue field is of characteristic p . Then there exists a unique exact sequence

$$0 \rightarrow G^0 \rightarrow G \rightarrow G^{\text{ét}} \rightarrow 0,$$

where G^0 is a (classical) finite locally free connected group scheme over $\mathfrak{X}$ and $G^{\text{ét}}$ is a (classical) finite étale group scheme over $\mathfrak{X}$. Moreover, if $G \in \text{BT}_n(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$, then G^0 and $G^{\text{ét}}$ belong to $\text{BT}_n(\mathfrak{X})$.

Proof. Lemma 6.3.3 allows us to assume that $\mathfrak{X}$ is a scheme, in which case the assertion follows from Proposition 3.2.22. $\square$

Lemma 6.3.7. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be an fs log formal scheme on which $p^n = 0$ for an integer $n \geq 1$. Let G be an object of $\text{wBT}_n(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. We define $\text{Lie}(G)$ as the sheaf on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{kfl}}$ given by

$$\text{Lie}(G)(Y, \mathcal{M}_Y) := \text{Ker}(G(Y[\epsilon], \mathcal{M}_{Y[\epsilon]}) \rightarrow G(Y, \mathcal{M}_Y))$$

for $(Y, \mathcal{M}_Y) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{kfl}}$, where the scheme $Y[\epsilon]$ is defined by $Y[\epsilon] := Y \otimes_{\mathbb{Z}} \mathbb{Z}[t]/(t^2)$ and $\mathcal{M}_{Y[\epsilon]}$ is the pullback log structure of $\mathcal{M}_Y$. Then $\text{Lie}(G)$ is a kfl vector bundle on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Moreover, for an integer $1 \leq m \leq n$ with $p^m \mathcal{O}_{\mathfrak{X}} = 0$, we have an equality

$$\text{Lie}(G) = \text{Lie}(G[p^m]).$$

Proof. Since the formation of $\mathrm{Lie}(G)$ is functorial, working kfl locally on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ allows us to assume that G is classical. In this case, $\mathrm{Lie}(G)$ is the classical kfl vector bundle by [Ill85, Proposition 2.2.1 (b)]. The last assertion follows from [Mes72, Chapter II, Corollary 3.3.16]. $\square$

Definition 6.3.8 (Lie algebras of weak log truncated Barsotti-Tate group schemes). Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$, n , and G be as in Lemma 6.3.7. The kfl vector bundle $\mathrm{Lie}(G)$ defined therein is called the *Lie algebra* of G .

We define the generic fiber of weak log finite group schemes over p -adic fs log formal schemes. To give a formulation over general p -adic fs log formal schemes, we use the formalism of log diamonds developed in [KY25, Chapter 7].

Construction 6.3.9 (Generic fibers of weak log finite group schemes). Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a p -adic fs log formal scheme. Let $\mathcal{C}_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})}$ denote the category of a kfl covering $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ such that the associated morphism $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})_{\eta}^{\diamond} \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta}^{\diamond}$ is a Kummer log étale covering. Since the functor associating fs log diamonds to p -adic fs log formal schemes preserves fiber products, the category $\mathcal{C}_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})}$ has finite products. For a weak log finite group scheme G over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$, there exists $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \in \mathcal{C}_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})}$ such that $G|_{(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})}$ is classical. Indeed, we may assume that $\mathfrak{X}$ is quasi-compact and we are given an fs chart $P \rightarrow \mathcal{M}_{\mathfrak{X}}$, in which case, a kfl covering

$$(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \otimes_{(\mathbb{Z}[P], P)^a} (\mathbb{Z}[P^{1/n}], P^{1/n})^a \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$$

is the desired object of $\mathcal{C}_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})}$ for a sufficiently large $n \geq 1$ by Lemma 6.3.2 and [KY25, Proposition 7.20].

We let $\mathrm{LC}(\mathcal{A})$ denote the category of locally constant sheaves on $\mathcal{A}$ for a site $\mathcal{A}$. Let G and $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ be as above. Let $(\mathfrak{Y}^{(\bullet)}, \mathcal{M}_{\mathfrak{Y}^{(\bullet)}})$ denote the Čech nerve of $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ in the category of p -adic fs log schemes. Taking the generic fiber of finite locally free group schemes $G|_{(\mathfrak{Y}^{(\bullet)}, \mathcal{M}_{\mathfrak{Y}^{(\bullet)}})}$ over $\mathfrak{Y}^{(\bullet)}$ defines an object of the category

$$\varprojlim_{\bullet \in \Delta} \mathrm{LC}((\mathfrak{Y}^{(\bullet)})_{\mathrm{pro\acute{e}t}}^{\diamond}).$$

There is a natural functor

$$(6.3.6) \quad \varprojlim_{\bullet \in \Delta} \mathrm{LC}((\mathfrak{Y}^{(\bullet)})_{\mathrm{pro\acute{e}t}}^{\diamond}) \rightarrow \varprojlim_{\bullet \in \Delta} \mathrm{LC}((\mathfrak{Y}^{(\bullet)}, \mathcal{M}_{\mathfrak{Y}^{(\bullet)}})_{\mathrm{prok\acute{e}t}}^{\diamond}).$$

Kummer log étale descent defines a fully faithful functor

$$(6.3.7) \quad \varprojlim_{\bullet \in \Delta} \mathrm{LC}((\mathfrak{Y}^{(\bullet)}, \mathcal{M}_{\mathfrak{Y}^{(\bullet)}})_{\mathrm{prok\acute{e}t}}^{\diamond}) \rightarrow \mathrm{LC}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\mathrm{prok\acute{e}t}}^{\diamond}).$$

Sending the object of $\varprojlim_{\bullet \in \Delta} \mathrm{LC}((\mathfrak{Y}^{(\bullet)})_{\mathrm{pro\acute{e}t}}^{\diamond})$ constructed above by the composite functor of (6.3.6) and (6.3.7), we obtain an object of $\mathrm{LC}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\mathrm{prok\acute{e}t}}^{\diamond})$, denoted by G_{η} . The sheaf G_{η} is independent of the choice of $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ and is called the *generic fiber* of G . The resulting functor

$$\mathrm{wFin}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow \mathrm{LC}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\mathrm{prok\acute{e}t}}^{\diamond})$$

given by $G \mapsto G_{\eta}$ is exact. When G is a weak log Barsotti-Tate group scheme of level n , the sheaf G_{η} is a $\mathbb{Z}/p^n$ -local system on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \mathrm{qprok\acute{e}t}}^{\diamond}$.

To formulate desired saturated descent properties for weak log finite group schemes (Proposition 6.3.15), we generalize the formalism of pullback functors for weak log finite group schemes to non-fs cases. Arguments proceed in a similar way to the case of kfl vector

bundles. In the rest of this subsection, let $f: (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be an adic morphism from a (not necessarily fs) saturated log formal scheme $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ to an fs log formal scheme $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$.

Lemma 6.3.10. Let G be a weak log finite group scheme over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$, and let $\mathcal{A}_G$ denote the Hopf object of $\text{Vect}_{\text{kfl}}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ corresponding to G via the equivalence in Proposition 6.3.5. Take a point $y \in \mathfrak{Y}$. Then the following statements are equivalent.

- (1) The underlying kfl vector bundle of $\mathcal{A}_G$ is f -classical at y .
- (2) There exist a strict fppf morphism $p: (\mathfrak{V}, \mathcal{M}_{\mathfrak{V}}) \rightarrow (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ with $y \in p(\mathfrak{V})$ and a factorization of fp

$$(\mathfrak{V}, \mathcal{M}_{\mathfrak{V}}) \xrightarrow{g} (\mathfrak{Z}, \mathcal{M}_{\mathfrak{Z}}) \xrightarrow{h} (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$$

such that $(\mathfrak{Z}, \mathcal{M}_{\mathfrak{Z}})$ is fs and $G|_{(\mathfrak{Z}, \mathcal{M}_{\mathfrak{Z}})}$ is classical.

Proof. This follows from Lemma 6.2.20 and Proposition 6.3.5. $\square$

Definition 6.3.11. For a weak log finite group scheme G over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ and a point $y \in \mathfrak{Y}$, we say that G is *f-classical at y* if the equivalent conditions in Lemma 6.3.10 are satisfied. Lemma 6.3.10 implies that, if G is f -classical at y , there exists an open neighborhood $\mathfrak{V}$ of y such that G is f -classical at every $y' \in \mathfrak{V}$. When G is f -classical at every $y \in \mathfrak{Y}$, we say that G is *f-classical*.

Let $\text{wFin}^{f\text{-cl}}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ denote the full subcategory of $\text{wFin}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ consisting of f -classical objects. One can define the pullback functor

$$f^*: \text{wFin}^{f\text{-cl}}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow \text{Fin}(\mathfrak{Y})$$

in a similar way to Definition 6.2.22.

Lemma 6.3.12. Consider a sequence of adic morphisms of log formal schemes

$$(\mathfrak{Z}, \mathcal{M}_{\mathfrak{Z}}) \xrightarrow{g} (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \xrightarrow{f} (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \xrightarrow{h} (\mathfrak{W}, \mathcal{M}_{\mathfrak{W}}),$$

where $(\mathfrak{Z}, \mathcal{M}_{\mathfrak{Z}})$ is saturated and $(\mathfrak{W}, \mathcal{M}_{\mathfrak{W}})$ is fs. Let G be a weak log finite group scheme over $(\mathfrak{W}, \mathcal{M}_{\mathfrak{W}})$ with h^*G being f -classical. Then G is hfg -classical, and there is a natural isomorphism of finite locally free group schemes over $\mathfrak{Z}$

$$(hfg)^*G \cong g^*(f^*(h^*G)).$$

Proof. This follows directly from the definition. $\square$

Lemma 6.3.13. Suppose that f admits a chart of the form $P \rightarrow P_{\mathbb{Q}_{\geq 0}}$ for an fs monoid P . Then every weak log finite group scheme G over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is f -classical.

Proof. This assertion follows from the same argument as Lemma 6.2.24. $\square$

Lemma 6.3.14. Let $n \geq 1$ be an integer with $p^n \mathcal{O}_{\mathfrak{X}} = 0$, and let G be an f -classical object of $\text{wBT}_n(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Then $\text{Lie}(G)$ is also f -classical, and there exists a natural isomorphism of vector bundles on $\mathfrak{Y}$

$$f^*\text{Lie}(G) \cong \text{Lie}(f^*G).$$

Proof. This follows directly from the definition. $\square$

Proposition 6.3.15 (Strongly pro-kfl descent for weak log finite group schemes). Let $f: (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a strongly pro-kfl covering, and $(\mathfrak{Y}^{(\bullet)}, \mathcal{M}_{\mathfrak{Y}^{(\bullet)}})$

be the Čech nerve of f in the category of saturated log formal schemes. Then there exists a natural bi-exact equivalence

$$\mathrm{wFin}^{f\text{-cl}}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \xrightarrow{\sim} \varprojlim_{\bullet \in \Delta} \mathrm{Fin}(\mathfrak{Y}^{(\bullet)})$$

Proof. By Lemma 6.3.12, the functor f^* induces the functors in the assertion, and the equivalence can be proved in the same way as the proof of Proposition 6.2.26. $\square$

6.3.2. Log p -divisible groups. In this subsection, we define the notion of (weak) log p -divisible groups over log formal schemes and summarize properties of them, which are immediately deduced from the results in the previous subsection.

Definition 6.3.16. Let G be a sheaf of abelian groups on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\mathrm{kff}}$. We call G a *weak log p -divisible group* if the following conditions are satisfied.

- (1) A map $\times p: G \rightarrow G$ is surjective.
- (2) For every $n \geq 1$, the sheaf $G[p^n] := \mathrm{Ker}(\times p^n: G \rightarrow G)$ is a weak log finite group scheme over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$.
- (3) $G = \bigcup_{n \geq 1} G[p^n]$.

The category of weak log p -divisible groups over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is denoted by $\mathrm{wBT}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. For a weak log p -divisible group G over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$, the sheaf $G^* := \varinjlim_{n \geq 1} G[p^n]^*$ is a weak log p -divisible group over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. We say that G^* is the *Cartier dual* of G .

A weak log p -divisible group G over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is called a *log p -divisible group* if $G[p^n]$ is a log finite group scheme for each $n \geq 1$. The category of log p -divisible groups over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is denoted by $\mathrm{BT}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. By Remark 6.2.9, the category $\mathrm{BT}(\mathfrak{X})$ is regarded as the full subcategory of $\mathrm{wBT}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. A weak log p -divisible group G over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is called *classical* if G belongs to $\mathrm{BT}(\mathfrak{X})$. Clearly, G is classical if and only if $G[p^n]$ is classical for each $n \geq 1$.

Lemma 6.3.17 (cf. [Kat23, Lemma 4.2]). Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be an fs log formal scheme. A weak log p -divisible group G over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ belongs to $\mathrm{BT}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ if $G[p^n]$ belongs to $\mathrm{Fin}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ for some $n \geq 1$.

Proof. Lemma 6.3.3 and the limit argument (Proposition 3.2.16) reduce us to the case where $\mathfrak{X}$ is a noetherian scheme. In this case, $G[p]$ belongs to $\mathrm{Fin}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ by Lemma 3.2.27, and so the assertion follows from [Kat23, Lemma 4.2]. $\square$

Lemma 6.3.18. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be an fs log formal scheme with a finitely generated ideal of definition $\mathcal{I}$. Let $(X_n, \mathcal{M}_{X_n})$ be the strict closed subscheme of $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ defined by $\mathcal{I}^{n+1}$ for $n \geq 0$. Then the natural functors

$$\begin{aligned} \mathrm{wBT}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) &\rightarrow \varprojlim_n \mathrm{wBT}(X_n, \mathcal{M}_{X_n}) \\ \mathrm{BT}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) &\rightarrow \varprojlim_n \mathrm{BT}(X_n, \mathcal{M}_{X_n}) \end{aligned}$$

give bi-exact equivalences.

Proof. This follows immediately from Lemma 6.3.3. $\square$

Lemma 6.3.19. Let G be a weak log p -divisible group over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. For a point $x \in \mathfrak{X}$ with residue field $k(x)$, we let $\mathcal{M}_{k(x)}$ denote the log structure on $\mathrm{Spec}(k(x))$ induced from $\mathcal{M}_{\mathfrak{X}}$.

- (1) Suppose that the pullback $G|_{(\mathrm{Spec}(k(x)), \mathcal{M}_{k(x)})}$ is a log p -divisible group for some point $x \in \mathfrak{X}$. Then there exists an open neighborhood $\mathfrak{U}$ of x such that $G|_{(\mathfrak{U}, \mathcal{M}_{\mathfrak{U}})}$ is a log p -divisible group. In particular, if $G|_{(\mathrm{Spec}(k(x)), \mathcal{M}_{k(x)})}$ is a log p -divisible group for every $x \in \mathfrak{X}$, then G itself is a log p -divisible group.
- (2) Suppose that the pullback $G|_{(\mathrm{Spec}(k(x)), \mathcal{M}_{k(x)})}$ is classical for every point $x \in \mathfrak{X}$. Then G itself is classical.

Proof. The assertions immediately follow from Lemma 6.3.4 and Lemma 6.3.17. $\square$

Lemma 6.3.20. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be as in Lemma 6.3.6, and let G be a log p -divisible group over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Then there exists a unique exact sequence

$$0 \rightarrow G^0 \rightarrow G \rightarrow G^{\mathrm{\acute{e}t}} \rightarrow 0,$$

where G^0 is a (classical) connected p -divisible group over $\mathfrak{X}$ and $G^{\mathrm{\acute{e}t}}$ is an étale p -divisible group over $\mathfrak{X}$.

Proof. Lemma 6.3.18 allows us to assume that $\mathfrak{X}$ is a scheme, in which case the assertion follows from Lemma 3.2.31. $\square$

Lemma 6.3.21. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be an fs log formal scheme on which p is topologically nilpotent and G be a weak log p -divisible group over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. We let $\mathrm{Lie}(G)$ denote the sheaf on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\mathrm{kfl}}$ given by

$$\mathrm{Lie}(G)(Y, \mathcal{M}_Y) := \mathrm{Ker}(G(Y[\epsilon], \mathcal{M}_{Y[\epsilon]}) \rightarrow G(Y, \mathcal{M}_Y))$$

for $(Y, \mathcal{M}_Y) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\mathrm{kfl}}$, where the scheme $Y[\epsilon]$ is defined by $Y[\epsilon] := Y \otimes_{\mathbb{Z}} \mathbb{Z}[t]/(t^2)$ and $\mathcal{M}_{Y[\epsilon]}$ is the pullback log structure of $\mathcal{M}_Y$. Then $\mathrm{Lie}(G)$ is a kfl vector bundle on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Furthermore, when $p^N = 0$ on $\mathfrak{X}$ for an integer $N \geq 1$, we have an equality

$$\mathrm{Lie}(G) = \mathrm{Lie}(G[p^n])$$

for every integer $n \geq N$.

Proof. Taking a reduction by a finitely generated ideal of definition, we may assume that $\mathfrak{X}$ is a scheme on which $p^N = 0$ for some integer $N \geq 1$. Then all assertions follow from Lemma 6.3.7. $\square$

Definition 6.3.22 (Lie algebras of weak log p -divisible groups). Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ and G be as in Lemma 6.3.21. The kfl vector bundle $\mathrm{Lie}(G)$ defined therein is called the *Lie algebra* of G .

Proposition 6.3.23 (cf. [Kat23, Proposition 7.3]). Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be an fs log formal scheme on which p is topologically nilpotent. Then a weak log p -divisible group G over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is a log p -divisible group if and only if both kfl vector bundles $\mathrm{Lie}(G)$ and $\mathrm{Lie}(G^*)$ are classical.

Proof. Lemma 6.2.16 and Lemma 6.3.19 (2) allow us to assume that $\mathfrak{X}$ is the spectrum of a field. Then the assertion follows from [Kat23, Proposition 7.3]. $\square$

Definition 6.3.24 (Generic fibers of weak log p -divisible groups). Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a p -adic fs log formal scheme and G be a weak log p -divisible group on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. By Construction 6.3.9, we have $\mathbb{Z}/p^n$ -local systems $G[p^n]_{\eta}$ on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \mathrm{qprok\acute{e}t}}^{\diamond}$ for each $n \geq 1$. Hence, $G_{\eta} := \varprojlim_{n \geq 1} G[p^n]_{\eta}$ is a $\mathbb{Z}_p$ -local system on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \mathrm{qprok\acute{e}t}}^{\diamond}$ and is called the *generic fiber* of G .

In the rest of this subsection, let $f: (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be an adic morphism from (not necessarily fs) saturated log formal scheme $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ to an fs log formal scheme $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$.

Definition 6.3.25. A weak log p -divisible group G over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is called *f -classical* if $G[p^n]$ is f -classical for each $n \geq 1$. Let $\mathrm{wBT}^{f\text{-cl}}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ denote the full subcategory of $\mathrm{wBT}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ consisting of f -classical objects. The pullback functor along f defined in Definition 6.3.11 naturally induces the pullback functor

$$f^*: \mathrm{wBT}^{f\text{-cl}}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow \mathrm{BT}(\mathfrak{Y}).$$

Lemma 6.3.26. Let G be an f -classical weak log p -divisible group over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Suppose that p is topologically nilpotent on $\mathfrak{X}$. Then $\mathrm{Lie}(G)$ is also f -classical, and there exists a natural isomorphism of vector bundles on $\mathfrak{Y}$

$$f^* \mathrm{Lie}(G) \cong \mathrm{Lie}(f^* G).$$

Proof. Taking a reduction by a finitely generated ideal of definition, we may assume that $\mathfrak{X}$ is a scheme on which $p^n = 0$ for an integer $n \geq 1$. Since we have $\mathrm{Lie}(G) = \mathrm{Lie}(G[p^n])$ by Lemma 6.3.21, all assertion follows from Lemma 6.3.14. $\square$

Proposition 6.3.27 (Strongly pro-kfl descent for weak log p -divisible groups). Let $f: (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a strongly pro-kfl covering, and let $(\mathfrak{Y}^{(\bullet)}, \mathcal{M}_{\mathfrak{Y}^{(\bullet)}})$ denote the Čech nerve of f in the category of saturated log formal schemes. Then there exists a natural bi-exact equivalence

$$\mathrm{wBT}^{f\text{-cl}}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow \varprojlim_{\bullet \in \Delta} \mathrm{BT}(\mathfrak{Y}^{(\bullet)}).$$

Proof. This follows from Proposition 6.3.15. $\square$

6.4. Kfl prismatic crystals

In this section, we introduce a category including the category of log prismatic (F -)crystals as a full subcategory. An object of this category is called a *kfl prismatic (F -)crystal* (Definition 6.4.1). Furthermore, we prove that this larger category satisfies a certain saturated descent property regarded as a log version of quasi-syntomic descent (Proposition 6.4.12). Even when our main interest is often in (not kfl) prismatic (F -)crystals, it is convenient to work in the category of this larger category of kfl prismatic (F -)crystals because the category of log prismatic (F -)crystals does not have good saturated descent properties. We also establish a similar framework for crystalline crystals in vector bundles.

6.4.1. Kfl prismatic F -crystals. Throughout this subsection, let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a bounded p -adic fs log formal scheme.

Definition 6.4.1 (Kfl prismatic F -crystals). We define an exact category

$$\mathrm{Vect}_{\mathrm{kfl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) := \varprojlim_{(A, I, \mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}} \mathrm{Vect}_{\mathrm{kfl}}((\mathrm{Spf}(A), \mathcal{M}_A)).$$

An object of $\mathrm{Vect}_{\mathrm{kfl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ is called a *kfl prismatic crystal* on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$.

For a log prism $(A, I, \mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}$ and integers $a \leq b$, we let

$$\mathrm{Vect}_{\mathrm{kfl}, [a, b]}^{\varphi}(A, I, \mathcal{M}_A)$$

denote the exact category of a kfl vector bundle $\mathcal{E}$ on $(\mathrm{Spf}(A), \mathcal{M}_A)$ equipped with maps of kfl vector bundles $I^b \otimes_A \mathcal{E} \rightarrow \phi_A^* \mathcal{E}$ and $\phi_A^* \mathcal{E} \rightarrow I^a \otimes_A \mathcal{E}$ such that the composition map $I^b \otimes_A \mathcal{E} \rightarrow I^a \otimes_A \mathcal{E}$ coincides with the map induced from the natural inclusion map of A -modules $I^b \hookrightarrow I^a$. We define an exact category

$$\mathrm{Vect}_{\mathrm{kfl}, [a, b]}^\varphi((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) := \varprojlim_{(A, I, \mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}} \mathrm{Vect}_{\mathrm{kfl}, [a, b]}^\varphi(A, I, \mathcal{M}_A).$$

Especially, we let $\mathrm{DM}_{\mathrm{kfl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ denote the category $\mathrm{Vect}_{\mathrm{kfl}, [0, 1]}^\varphi((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$, and an object of the category $\mathrm{DM}_{\mathrm{kfl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ is called a *kfl prismatic Dieudonné crystal* on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$.

For a log prism $(A, I, \mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}$, we define an exact category

$$\mathrm{Vect}_{\mathrm{kfl}}^\varphi(A, I, \mathcal{M}_A) := \varinjlim_{[a, b]} \mathrm{Vect}_{\mathrm{kfl}, [a, b]}^\varphi(A, I, \mathcal{M}_A).$$

Using this, we define an exact category

$$\mathrm{Vect}_{\mathrm{kfl}}^\varphi((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) := \varprojlim_{(A, I, \mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}} \mathrm{Vect}_{\mathrm{kfl}}^\varphi(A, I, \mathcal{M}_A).$$

An object of $\mathrm{Vect}_{\mathrm{kfl}}^\varphi((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ is called a *kfl prismatic F -crystal* on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$.

Remark 6.4.2. We cannot define a kfl version of analytic log prismatic F -crystals in the same way because, for a log prism $(A, I, \mathcal{M}_A)$, we have no canonical log structure on $U(A, I)$.

Lemma 6.4.3. There are natural functors giving bi-exact equivalences to the essential images

$$\begin{aligned} \mathrm{Vect}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) &\rightarrow \mathrm{Vect}_{\mathrm{kfl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}), \\ \mathrm{Vect}_{[a, b]}^\varphi((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) &\rightarrow \mathrm{Vect}_{\mathrm{kfl}, [a, b]}^\varphi((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}), \\ \mathrm{Vect}^\varphi((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) &\rightarrow \mathrm{Vect}_{\mathrm{kfl}}^\varphi((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}). \end{aligned}$$

Proof. Lemma 5.5.12 allows us to replace $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}$ in the source categories with the strict site $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}$. Then, via the equivalences in Remark 5.5.10, the functor ι in Definition 6.2.13 induces the desired functors. It follows from Lemma 6.2.14 that the resulting functors give equivalences to the essential images. $\square$

We generalize the functoriality of kfl prismatic crystals to non-fs log formal schemes. In the rest of this subsection, let $f: (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a morphism from a bounded p -adic (not necessarily fs) saturated log formal scheme $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ to a bounded p -adic fs log formal scheme $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$.

Lemma 6.4.4. Let $\mathcal{E}$ be a kfl prismatic crystal on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. The following statements are equivalent.

- (1) For any log prisms $(A, I, \mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}$ and $(B, J, \mathcal{M}_B) \in (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})_{\Delta}^{\mathrm{str}}$ with a log prism map $s: (A, I, \mathcal{M}_A) \rightarrow (B, J, \mathcal{M}_B)$ in $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}$, the kfl vector bundle $\mathcal{E}_{(A, I, \mathcal{M}_A)}$ is s -classical.
- (2) For any log prism $(B, J, \mathcal{M}_B) \in (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})_{\Delta}^{\mathrm{str}}$, if we let $(B, J, \mathcal{N}_B)$ denote its image via the functor $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta} \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}$ in Lemma 5.5.12 and $\pi: (B, J, \mathcal{N}_B) \rightarrow (B, J, \mathcal{M}_B)$ denote the natural map, the kfl vector bundle $\mathcal{E}_{(B, J, \mathcal{N}_B)}$ is π -classical.

Proof. It is enough to prove the implication (2) $\Rightarrow$ (1). By Lemma 5.5.12, the log prism map s uniquely factors as

$$(A, I, \mathcal{M}_A) \xrightarrow{t} (B, J, \mathcal{N}_B) \xrightarrow{\pi} (B, J, \mathcal{M}_B),$$

where $(B, J, \mathcal{N}_B)$ is the image of $(B, J, \mathcal{M}_B)$ via the functor $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta} \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\text{str}}$ therein. Since $\mathcal{E}_{(B, J, \mathcal{N}_B)} \cong t^*(\mathcal{E}_{(A, I, \mathcal{M}_A)})$ is π -classical, $\mathcal{E}_{(A, I, \mathcal{M}_A)}$ is s -classical by Lemma 6.2.23. $\square$

Definition 6.4.5. A kfl prismatic crystal $\mathcal{E}$ on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is called *f-classical* if the equivalent conditions in Lemma 6.4.4 are satisfied. Let $\text{Vect}_{\text{kfl}}^{f\text{-cl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ be the full subcategory of $\text{Vect}_{\text{kfl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ consisting of *f-classical* objects. We define an exact functor

$$f^*: \text{Vect}_{\text{kfl}}^{f\text{-cl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \text{Vect}((\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})_{\Delta}^{\text{str}})$$

as follows. Let $\mathcal{E} \in \text{Vect}_{\text{kfl}}^{f\text{-cl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$. For $(B, J, \mathcal{M}_B) \in (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})_{\Delta}^{\text{str}}$, we choose $(A, I, \mathcal{M}_A)$ and s as in Lemma 6.4.4 (1) and set $(f^*\mathcal{E})_{(B, J, \mathcal{M}_B)} := s^*\mathcal{E}_{(A, I, \mathcal{M}_A)}$. If we let $(B, J, \mathcal{N}_B)$ and π as in Lemma 6.4.4 (2), every s uniquely factors through π . Hence, by Lemma 6.2.23, the formation of $(f^*\mathcal{E})_{(B, J, \mathcal{M}_B)}$ is independent of a choice of $(A, I, \mathcal{M}_A)$, and the assignment of $(f^*\mathcal{E})_{(B, J, \mathcal{M}_B)}$ to $(B, J, \mathcal{M}_B)$ defines an object $f^*\mathcal{E} \in \text{Vect}((\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})_{\Delta}^{\text{str}})$.

We say that a kfl prismatic F -crystal on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is *f-classical* if its underlying kfl prismatic crystal is *f-classical*. We write $\text{Vect}_{\text{kfl}, [a, b]}^{\varphi, f\text{-cl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ and $\text{Vect}_{\text{kfl}}^{\varphi, f\text{-cl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ for the full subcategory of $\text{Vect}_{\text{kfl}, [a, b]}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ and $\text{Vect}_{\text{kfl}}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ consisting of an object whose underlying kfl prismatic crystal is *f-classical*, respectively. In the same way as the case of kfl prismatic crystals, one can define the pullback functors

$$\begin{aligned} f^*: \text{Vect}_{\text{kfl}, [a, b]}^{\varphi, f\text{-cl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) &\rightarrow \text{Vect}_{[a, b]}^{\varphi}((\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})_{\Delta}^{\text{str}}), \\ f^*: \text{Vect}_{\text{kfl}}^{\varphi, f\text{-cl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) &\rightarrow \text{Vect}^{\varphi}((\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})_{\Delta}^{\text{str}}). \end{aligned}$$

Lemma 6.4.6. Consider a sequence of morphisms of p -adic log formal schemes

$$(\mathfrak{Z}, \mathcal{M}_{\mathfrak{Z}}) \xrightarrow{g} (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \xrightarrow{f} (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \xrightarrow{h} (\mathfrak{W}, \mathcal{M}_{\mathfrak{W}}),$$

where $(\mathfrak{Z}, \mathcal{M}_{\mathfrak{Z}})$ is saturated and $(\mathfrak{W}, \mathcal{M}_{\mathfrak{W}})$ is fs. Let $\mathcal{E}$ be a kfl prismatic crystal on $(\mathfrak{W}, \mathcal{M}_{\mathfrak{W}})$ with $h^*\mathcal{E}$ being *f-classical*. Then $\mathcal{E}$ is *hfg-classical*, and there is a natural isomorphism of prismatic crystals on $\mathfrak{Z}$

$$(hfg)^*\mathcal{E} \cong g^*(f^*(h^*\mathcal{E})).$$

Conversely, if $\mathcal{E}$ is *hfg-classical*, and if g is a strict quasi-compact quasi-syntomic covering, then $h^*\mathcal{E}$ is *f-classical*.

Proof. The former assertion follows from Lemma 6.2.23, and the latter assertion follows from the same lemma and [BS22, Proposition 7.11]. $\square$

Lemma 6.4.7. Suppose that f admits a chart of the form $P \rightarrow P_{\mathbb{Q}_{\geq 0}}$ for an fs monoid P . Then every kfl prismatic crystal $\mathcal{E}$ on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is *f-classical*.

Proof. Let $\mathcal{E}$ be a kfl prismatic crystal on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Let $(B, J, \mathcal{M}_B) \in (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})_{\Delta}^{\text{str}}$ and $\pi: (B, J, \mathcal{N}_B) \rightarrow (B, J, \mathcal{M}_B)$ be as in Lemma 6.4.4 (1), and let $\bar{\pi}: (\text{Spf}(B/J), \mathcal{M}_{B/J}) \rightarrow (\text{Spf}(B/J), \mathcal{N}_{B/J})$ be the induced map from π . By Lemma 6.2.24, the kfl vector bundle $(\mathcal{E}_{(B, J, \mathcal{M}_B)})|_{(\text{Spf}(B/J), \mathcal{M}_{B/J})}$ is $\bar{\pi}$ -classical. Hence, Lemma 6.2.23 implies that $\mathcal{E}_{(B, J, \mathcal{M}_B)}$ is π -classical. This proves the assertion. $\square$

The goal in this subsection is to prove certain saturated descent properties for kfl prismatic (F -)crystals (Proposition 6.4.12). To this end, we start with some technical definitions and lemmas.

Lemma 6.4.8. Let $f: (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be an integral Kummer morphism of p -adic fs log formal schemes. Let $(A, I, \mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\text{str}}$ and $(B_i, IB_i, \mathcal{M}_{B_i}) \in (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})_{\Delta}^{\text{str}}$ be log prisms with log prism maps $q_i: (A, I, \mathcal{M}_A) \rightarrow (B_i, IB_i, \mathcal{M}_{B_i})$ in $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}$ for $i = 1, 2$. We set $(\text{Spf}(C), \mathcal{M}_C)$ to be the saturated fiber product of $(\text{Spf}(B_i), \mathcal{M}_{B_i})$ for $i = 1, 2$ over $(\text{Spf}(A), \mathcal{M}_A)$. Suppose that q_1 is kfl. Then the triple $(C, IC, \mathcal{M}_C)$ has a unique log prism structure such that the natural maps $(B_i, IB_i, \mathcal{M}_{B_i}) \rightarrow (C, IC, \mathcal{M}_C)$ are log prism maps for $i = 1, 2$. Moreover, the log prism map $(B_2, IB_2, \mathcal{M}_{B_2}) \rightarrow (C, IC, \mathcal{M}_C)$ is strict flat.

Proof. We set $(\text{Spf}(C'), \mathcal{M}_{C'})$ to be the fiber product of fs log formal schemes $(\text{Spf}(B_i), \mathcal{M}_{B_i})$ for $i = 1, 2$ over $(\text{Spf}(A), \mathcal{M}_A)$ in the category of (not necessarily saturated) log formal schemes. Since f is integral, the maps $q_i: (\text{Spf}(B_i), \mathcal{M}_{B_i}) \rightarrow (\text{Spf}(A), \mathcal{M}_A)$ are integral for $i = 1, 2$, and so $(\text{Spf}(C'), \mathcal{M}_{C'})$ is fine. By [Kos22, Remark 2.6], $(\text{Spf}(C'), \mathcal{M}_{C'})$ is equipped with the unique $\delta_{\log}$ -structure such that the projection maps to $(\text{Spf}(B_i), \mathcal{M}_{B_i})$ are compatible with $\delta_{\log}$ -structures for $i = 1, 2$, and so its saturation $(\text{Spf}(C), \mathcal{M}_C)$ is uniquely equipped with the $\delta_{\log}$ -structure by [Kos22, Proposition 2.16].

Let $(\mathfrak{Y}^{(1)}, \mathcal{M}_{\mathfrak{Y}^{(1)}})$ be the self-product of $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ in the category of saturated log formal schemes over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. The natural map $(\text{Spf}(C/IC), \mathcal{M}_{C/IC}) \rightarrow (\mathfrak{Y}^{(1)}, \mathcal{M}_{\mathfrak{Y}^{(1)}})$ is strict. Since f is Kummer, both projection maps $(\mathfrak{Y}^{(1)}, \mathcal{M}_{\mathfrak{Y}^{(1)}}) \rightarrow (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ are strict. Hence, the projection map $(\text{Spf}(C), \mathcal{M}_C) \rightarrow (\text{Spf}(B_2), \mathcal{M}_{B_2})$ is strict kfl. By Lemma 6.2.5, this is strict flat. Therefore, by [BS22, Lemma 3.7 (3)], $(C, IC, \mathcal{M}_C)$ is a log prism. $\square$

Lemma 6.4.9. Let $(A, I, \mathcal{M}_A)$ be a log prism with a finite free chart $\alpha: \mathbb{N}^r \rightarrow \mathcal{M}_A$ with $\delta_{\log}(\alpha(m)) = 0$ for every $m \in \mathbb{N}^r$. Consider a Kummer map $\mathbb{N}^r \rightarrow Q$ of saturated monoids. We set

$$(\text{Spf}(A'), \mathcal{M}_{A'}) := (\text{Spf}(A), \mathcal{M}_A) \otimes_{(\mathbb{Z}[\mathbb{N}^r], \mathbb{N}^r)^a} (\mathbb{Z}[Q], Q)^a.$$

Then the triple $(A', IA', \mathcal{M}_{A'})$ has a unique log prism structure such that the natural map $(A, I, \mathcal{M}_A) \rightarrow (A', IA', \mathcal{M}_{A'})$ is a log prism map and that the composite

$$Q \rightarrow \Gamma(\text{Spf}(A'), \mathcal{M}_{A'}) \xrightarrow{\delta_{\log}} A'$$

is the zero map.

Proof. We regard the prelog ring $(\mathbb{Z}[\mathbb{N}^r], \mathbb{N}^r)$ (resp. $(\mathbb{Z}[Q], Q)$) as a $\delta_{\log}$ -ring by setting $\delta(m) = \delta_{\log}(m) = 0$ for every $m \in \mathbb{N}^r$ (resp. $m \in Q$). Since the map of prelog rings $(\mathbb{Z}[\mathbb{N}^r], \mathbb{N}^r) \rightarrow (A, \Gamma(\text{Spf}(A), \mathcal{M}_A))$ induced from α is compatible with $\delta_{\log}$ -structures by the assumption on α , the log formal scheme $(\text{Spf}(A'), \mathcal{M}_{A'})$ is equipped with a unique $\delta_{\log}$ -structure such that the natural morphisms from $(\text{Spf}(A'), \mathcal{M}_{A'})$ to $(\text{Spf}(A), \mathcal{M}_A)$ and $(\text{Spec}(\mathbb{Z}[Q]), Q)^a$ are compatible with $\delta_{\log}$ -structures by [Kos22, Remark 2.6 and Lemma 2.9]. Hence, it remains to prove that (A', IA') is a prism. By [DA25, Lemma 2.7], the ring map $\mathbb{Z}[\mathbb{N}^r] \rightarrow \mathbb{Z}[Q]$ is faithfully flat. Note that, although it is assumed that Q is fs in *loc. cit.*, the problem is reduced to this case because Q is written as a union of fs submonoids containing $\mathbb{N}^r$. Therefore, $A \rightarrow A'$ is (p, I) -completely faithfully flat, and so (A', IA') is a prism by [BS22, Lemma 3.7 (3)]. $\square$

Lemma 6.4.10. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a bounded p -adic log formal scheme with a finite free chart $\alpha: \mathbb{N}^r \rightarrow \mathcal{M}_{\mathfrak{X}}$. Then, for every log prism $(A, I, \mathcal{M}_A)$ in $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\text{str}}$, there exist a strict flat cover $(A, I, \mathcal{M}_A) \rightarrow (B, J, \mathcal{M}_B)$ and a monoid map $\tilde{\alpha}: \mathbb{N}^r \rightarrow \Gamma(\text{Spf}(B), \mathcal{M}_B)$ with $\delta_{\log}(\tilde{\alpha}(m)) = 0$ for every $m \in \mathbb{N}^r$ fitting into the following commutative diagram:

$$\begin{array}{ccc} & & \Gamma(\text{Spf}(B), \mathcal{M}_B) \\ & \nearrow \tilde{\alpha} & \downarrow \\ \mathbb{N}^r & \xrightarrow{\alpha} \Gamma(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \longrightarrow \Gamma(\text{Spf}(B/J), \mathcal{M}_{B/J}). \end{array}$$

Proof. The given chart α induces a strict map $f: (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow (\text{Spf}(\mathbb{Z}_p\langle\mathbb{N}^r\rangle), \mathbb{N}^r)^a$. The log prism $(A, I, \mathcal{M}_A)$ can be regarded as a log prism in $(\text{Spf}(\mathbb{Z}_p\langle\mathbb{N}^r\rangle), \mathbb{N}^r)_{\Delta}^{\text{str}}$ via f . Hence, we may assume that $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) = (\text{Spf}(\mathbb{Z}_p\langle\mathbb{N}^r\rangle), \mathbb{N}^r)^a$ and α is the tautological chart.

Note that $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is small affine log formal scheme over $\mathbb{Z}_p$. Write $R := \mathbb{Z}_p\langle\mathbb{N}^r\rangle$. By Lemma 5.5.16, there exist a strict flat cover $(A, I, \mathcal{M}_A) \rightarrow (B, J, \mathcal{M}_B)$ and a log prism map

$$(\mathfrak{S}_R, (E), \mathbb{N}^r)^a = (\mathfrak{S}_R, (E), \mathcal{M}_{\mathfrak{S}_R}) \rightarrow (B, J, \mathcal{M}_B).$$

Since the $\delta_{\log}$ -ring $(\mathfrak{S}_R, \mathbb{N}^r)$ is of rank 1 in the sense of [Kos22, Definition 2.2], the composite

$$\mathbb{N}^r \rightarrow \Gamma(\text{Spf}(\mathfrak{S}_R), \mathcal{M}_{\mathfrak{S}_R}) \rightarrow \Gamma(\text{Spf}(B), \mathcal{M}_B)$$

is a desired lift $\tilde{\alpha}$. □

Definition 6.4.11. We say that the morphism f is a *strongly pro-kfl quasi-syntomic* if there exists a chart $\mathbb{N}^r \rightarrow Q$ of f such that Q is saturated, $\mathbb{N}^r \rightarrow Q$ is Kummer, and the induced map

$$(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \otimes_{(\mathbb{Z}[\mathbb{N}^r], \mathbb{N}^r)^a} (\mathbb{Z}[Q], Q)^a$$

is strict quasi-syntomic. The morphism f is called a *strongly pro-kfl quasi-syntomic covering* if f is strongly pro-kfl quasi-syntomic, quasi-compact, and surjective.

Proposition 6.4.12 (Strongly pro-kfl quasi-syntomic descent for kfl prismatic crystals). Let $f: (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a strongly pro-kfl quasi-syntomic covering. We let $(\mathfrak{Y}^{(\bullet)}, \mathcal{M}_{\mathfrak{Y}^{(\bullet)}})$ denote the saturated Čech nerve of f . Then there exist natural bi-exact equivalences

$$(6.4.1) \quad \text{Vect}_{\text{kfl}}^{f\text{-cl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \simeq \varprojlim_{\bullet \in \Delta} \text{Vect}((\mathfrak{Y}^{(\bullet)}, \mathcal{M}_{\mathfrak{Y}^{(\bullet)}})_{\Delta}^{\text{str}}),$$

$$(6.4.2) \quad \text{Vect}_{\text{kfl}}^{\varphi, f\text{-cl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \simeq \varprojlim_{\bullet \in \Delta} \text{Vect}^{\varphi}((\mathfrak{Y}^{(\bullet)}, \mathcal{M}_{\mathfrak{Y}^{(\bullet)}})_{\Delta}^{\text{str}}),$$

$$(6.4.3) \quad \text{Vect}_{\text{kfl}, [a, b]}^{\varphi, f\text{-cl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \simeq \varprojlim_{\bullet \in \Delta} \text{Vect}_{[a, b]}^{\varphi}((\mathfrak{Y}^{(\bullet)}, \mathcal{M}_{\mathfrak{Y}^{(\bullet)}})_{\Delta}^{\text{str}}).$$

Proof. We prove only the equivalence (6.4.1) because other equivalences can be proved by the same argument. By Lemma 6.4.6, the pullback functor f^* induces an exact functor

$$\text{Vect}_{\text{kfl}}^{f\text{-cl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \varprojlim_{\bullet \in \Delta} \text{Vect}((\mathfrak{Y}^{(\bullet)}, \mathcal{M}_{\mathfrak{Y}^{(\bullet)}})_{\Delta}^{\text{str}}).$$

We shall prove that this functor is a bi-exact equivalence. Fix a chart $\mathbb{N}^r \rightarrow Q$ of f as in Definition 6.4.11. We set

$$(\mathfrak{Z}, \mathcal{M}_{\mathfrak{Z}}) := (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \otimes_{(\mathbb{Z}[\mathbb{N}^r], \mathbb{N}^r)^a} (\mathbb{Z}[Q], Q)^a,$$

and let $(\mathfrak{Z}^{(\bullet)}, \mathcal{M}_{\mathfrak{Z}^{(\bullet)}})$ denote its saturated Čech nerve over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Consider the following Cartesian diagram in the category of p -adic saturated log formal schemes:

$$\begin{array}{ccc} (\mathfrak{W}, \mathcal{M}_{\mathfrak{W}}) & \longrightarrow & (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \\ \downarrow & & \downarrow f \\ (\mathfrak{Z}, \mathcal{M}_{\mathfrak{Z}}) & \xrightarrow{g} & (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}). \end{array}$$

By the same argument as in the second paragraph in the proof of Proposition 6.2.26, both the upper horizontal map and the left vertical map are strict quasi-compact quasi-syntomic coverings. Hence, a kfl prismatic crystal on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is f -classical if and only if g -classical by Lemma 6.4.6, and so quasi-syntomic descent for prismatic crystals reduces us to proving that the functor

$$\mathrm{Vect}_{\mathrm{kfl}}^{q\text{-cl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \xrightarrow{\sim} \varprojlim_{\bullet \in \Delta} \mathrm{Vect}((\mathfrak{Z}^{(\bullet)}, \mathcal{M}_{\mathfrak{Z}^{(\bullet)}})_{\Delta}^{\mathrm{str}})$$

induced from the pullback functor g^* is a bi-exact equivalence.

To construct an exact quasi-inverse functor, we shall construct an exact functor

$$\varprojlim_{\bullet \in \Delta} \mathrm{Vect}((\mathfrak{Z}^{(\bullet)}, \mathcal{M}_{\mathfrak{Z}^{(\bullet)}})_{\Delta}^{\mathrm{str}}) \rightarrow \mathrm{Vect}_{\mathrm{kfl}}(\mathrm{Spf}(A), \mathcal{M}_A)$$

for every log prism $(A, I, \mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}$. Working strict flat locally on $(A, I, \mathcal{M}_A)$, we may assume that there is a finite free chart $\alpha: \mathbb{N}^r \rightarrow \mathcal{M}_A$ with $\delta_{\log}(\alpha(m)) = 0$ for every $m \in \mathbb{N}^r$. If we set

$$(\mathrm{Spf}(A'), \mathcal{M}_{A'}) := (\mathrm{Spf}(A), \mathcal{M}_A) \otimes_{(\mathbb{Z}[\mathbb{N}^r], \mathbb{N}^r)^a} (\mathbb{Z}[Q], Q)^a,$$

the triple $(A', IA', \mathcal{M}_{A'})$ is naturally a log prism in $(\mathfrak{Z}, \mathcal{M}_{\mathfrak{Z}})_{\Delta}^{\mathrm{str}}$ by Lemma 6.4.9. Let $q: (A, I, \mathcal{M}_A) \rightarrow (A', IA', \mathcal{M}_{A'})$ be a natural log prism map.

We let $\mathcal{C}_{(A, I, \mathcal{M}_A)}$ denote the category of a log prism $(B, J, \mathcal{M}_B)$ belonging to the essential image of the functor $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})_{\Delta}^{\mathrm{str}} \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}$ equipped with a log prism map $(A, I, \mathcal{M}_A) \rightarrow (B, J, \mathcal{M}_B)$. We claim that there is a natural bi-exact equivalence

$$(6.4.4) \quad \mathrm{Vect}_{\mathrm{kfl}}^{q\text{-cl}}(\mathrm{Spf}(A), \mathcal{M}_A) \xrightarrow{\sim} \varprojlim_{(B, J, \mathcal{M}_B) \in \mathcal{C}_{(A, I, \mathcal{M}_A)}} \mathrm{Vect}(B).$$

To show this, it suffices to prove the following two claims.

- (a) For every $(B, IB, \mathcal{M}_B) \in \mathcal{C}_{(A, I, \mathcal{M}_A)}$, the saturated fiber product $(\mathrm{Spf}(B'), \mathcal{M}_{B'})$ of $(\mathrm{Spf}(B), \mathcal{M}_B)$ and $(\mathrm{Spf}(A'), \mathcal{M}_{A'})$ over $(\mathrm{Spf}(A), \mathcal{M}_A)$ exists, the natural map $(\mathrm{Spf}(B'), \mathcal{M}_{B'}) \rightarrow (\mathrm{Spf}(B), \mathcal{M}_B)$ is strict flat cover, and the triple $(B', IB', \mathcal{M}_{B'})$ is a log prism which is the pushout of $(B, IB, \mathcal{M}_B)$ and $(A', IA', \mathcal{M}_{A'})$ under $(A, I, \mathcal{M}_A)$ in the category of saturated log prisms.
- (b) For an object $r: (A, I, \mathcal{M}_A) \rightarrow (B, IB, \mathcal{M}_B)$, every q -classical kfl vector bundle on $(\mathrm{Spf}(A), \mathcal{M}_A)$ is r -classical.

Indeed, by the claim (b), pullback functors gives the functor (6.4.4), and Proposition 6.2.26 and the claim (a) imply the equivalences

$$\mathrm{Vect}_{\mathrm{kfl}}^{q\text{-cl}}(\mathrm{Spf}(A), \mathcal{M}_A) \simeq \varprojlim_{\bullet \in \Delta} \mathrm{Vect}(A'^{(\bullet)}) \simeq \varprojlim_{(B, J, \mathcal{M}_B) \in \mathcal{C}_{(A, I, \mathcal{M}_A)}} \mathrm{Vect}(B),$$

where $(\mathrm{Spf}(A'^{(\bullet)}), \mathcal{M}_{A'^{(\bullet)}})$ denotes the saturated Čech nerve of $(\mathrm{Spf}(A'), \mathcal{M}_{A'})$ over $(\mathrm{Spf}(A), \mathcal{M}_A)$.

Since the claim (b) follows from the claim (a) and Lemma 6.2.23, the proof of the equivalence (6.4.4) is reduced to showing the claim (a). Let $(B, IB, \mathcal{M}_B) \in \mathcal{C}_{(A, I, \mathcal{M}_A)}$, and fix a strict map $(\mathrm{Spf}(B/IB), \mathcal{M}_{B/IB}) \rightarrow (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$. Write Q as a union $\bigcup_{i \in I} P_i$ where P_i runs over all fs submonoids of Q containing $\mathbb{N}^r$. We set

$$(\mathrm{Spf}(A'_i), \mathcal{M}_{A'_i}) := (\mathrm{Spf}(A), \mathcal{M}_A) \otimes_{(\mathbb{Z}[\mathbb{N}^r], \mathbb{N}^r)^a} (\mathbb{Z}[P_i], P_i)^a$$

for each $i \in I$. By Lemma 6.4.9, the triple $(A'_i, IA'_i, \mathcal{M}_{A'_i})$ is a log prism. We let $\mathcal{M}_{i, B/IB}$ denote the log structure on $\mathrm{Spf}(B/IB)$ associated with the composite

$$P_i \rightarrow Q \rightarrow \Gamma(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow \Gamma(\mathrm{Spf}(B/IB), \mathcal{M}_{B/IB}),$$

and define the log structure $\mathcal{M}_{i, B} := \mathcal{M}_B \times_{\mathcal{M}_{B/IB}} \mathcal{M}_{i, B/IB}$ on $\mathrm{Spf}(B)$ under the identification $\mathrm{Spf}(B)_{\text{ét}} \simeq \mathrm{Spf}(B/IB)_{\text{ét}}$. If we put

$$(\mathfrak{Z}_i, \mathcal{M}_{\mathfrak{Z}_i}) := (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \otimes_{(\mathbb{Z}[\mathbb{N}^r], \mathbb{N}^r)^a} (\mathbb{Z}[P_i], P_i)^a,$$

the log prisms $(A'_i, IA'_i, \mathcal{M}_{A'_i})$ and $(B, IB, \mathcal{M}_{i, B})$ are regarded as objects of $(\mathfrak{Z}_i, \mathcal{M}_{\mathfrak{Z}_i})_{\Delta}^{\text{str}}$. We set

$$(\mathrm{Spf}(B'_i), \mathcal{M}_{B'_i}) := (\mathrm{Spf}(A'_i), \mathcal{M}_{A'_i}) \times_{(\mathrm{Spf}(A), \mathcal{M}_A)}^{\text{sat}} (\mathrm{Spf}(B), \mathcal{M}_{i, B}).$$

Since the morphism $(\mathfrak{Z}_i, \mathcal{M}_{\mathfrak{Z}_i}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is integral and Kummer by [DA25, Lemma 2.7], the triple $(B'_i, IB'_i, \mathcal{M}_{B'_i})$ is equipped with a natural log prism structure by Lemma 6.4.8, and $(B, IB, \mathcal{M}_{i, B}) \rightarrow (B'_i, IB'_i, \mathcal{M}_{B'_i})$ is a strict flat cover. Since $(\mathrm{Spf}(A'), \mathcal{M}_{A'})$ and $(\mathrm{Spf}(B), \mathcal{M}_B)$ are isomorphic to the inverse limits of $(\mathrm{Spf}(A'_i), \mathcal{M}_{A'_i})$ and $(\mathrm{Spf}(B), \mathcal{M}_{i, B})$ for $i \in I$ respectively, if we set $(\mathrm{Spf}(B'), \mathcal{M}_{B'})$ to be the inverse limit of $(\mathrm{Spf}(B'_i), \mathcal{M}_{B'_i})$, the log formal scheme $(\mathrm{Spf}(B'), \mathcal{M}_{B'})$ is the saturated fiber product of $(\mathrm{Spf}(B), \mathcal{M}_B)$ and $(\mathrm{Spf}(A'), \mathcal{M}_{A'})$ over $(\mathrm{Spf}(A), \mathcal{M}_A)$ and is equipped with the unique $\delta_{\log}$ -structure such that both of the projection maps to $(\mathrm{Spf}(A'), \mathcal{M}_{A'})$ and $(\mathrm{Spf}(B), \mathcal{M}_B)$ are compatible with $\delta_{\log}$ -structures. Since $(\mathrm{Spf}(B'), \mathcal{M}_{B'})$ is isomorphic to the inverse limit of $(\mathrm{Spf}(B'_i), \mathcal{M}_{B'_i}) \times_{(\mathrm{Spf}(B), \mathcal{M}_{i, B})} (\mathrm{Spf}(B), \mathcal{M}_B)$ (which is a strict flat cover of $(\mathrm{Spf}(B), \mathcal{M}_B)$), the natural map $(\mathrm{Spf}(B'), \mathcal{M}_{B'}) \rightarrow (\mathrm{Spf}(B), \mathcal{M}_B)$ is a strict flat cover, and so $(B', IB', \mathcal{M}_{B'})$ is a log prism by [BS22, Lemma 3.7 (3)]. This proves the claim (a).

Using the equivalence (6.4.4), we get exact functors

$$\begin{aligned} \varprojlim_{\bullet \in \Delta} \mathrm{Vect}((\mathfrak{Y}^{(\bullet)}, \mathcal{M}_{\mathfrak{Y}^{(\bullet)}})_{\Delta}^{\text{str}}) &\rightarrow \varprojlim_{(B, J, \mathcal{M}_B) \in \mathcal{C}_{(A, I, \mathcal{M}_A)}} \mathrm{Vect}(B) \\ &\simeq \mathrm{Vect}_{\text{kfl}}^{q\text{-cl}}(\mathrm{Spf}(A), \mathcal{M}_A) \\ &\rightarrow \mathrm{Vect}_{\text{kfl}}(\mathrm{Spf}(A), \mathcal{M}_A). \end{aligned}$$

Taking a limit with respect to $(A, I, \mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\text{str}}$, we obtain an exact functor

$$\varprojlim_{\bullet \in \Delta} \mathrm{Vect}((\mathfrak{Y}^{(\bullet)}, \mathcal{M}_{\mathfrak{Y}^{(\bullet)}})_{\Delta}^{\text{str}}) \rightarrow \mathrm{Vect}_{\text{kfl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}).$$

The essential image of this functor is contained in $\mathrm{Vect}_{\text{kfl}}^{f\text{-cl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ by the claim (b), and the induced functor

$$\varprojlim_{\bullet \in \Delta} \mathrm{Vect}((\mathfrak{Y}^{(\bullet)}, \mathcal{M}_{\mathfrak{Y}^{(\bullet)}})_{\Delta}^{\text{str}}) \rightarrow \mathrm{Vect}_{\text{kfl}}^{f\text{-cl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$$

is a desired quasi-inverse functor. $\square$

Definition 6.4.13. A bounded p -adic fs log formal scheme $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is called *strongly quasi-syntomic* if $\mathfrak{X}$ is quasi-syntomic and $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ admits finite free charts étale locally.

Remark 6.4.14. Koshikawa-Yao introduced the notion of quasi-syntomic prelog rings in [KY25, Definition 3.2]. A strongly quasi-syntomic log formal scheme has an étale cover consisting of log formal schemes associated with quasi-syntomic prelog rings because a prelog ring (R, P) is quasi-syntomic if R is quasi-syntomic and P is free. Of course, the converse does not hold. In other words, a log formal scheme associated with a quasi-syntomic prelog ring is not necessarily strongly quasi-syntomic (not even fs).

Definition 6.4.15. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a bounded p -adic fs log formal scheme. We define $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{pkqsyn}}$ as the category of a p -adic saturated log formal scheme $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ such that the structure morphism $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ admits a factorization

$$(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow (\mathfrak{U}_{\infty}, \mathcal{M}_{\mathfrak{U}_{\infty}}) \rightarrow (\mathfrak{U}, \mathcal{M}_{\mathfrak{U}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}),$$

where the morphism $(\mathfrak{U}, \mathcal{M}_{\mathfrak{U}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is strict étale, $(\mathfrak{U}_{\infty}, \mathcal{M}_{\mathfrak{U}_{\infty}})$ is defined as

$$(\mathfrak{U}, \mathcal{M}_{\mathfrak{U}}) \otimes_{(\mathbb{Z}[\mathbb{N}^r, \mathbb{N}^r])^a} (\mathbb{Z}[\mathbb{Q}_{\geq 0}^r], \mathbb{Q}_{\geq 0}^r)^a$$

for a finite free chart $\mathbb{N}^r \rightarrow \mathcal{M}_{\mathfrak{U}}$, and the morphism $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow (\mathfrak{U}, \mathcal{M}_{\mathfrak{U}})$ is strict quasi-syntomic. Note that, when $\mathfrak{X}$ is quasi-syntomic, $\mathfrak{Y}$ is also quasi-syntomic for every $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{pkqsyn}}$ because $\mathbb{Z}_p\langle\mathbb{N}^r\rangle \rightarrow \mathbb{Z}_p\langle\mathbb{Q}_{\geq 0}^r\rangle$ is quasi-syntomic.

When $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is strongly quasi-syntomic, we let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{pkqrsp}}$ denote the full subcategory of $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{pkqsyn}}$ consisting of $(\text{Spf}(S), \mathcal{M}_S)$ with S being qrsp.

The following proposition can be regarded as an analogous statement to [BS23, Proposition 2.14].

Proposition 6.4.16. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a strongly quasi-syntomic log formal scheme. Then there exist natural bi-exact equivalences

$$\begin{aligned} \text{Vect}_{\text{kfl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) &\simeq \varprojlim_{(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{pkqsyn}}} \text{Vect}((\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})_{\Delta}^{\text{str}}) \\ &\simeq \varprojlim_{(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{pkqsyn}}} \text{Vect}(\mathfrak{Y}_{\Delta}) \\ &\simeq \varprojlim_{(S, \mathcal{M}_S) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{pkqrsp}}} \text{Vect}(S_{\Delta}). \end{aligned}$$

The analogous assertion also holds for $\text{Vect}_{\text{kfl}}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ and $\text{Vect}_{\text{kfl}, [a, b]}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$.

Proof. We shall prove only the statement for $\text{Vect}_{\text{kfl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ because the other statements can be proved in the same way. The second equivalence follows from Lemma 5.5.13. Since $\mathfrak{Y}$ is quasi-syntomic for every $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{pkqsyn}}$, the third equivalence follows from quasi-syntomic descent for prismatic crystals.

It remains to prove the first equivalence. For any $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{pkqsyn}}$ with structure morphism $f: (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$, every kfl prismatic crystal on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is f -classical by Lemma 6.4.6 and Lemma 6.4.7, and so we have the pullback functor

$$f^*: \text{Vect}_{\text{kfl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \text{Vect}((\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})_{\Delta}^{\text{str}}),$$

which induces an exact functor

$$\text{Vect}_{\text{kfl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \varprojlim_{(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{pkqsyn}}} \text{Vect}((\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})_{\Delta}^{\text{str}})$$

by Lemma 6.4.6.

We shall prove that this functor gives a bi-exact equivalence. By working étale locally on $\mathfrak{X}$, we may assume that there exists a finite free chart $\mathbb{N}^r \rightarrow \mathcal{M}_{\mathfrak{X}}$. We set

$$(\mathfrak{X}_{\infty}, \mathcal{M}_{\mathfrak{X}_{\infty}}) := (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \otimes_{(\mathbb{Z}[\mathbb{N}^r], \mathbb{N}^r)^a} (\mathbb{Z}[\mathbb{Q}_{\geq 0}^r], \mathbb{Q}_{\geq 0}^r)^a,$$

and let $(\mathfrak{X}_{\infty}^{(\bullet)}, \mathcal{M}_{\mathfrak{X}_{\infty}^{(\bullet)}})$ denote its saturated Čech nerve over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. By strongly pro-kfl quasi-syntomic descent (Proposition 6.4.12), we have a natural bi-exact equivalence

$$\mathrm{Vect}_{\mathrm{kfl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \xrightarrow{\sim} \varprojlim_{\bullet \in \Delta} \mathrm{Vect}((\mathfrak{X}_{\infty}^{(\bullet)}, \mathcal{M}_{\mathfrak{X}_{\infty}^{(\bullet)}})_{\Delta}).$$

To deduce the statement from this, it is enough to prove the following claim: for any $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\mathrm{pkqsyn}}$, the saturated fiber product $(\mathfrak{Z}, \mathcal{M}_{\mathfrak{Z}})$ of $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ and $(\mathfrak{X}_{\infty}, \mathcal{M}_{\mathfrak{X}_{\infty}})$ over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ exists, and the natural map $(\mathfrak{Z}, \mathcal{M}_{\mathfrak{Z}}) \rightarrow (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ is strict quasi-compact quasi-syntomic covering. This claim follows from Lemma 6.A.2 (2). $\square$

Using a similar method, we prove “pro-kfl quasi-syntomic” descent for morphisms of analytic prismatic F -crystals.

Proposition 6.4.17. In the setting of Proposition 6.4.12, the natural functor

$$\mathrm{Vect}^{\mathrm{an}, \star}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \varprojlim_{\bullet \in \Delta} \mathrm{Vect}^{\mathrm{an}, \star}((\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})^{(\bullet)})_{\Delta}$$

is fully faithful for $\star \in \{\emptyset, \varphi\}$. Furthermore, when $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is a semi-stable log formal scheme over $\mathcal{O}_K$, the above functor gives a bi-exact equivalence to the essential image.

Proof. We use the notation in the proof of Proposition 6.4.12. To prove the former assertion, it is enough to show that the natural functor

$$\mathrm{Vect}^{\mathrm{an}, \star}(A, I) \rightarrow \varprojlim_{(B, J, \mathcal{M}_B) \in \mathcal{C}_{(A, I, \mathcal{M}_A)}} \mathrm{Vect}^{\mathrm{an}, \star}(B, J).$$

To do this, the same argument as the proof of Proposition 6.4.12 reduces us to proving that the natural functor

$$\mathrm{Vect}^{\mathrm{an}}(A, I) \rightarrow \varprojlim_{\bullet \in \Delta} \mathrm{Vect}^{\mathrm{an}}(A'^{\bullet}, IA'^{\bullet})$$

is fully faithful. Replacing $(A, I, \mathcal{M}_A)$ with an étale cover, we may assume that I is principal. Let $I = (d)$. Considering internal homomorphisms, we are reduced to showing that, for $\mathcal{E} \in \mathrm{Vect}^{\mathrm{an}}(A, I)$, the sequence

$$(6.4.5) \quad 0 \rightarrow \Gamma(U(A, I), \mathcal{E}) \rightarrow \Gamma(U(A', IA'), \mathcal{E}_{A'}) \rightarrow \Gamma(U(A'^{(1)}, IA'^{(1)}), \mathcal{E}_{A'^{(1)}})$$

is exact. Since $(\mathrm{Spf}(A'), \mathcal{M}_{A'}) \rightarrow (\mathrm{Spf}(A), \mathcal{M}_A)$ is a strongly pro-kfl cover, we have an exact sequence

$$0 \rightarrow A \rightarrow A' \rightarrow A'^{(1)}$$

by Proposition 6.2.26. For $f \in \{p, d, pd\}$, taking tensor products with $\Gamma(\mathrm{Spec}(A[1/f]), \mathcal{E})$ over A , we obtain an exact sequence

$$0 \rightarrow \Gamma(\mathrm{Spec}(A[1/f]), \mathcal{E}) \rightarrow \Gamma(\mathrm{Spec}(A'[1/f]), \mathcal{E}_{A'}) \rightarrow \Gamma(\mathrm{Spec}(A'^{(1)}[1/f]), \mathcal{E}_{A'^{(1)}}).$$

Since $U(A, I)$ has a covering $\mathrm{Spec}(A[1/p]) \cap \mathrm{Spec}(A[1/d])$ with the intersection $\mathrm{Spec}(A[1/pd])$, this implies the exactness of (6.4.5).

It remains to prove the latter assertion. Take a chart $c: \mathbb{N}^r \rightarrow Q$ of f as in Definition 6.4.11. Note that the natural map $\mathbb{Q}_{\geq 0}^r \rightarrow Q_{\mathbb{Q}_{\geq 0}}$ is an isomorphism. Replacing $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ with $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \otimes_{(\mathbb{Z}[Q], Q)^a} (\mathbb{Z}[\mathbb{Q}_{\geq 0}^r], \mathbb{Q}_{\geq 0}^r)^a$, we may assume that c is the natural map $\mathbb{N}^r \rightarrow \mathbb{Q}_{\geq 0}^r$.

Working étale locally on $\mathfrak{X}$, we may assume that $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is small affine. Fix a framing on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$, and let $\alpha: \mathbb{N}^s \rightarrow \mathcal{M}_{\mathfrak{X}}$ denote the chart induced from the framing. If we set

$$(\mathfrak{Z}, \mathcal{M}_{\mathfrak{Z}}) := (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \otimes_{\alpha, (\mathbb{Z}[\mathbb{N}^s], \mathbb{N}^s)^a} (\mathbb{Z}[\mathbb{Q}_{\geq 0}^s], \mathbb{Q}_{\geq 0}^s)^a,$$

we have a Cartesian diagram in the category of saturated log formal schemes

$$\begin{array}{ccc} (\mathfrak{W}, \mathcal{M}_{\mathfrak{W}}) & \longrightarrow & (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \\ \downarrow & & \downarrow \\ (\mathfrak{Z}, \mathcal{M}_{\mathfrak{Z}}) & \longrightarrow & (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \end{array}$$

where the left vertical map is a strict quasi-compact quasi-syntomic covering by Lemma 6.A.2 (2). Hence, a sequence of analytic log prismatic crystals on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ that is exact after being pulled back to $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ is exact after being pulled back to $(\mathfrak{Z}, \mathcal{M}_{\mathfrak{Z}})$. Lemma 5.5.16 implies that the exactness of a sequence of analytic log prismatic crystals on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ can be checked by evaluating at the Breuil-Kisin log prism $(\mathfrak{S}_R, (E), \mathbb{N}^s)^a$. We put $\mathfrak{S}_{R,\infty} := \mathfrak{S}_R \widehat{\otimes}_{\mathbb{Z}[\mathbb{N}^s]} \mathbb{Z}[\mathbb{Q}_{\geq 0}^s]$. Since $\mathfrak{S}_R$ is noetherian, the (p, I) -completely faithfully flat map $\mathfrak{S}_R \rightarrow \mathfrak{S}_{R,\infty}$ is a faithfully flat ring map, and so the morphism of schemes $U(A_{\infty, \tilde{\alpha}}, IA_{\infty, \tilde{\alpha}}) \rightarrow U(A, I)$ is also faithfully flat. Since a log prism $(\mathfrak{S}_{R,\infty}, I\mathfrak{S}_{R,\infty}, \mathbb{Q}_{\geq 0}^s)^a$ is naturally an object of $(\mathfrak{Z}, \mathcal{M}_{\mathfrak{Z}})_{\Delta}^{\text{str}}$, a sequence of analytic log prismatic crystals on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ that is exact after being pulled back to $(\mathfrak{Z}, \mathcal{M}_{\mathfrak{Z}})$ is exact. This proves the statement. $\square$

6.4.2. Kfl crystalline crystals. We introduce the notion of the kfl version of crystalline crystals. Let $(X, \mathcal{M}_X)$ be a log scheme over $\mathbb{F}_p$.

Lemma 6.4.18. Suppose that $(X, \mathcal{M}_X)$ is integral and there is a chart $M \rightarrow \mathcal{M}_X$ where M is uniquely p -divisible (i.e. the monoid map $M \rightarrow M$ given by $m \mapsto m^p$ is an isomorphism). Then a natural functor

$$(X, \mathcal{M}_X)_{\text{crys}}^{\text{str}} \rightarrow X_{\text{crys}}$$

sending $(U, T, \mathcal{M}_T)$ to (U, T) gives an equivalence.

In particular, natural functors

$$\begin{aligned} \text{Crys}(X_{\text{crys}}) &\rightarrow \text{Crys}((X, \mathcal{M}_X)_{\text{crys}}), \\ \text{Vect}^*(X_{\text{crys}}) &\rightarrow \text{Vect}^*((X, \mathcal{M}_X)_{\text{crys}}), \\ \text{Isoc}^*(X_{\text{crys}}) &\rightarrow \text{Isoc}^*((X, \mathcal{M}_X)_{\text{crys}}), \\ \text{DM}(X_{\text{crys}}) &\rightarrow \text{DM}((X, \mathcal{M}_X)_{\text{crys}}), \end{aligned}$$

give equivalences for $\star \in \{\emptyset, \varphi\}$.

Proof. This can be proved in the same way as Lemma 5.5.13. $\square$

We give a definition of a kfl version of crystalline crystals. Let $(X, \mathcal{M}_X)$ be an fs log scheme over $\mathbb{F}_p$.

Definition 6.4.19 (Kfl crystalline crystals in vector bundles). We define an exact category

$$\text{Vect}_{\text{kfl}}((X, \mathcal{M}_X)_{\text{crys}}) := \varprojlim_{(T, \mathcal{M}_T) \in (X, \mathcal{M}_X)_{\text{crys}}^{\text{str}}} \text{Vect}_{\text{kfl}}(T, \mathcal{M}_T).$$

An object of $\text{Vect}_{\text{kfl}}((X, \mathcal{M}_X)_{\text{crys}})$ is called a *kfl crystalline crystal in vector bundles* on $(X, \mathcal{M}_X)$. For a map $f: (X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y)$ of fs log schemes over $\mathbb{F}_p$, the morphism of sites $(Y, \mathcal{M}_Y)_{\text{crys}}^{\text{str}} \rightarrow (X, \mathcal{M}_X)_{\text{crys}}^{\text{str}}$ induces an exact functor

$$f^*: \text{Vect}_{\text{kfl}}((Y, \mathcal{M}_Y)_{\text{crys}}) \rightarrow \text{Vect}_{\text{kfl}}((X, \mathcal{M}_X)_{\text{crys}}).$$

In particular, the Frobenius morphism $F: (X, \mathcal{M}_X) \rightarrow (X, \mathcal{M}_X)$ induces

$$F^*: \text{Vect}_{\text{kfl}}((X, \mathcal{M}_X)_{\text{crys}}) \rightarrow \text{Vect}_{\text{kfl}}((X, \mathcal{M}_X)_{\text{crys}}).$$

Let $\text{Vect}_{\text{kfl}}((X, \mathcal{M}_X)_{\text{crys}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ denote the isogeny category. For an object $\mathcal{E}^+ \in \text{Vect}_{\text{kfl}}((X, \mathcal{M}_X)_{\text{crys}})$, the image of $\mathcal{E}^+$ by the natural functor

$$\text{Vect}_{\text{kfl}}((X, \mathcal{M}_X)_{\text{crys}}) \rightarrow \text{Vect}_{\text{kfl}}((X, \mathcal{M}_X)_{\text{crys}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is denoted by $\mathcal{E}^+[1/p]$. We let $\text{Vect}_{\text{kfl}}^{\varphi}((X, \mathcal{M}_X)_{\text{crys}})$ denote the exact category of pairs $(\mathcal{E}^+, \varphi_{\mathcal{E}^+})$ consisting of an object $\mathcal{E}^+ \in \text{Vect}_{\text{kfl}}((X, \mathcal{M}_X)_{\text{crys}})$ and an isomorphism $\varphi_{\mathcal{E}^+}: F^*\mathcal{E}^+[1/p] \xrightarrow{\sim} \mathcal{E}^+[1/p]$ in the isogeny category $\text{Vect}_{\text{kfl}}((X, \mathcal{M}_X)_{\text{crys}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Morphisms $(\mathcal{E}_1^+, \varphi_{\mathcal{E}_1^+}) \rightarrow (\mathcal{E}_2^+, \varphi_{\mathcal{E}_2^+})$ are maps $\mathcal{E}_1^+ \rightarrow \mathcal{E}_2^+$ that are compatible with $\varphi_{\mathcal{E}_1^+}$ and $\varphi_{\mathcal{E}_2^+}$.

Definition 6.4.20 (Kfl crystalline Dieudonné crystals). We define an exact category $\text{DM}_{\text{kfl}}((X, \mathcal{M}_X)_{\text{crys}})$ as the category of a triple $(\mathcal{E}, \varphi_{\mathcal{E}}, \psi_{\mathcal{E}})$ consisting of $\mathcal{E} \in \text{Vect}_{\text{kfl}}((X, \mathcal{M}_X)_{\text{crys}})$ and maps $\varphi_{\mathcal{E}}: \mathcal{E} \rightarrow F^*\mathcal{E}$, $\psi_{\mathcal{E}}: \mathcal{E} \rightarrow F^*\mathcal{E}$ such that both $\psi_{\mathcal{E}} \circ \varphi_{\mathcal{E}}$ and $\varphi_{\mathcal{E}} \circ \psi_{\mathcal{E}}$ coincide with the multiplication by p .

Lemma 6.4.21. For $\star \in \{\emptyset, \varphi\}$, there are natural functors

$$\begin{aligned} \text{Vect}^{\star}((X, \mathcal{M}_X)_{\text{crys}}) &\rightarrow \text{Vect}_{\text{kfl}}^{\star}((X, \mathcal{M}_X)_{\text{crys}}) \\ \text{DM}((X, \mathcal{M}_X)_{\text{crys}}) &\rightarrow \text{DM}_{\text{kfl}}((X, \mathcal{M}_X)_{\text{crys}}) \end{aligned}$$

giving a bi-exact equivalence to the essential image.

Proof. It is enough to the assertion for $\text{Vect}((X, \mathcal{M}_X)_{\text{crys}})$. Due to Remark 5.3.6, we may replace $(X, \mathcal{M}_X)_{\text{crys}}$ in the source category with the strict site $(X, \mathcal{M}_X)_{\text{crys}}^{\text{str}}$. Then, by Remark 5.3.5, the functor ι in Definition 6.2.13 induces the functor in the statement, which gives a bi-exact equivalence to the essential image by Lemma 6.2.14. $\square$

Before stating the descent property for kfl crystalline crystals in vector bundles, we briefly recall p -root descent established in [Lau18, §7.1] and introduce a mild generalization of it.

Definition 6.4.22 (Root morphisms, (cf. [Lau18, §7.1])).

- (1) Let $R \rightarrow S$ be a ring map. We say that S is an *extraction of a p -th root* (resp. an *extraction of a root*) if $S \cong R[a^{1/p}] := R[x]/(x^p - a)$ for some $a \in R$ (resp. $S \cong R[a^{1/n}] := R[x]/(x^n - a)$ for some $a \in R$ and integer $n \geq 1$). More generally, the map $R \rightarrow S$ is called a *p -root extension* (resp. a *root extension*) if we can write $R \rightarrow S$ as the transfinite composition of extractions of a p -th root (resp. extractions of a root).
- (2) Let $f: X \rightarrow Y$ be a map of schemes. We say that f is a *p -root morphism* (resp. *root morphism*) if, for each $x \in X$, there exist an affine étale neighborhood U of x and an affine étale neighborhood V of $f(x)$ such that f restricts to a morphism $U \rightarrow V$ corresponding to a p -root extension ring map (resp. a root extension ring map). A family of maps $\{U_i \rightarrow X\}_{i \in I}$ is called a *p -root covering* (resp. a *root covering*) if $U_i \rightarrow X$ is a p -root morphism (resp. root morphism) for each $i \in I$ and the

induced morphism $\bigsqcup_{i \in I} U_i \rightarrow X$ is surjective. The Grothendieck topology on the category of schemes defined by p -root coverings (resp. root coverings) is called *p -root topology* (resp. *root topology*).

Lemma 6.4.23 (cf. [Lau18, Lemma 7.5]). Let $f: X \rightarrow Y$ be a quasi-compact root morphism of schemes over $\mathbb{F}_p$ and (U, T) be an affine PD-thickening in Y_{crys} . Then there exist an affine PD-thickening $(U', T') \in X_{\text{crys}}$ and a Cartesian flat map $(U', T') \rightarrow (U, T)$ of affine PD-thickenings such that the natural map $U' \rightarrow X \times_Y U$ is an étale covering. In particular, for $\star \in \{\emptyset, \varphi\}$, the fibered categories over the category of schemes over $\mathbb{F}_p$ given by $X \mapsto \text{Vect}^\star(X_{\text{crys}})$ is a 2-sheaf for root topology.

Proof. Replacing f with $X \times_Y U \rightarrow U$, we may assume that $Y = U$ (in particular, Y is affine). Take an étale covering $\{V_i \rightarrow X\}_{1 \leq i \leq n}$ of X and a family of étale maps $\{W_i \rightarrow Y\}_{1 \leq i \leq n}$ such that V_i and W_i are affine and f restricts to a map $V_i \rightarrow W_i$ corresponding to a root extension ring map for each $1 \leq i \leq n$. The étale map $W_i \rightarrow Y$ extends uniquely to an étale map $T_i \rightarrow T$, and the closed immersion $W_i \hookrightarrow T_i$ has a unique PD-structure extending the PD-structure of $Y \hookrightarrow T$. Since the map $V_i \rightarrow W_i$ corresponds to a root extension ring map, this extends to a Cartesian flat map of affine PD-thickenings

$$(V_i \hookrightarrow T'_i) \rightarrow (W_i \hookrightarrow T_i)$$

by the same argument as [Lau18, Lemma 7.5]. Then, if we set

$$(U', T') := \left(\bigsqcup_{i=1}^n V_i, \bigsqcup_{i=1}^n T'_i \right),$$

the assertion holds. $\square$

We generalize the functoriality of kfl crystalline crystals in vector bundles to non-fs bases. In the rest of this section, let $f: (Y, \mathcal{M}_Y) \rightarrow (X, \mathcal{M}_X)$ be a morphism from a (not necessarily fs) saturated log scheme $(Y, \mathcal{M}_Y)$ to an fs log scheme $(X, \mathcal{M}_X)$ over $\mathbb{F}_p$.

Lemma 6.4.24. Let $\mathcal{E} \in \text{Vect}_{\text{kfl}}((X, \mathcal{M}_X)_{\text{crys}})$. The following statements are equivalent.

- (1) For any log PD thickening $(T, \mathcal{M}_T) \in (X, \mathcal{M}_X)_{\text{crys}}^{\text{str}}$ and $(V, \mathcal{M}_V) \in (Y, \mathcal{M}_Y)_{\text{crys}}^{\text{str}}$ with a map $s: (V, \mathcal{M}_V) \rightarrow (T, \mathcal{M}_T)$ in $(X, \mathcal{M}_X)_{\text{crys}}$, the kfl vector bundle $\mathcal{E}_{(T, \mathcal{M}_T)}$ is s -classical.
- (2) For any log PD-thickening $(V, \mathcal{M}_V) \in (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})_{\Delta}^{\text{str}}$, if we let $(V, \mathcal{N}_V)$ denote its image via the functor $(X, \mathcal{M}_X)_{\text{crys}} \rightarrow (X, \mathcal{M}_X)_{\text{crys}}^{\text{str}}$ in Lemma 5.3.4 and $\pi: (V, \mathcal{M}_V) \rightarrow (V, \mathcal{N}_V)$ denote the natural map, the kfl vector bundle $\mathcal{E}_{(V, \mathcal{N}_V)}$ is π -classical.

Proof. This follows from Lemma 5.3.4 and the same argument as Lemma 6.4.4. $\square$

Definition 6.4.25. A kfl crystalline crystal in vector bundles on $(X, \mathcal{M}_X)$ is called *f -classical* if the equivalent conditions in Lemma 6.4.24 are satisfied. Let $\text{Vect}_{\text{kfl}}^{f\text{-cl}}((X, \mathcal{M}_X)_{\text{crys}})$ denote the full subcategory of $\text{Vect}_{\text{kfl}}((X, \mathcal{M}_X)_{\text{crys}})$ consisting of f -classical objects. For $(V, \mathcal{M}_V) \in (Y, \mathcal{M}_Y)_{\text{crys}}^{\text{str}}$, we choose $(T, \mathcal{M}_T)$ and s as in Lemma 6.4.24 (1) and set $(f^*\mathcal{E})_{(V, \mathcal{M}_V)} := s^*\mathcal{E}_{(T, \mathcal{M}_T)}$. If we let $(V, \mathcal{N}_V)$ and π as in Lemma 6.4.24 (2), every s uniquely factors through π . Hence, by Lemma 6.2.23, the formation of $(f^*\mathcal{E})_{(V, \mathcal{M}_V)}$ is independent of a choice of $(T, \mathcal{M}_T)$, and the assignment of $(f^*\mathcal{E})_{(V, \mathcal{M}_V)}$ to $(V, \mathcal{M}_V)$ defines an object $f^*\mathcal{E} \in \text{Vect}((Y, \mathcal{M}_Y)_{\text{crys}}^{\text{str}})$.

We say that an object of $\mathrm{Vect}_{\mathrm{kfl}}^\varphi((X, \mathcal{M}_X)_{\mathrm{crys}})$ or $\mathrm{DM}_{\mathrm{kfl}}((X, \mathcal{M}_X)_{\mathrm{kfl}})$ is called *f-classical* if its underlying kfl crystalline crystal in vector bundles is *f*-classical. We write $\mathrm{Vect}_{\mathrm{kfl}}^{\varphi, f\text{-cl}}((X, \mathcal{M}_X)_{\mathrm{crys}})$ (resp. $\mathrm{DM}_{\mathrm{kfl}}^{f\text{-cl}}((X, \mathcal{M}_X)_{\mathrm{crys}})$) for the full subcategory of $\mathrm{Vect}_{\mathrm{kfl}}^\varphi((X, \mathcal{M}_X)_{\mathrm{crys}})$ (resp. $\mathrm{DM}_{\mathrm{kfl}}((X, \mathcal{M}_X)_{\mathrm{crys}})$) consisting of *f*-classical objects. In the same way as the case of kfl crystalline crystals in vector bundles, one can define the pull-back functors

$$\begin{aligned} f^*: \mathrm{Vect}_{\mathrm{kfl}}^{\varphi, f\text{-cl}}((X, \mathcal{M}_X)_{\mathrm{crys}}) &\rightarrow \mathrm{Vect}^\varphi((Y, \mathcal{M}_Y)_{\mathrm{crys}}^{\mathrm{str}}), \\ f^*: \mathrm{DM}_{\mathrm{kfl}}^{f\text{-cl}}((X, \mathcal{M}_X)_{\mathrm{crys}}) &\rightarrow \mathrm{DM}((Y, \mathcal{M}_Y)_{\mathrm{crys}}^{\mathrm{str}}). \end{aligned}$$

Lemma 6.4.26. Consider a sequence of morphisms of log formal schemes over $\mathbb{F}_p$

$$(Z, \mathcal{M}_Z) \xrightarrow{g} (Y, \mathcal{M}_Y) \xrightarrow{f} (X, \mathcal{M}_X) \xrightarrow{h} (W, \mathcal{M}_W),$$

where $(Z, \mathcal{M}_Z)$ is saturated and $(W, \mathcal{M}_W)$ is fs. Let $\mathcal{E}$ be a kfl crystalline crystal in vector bundles on $(W, \mathcal{M}_W)$ with $h^*\mathcal{E}$ being *f*-classical. Then $\mathcal{E}$ is *hfg*-classical, and there is a natural isomorphism of crystalline crystals in vector bundles on Z

$$(hfg)^*\mathcal{E} \cong g^*(f^*(h^*\mathcal{E})).$$

Conversely, if $\mathcal{E}$ is *hfg*-classical, and if g is a strict root covering, then $h^*\mathcal{E}$ is *f*-classical.

Proof. The former assertion follows from Lemma 6.2.23, and the latter assertion follows from the same lemma and Lemma 6.4.23. $\square$

Lemma 6.4.27. Suppose that f admits a chart of the form $P \rightarrow P_{\mathbb{Q}_{\geq 0}}$ for an fs monoid P . Then every kfl prismatic crystal $\mathcal{E}$ on $(X, \mathcal{M}_X)$ is *f*-classical.

Proof. This follows from the same argument as Lemma 6.4.7. $\square$

Our goal in this subsection is to prove strongly pro-kfl root descent for kfl crystalline crystals in vector bundles (Proposition 6.4.30). To this end, we start with a technical lemma.

Lemma 6.4.28. Let $f: (Y, \mathcal{M}_Y) \rightarrow (X, \mathcal{M}_X)$ be an Kummer morphism of fs log schemes over $\mathbb{F}_p$. Let $(U, T, \mathcal{M}_T) \in (X, \mathcal{M}_X)_{\mathrm{crys}}^{\mathrm{str}}$ and $(U'_i, T'_i, \mathcal{M}_{T'_i}) \in (Y, \mathcal{M}_Y)_{\mathrm{crys}}^{\mathrm{str}}$ be log PD-thickenings with maps $q_i: (U'_i, T'_i, \mathcal{M}_{T'_i}) \rightarrow (U, T, \mathcal{M}_T)$ in $(X, \mathcal{M}_X)_{\mathrm{crys}}$ for $i = 1, 2$. We set

$$\begin{aligned} (T'', \mathcal{M}_{T''}) &:= (T'_1, \mathcal{M}_{T'_1}) \times_{(T, \mathcal{M}_T)}^{\mathrm{sat}} (T'_2, \mathcal{M}_{T'_2}), \\ (U'', \mathcal{M}_{U''}) &:= (U'_1, \mathcal{M}_{U'_1}) \times_{(U, \mathcal{M}_U)}^{\mathrm{sat}} (U'_2, \mathcal{M}_{U'_2}). \end{aligned}$$

Suppose that q_1 is Cartesian and kfl. Then the strict closed immersion $(U'', \mathcal{M}_{U''}) \hookrightarrow (T'', \mathcal{M}_{T''})$ has a unique PD structure extending the PD structures on $(T'_i, \mathcal{M}_{T'_i})$ for $i = 1, 2$. Moreover, the natural map $(T'', \mathcal{M}_{T''}) \rightarrow (T'_2, \mathcal{M}_{T'_2})$ is strict flat.

Proof. Let $(Y^{(1)}, \mathcal{M}_{Y^{(1)}})$ be the self-product of $(Y, \mathcal{M}_Y)$ in the category of saturated log schemes over $(X, \mathcal{M}_X)$. The natural map $(T'', \mathcal{M}_{T''}) \rightarrow (Y^{(1)}, \mathcal{M}_{Y^{(1)}})$ is strict. Since f is Kummer, both projection maps $(Y^{(1)}, \mathcal{M}_{Y^{(1)}}) \rightarrow (Y, \mathcal{M}_Y)$ are strict. Hence, the projection map $(T'', \mathcal{M}_{T''}) \rightarrow (T'_2, \mathcal{M}_{T'_2})$ is strict kfl, which implies that this map is strict flat. Therefore, the PD structure on $(T'_2, \mathcal{M}_{T'_2})$ extends to $(T'', \mathcal{M}_{T''})$. $\square$

Definition 6.4.29. We say that a morphism $f: (Y, \mathcal{M}_Y) \rightarrow (X, \mathcal{M}_X)$ is a *strongly pro-kfl root morphism* if there exists a chart $\mathbb{N}^r \rightarrow Q$ of f such that the induced morphism

$$(Y, \mathcal{M}_Y) \rightarrow (X, \mathcal{M}_X) \otimes_{(\mathbb{Z}[\mathbb{N}^r], \mathbb{N}^r)^a} (\mathbb{Z}[Q], Q)^a$$

is a strict root morphism. The morphism f is called a *strongly pro-kfl root covering* if f is strongly pro-kfl root, quasi-compact, and surjective.

Proposition 6.4.30 (Strongly pro-kfl root descent for kfl crystalline crystals in vector bundles). Let $f: (Y, \mathcal{M}_Y) \rightarrow (X, \mathcal{M}_X)$ be a strongly pro-kfl root covering of saturated log schemes over $\mathbb{F}_p$. We let $(Y^{(\bullet)}, \mathcal{M}_{Y^{(\bullet)}})$ be the saturated Čech nerve of f . Then there exist natural bi-exact equivalences

$$(6.4.6) \quad \mathrm{Vect}_{\mathrm{kfl}}^{f\text{-cl}}((X, \mathcal{M}_X)_{\mathrm{crys}}) \simeq \varprojlim_{\bullet \in \Delta} \mathrm{Vect}((Y^{(\bullet)}, \mathcal{M}_{Y^{(\bullet)}})_{\mathrm{crys}}^{\mathrm{str}}),$$

$$(6.4.7) \quad \mathrm{Vect}_{\mathrm{kfl}}^{\varphi, f\text{-cl}}((X, \mathcal{M}_X)_{\mathrm{crys}}) \simeq \varprojlim_{\bullet \in \Delta} \mathrm{Vect}^{\varphi}((Y^{(\bullet)}, \mathcal{M}_{Y^{(\bullet)}})_{\mathrm{crys}}^{\mathrm{str}}),$$

$$(6.4.8) \quad \mathrm{DM}_{\mathrm{kfl}}^{f\text{-cl}}((X, \mathcal{M}_X)_{\mathrm{crys}}) \simeq \varprojlim_{\bullet \in \Delta} \mathrm{DM}((Y^{(\bullet)}, \mathcal{M}_{Y^{(\bullet)}})_{\mathrm{crys}}^{\mathrm{str}}).$$

Proof. We prove only the equivalence (6.4.6) because other equivalences can be proved by the same argument. By Lemma 6.4.26, the pullback functor f^* induces an exact functor

$$\mathrm{Vect}_{\mathrm{kfl}}^{f\text{-cl}}((X, \mathcal{M}_X)_{\mathrm{crys}}) \rightarrow \varprojlim_{\bullet \in \Delta} \mathrm{Vect}((Y^{(\bullet)}, \mathcal{M}_{Y^{(\bullet)}})_{\mathrm{crys}}^{\mathrm{str}}).$$

We shall prove that this functor is a bi-exact equivalence. Fix a chart $\mathbb{N}^r \rightarrow Q$ of f as in Definition 6.4.29. We set

$$(Z, \mathcal{M}_Z) := (X, \mathcal{M}_X) \otimes_{(\mathbb{Z}[\mathbb{N}^r], \mathbb{N}^r)^a} (\mathbb{Z}[Q], Q)^a,$$

and let $(Z^{(\bullet)}, \mathcal{M}_{Z^{(\bullet)}})$ denote its saturated Čech nerve over $(X, \mathcal{M}_X)$. Consider the following Cartesian diagram in the category of saturated log schemes:

$$\begin{array}{ccc} (W, \mathcal{M}_W) & \longrightarrow & (Y, \mathcal{M}_Y) \\ \downarrow & & \downarrow f \\ (Z, \mathcal{M}_Z) & \xrightarrow{g} & (X, \mathcal{M}_X). \end{array}$$

Using Lemma 6.A.2 (3) and the same argument as in the second paragraph in the proof of Proposition 6.2.26, both the upper horizontal map and the left vertical map are strict root coverings. Hence, a kfl crystalline crystal in vector bundles on $(X, \mathcal{M}_X)$ is f -classical if and only if g -classical by Lemma 6.4.26, and so root descent for crystalline crystals in vector bundles (Lemma 6.4.23) reduces us to proving that the functor

$$\mathrm{Vect}_{\mathrm{kfl}}^{g\text{-cl}}((X, \mathcal{M}_X)_{\mathrm{crys}}) \xrightarrow{\sim} \varprojlim_{\bullet \in \Delta} \mathrm{Vect}((Z^{(\bullet)}, \mathcal{M}_{Z^{(\bullet)}})_{\mathrm{crys}}^{\mathrm{str}})$$

induced from the pullback functor g^* is a bi-exact equivalence.

To construct an exact quasi-inverse functor, we shall construct an exact functor

$$\varprojlim_{\bullet \in \Delta} \mathrm{Vect}((Z^{(\bullet)}, \mathcal{M}_{Z^{(\bullet)}})_{\mathrm{crys}}^{\mathrm{str}}) \rightarrow \mathrm{Vect}_{\mathrm{kfl}}(T, \mathcal{M}_T)$$

for every affine log PD-thickening $(U, T, \mathcal{M}_T) \in (X, \mathcal{M}_X)_{\mathrm{crys}}^{\mathrm{str}}$. Since the natural monoid map $\Gamma(T, \mathcal{M}_T) \rightarrow \Gamma(U, \mathcal{M}_U)$ is surjective by Lemma 5.A.1, there exists a monoid map $\mathbb{N}^r \rightarrow \mathcal{M}_T$ fitting into the following commutative diagram:

$$\begin{array}{ccccc} & & \Gamma(T, \mathcal{M}_T) & & \\ & \nearrow & \downarrow & & \\ \mathbb{N}^r & \longrightarrow & \Gamma(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) & \longrightarrow & \Gamma(U, \mathcal{M}_U). \end{array}$$

We set

$$(T', \mathcal{M}_{T'}) := (T, \mathcal{M}_T) \otimes_{(\mathbb{Z}[\mathbb{N}^r], \mathbb{N}^r)^a} (\mathbb{Z}[Q], Q)^a.$$

Since $T' \rightarrow T$ is flat by [DA25, Lemma 2.7], the PD structure on T extends to T' , and so $(T', \mathcal{M}_{T'})$ is naturally a log PD-thickening in $(Z, \mathcal{M}_Z)_{\text{cris}}^{\text{str}}$. Let $q: (T', \mathcal{M}_{T'}) \rightarrow (T, \mathcal{M}_T)$ be a natural map of log PD-thickenings.

We let $\mathcal{C}_{(T, \mathcal{M}_T)}$ denote the category of a log PD-thickening $(V, \mathcal{M}_V)$ belonging to the essential image of the functor $(Y, \mathcal{M}_Y)_{\text{cris}}^{\text{str}} \rightarrow (X, \mathcal{M}_X)_{\text{cris}}$ equipped with a map of log PD-thickenings $(V, \mathcal{M}_V) \rightarrow (T, \mathcal{M}_T)$. We claim that there is a natural bi-exact equivalence

$$(6.4.9) \quad \text{Vect}_{\text{kfl}}^{q\text{-cl}}(T, \mathcal{M}_T) \xrightarrow{\sim} \varprojlim_{(V, \mathcal{M}_V) \in \mathcal{C}_{(T, \mathcal{M}_T)}} \text{Vect}(V).$$

To show this, it suffices to prove the following two claims.

- (a) For every $(W, V, \mathcal{M}_V) \in \mathcal{C}_{(T, \mathcal{M}_T)}$, the saturated fiber product $(V', \mathcal{M}_{V'})$ of $(V, \mathcal{M}_V)$ and $(T', \mathcal{M}_{T'})$ over $(T, \mathcal{M}_T)$ exists, and the natural map $(V', \mathcal{M}_{V'}) \rightarrow (V, \mathcal{M}_V)$ is a strict flat cover. In particular, the PD structure on V extends to V' , and the log PD-thickening $(W', V', \mathcal{M}_{V'})$ is a fiber product of $(W, V, \mathcal{M}_V)$ and $(U', T', \mathcal{M}_{T'})$ over $(U, T, \mathcal{M}_T)$ in the category of saturated PD-thickenings.
- (b) For an object $r: (V, \mathcal{M}_V) \rightarrow (T, \mathcal{M}_T)$, every q -classical kfl vector bundle on $(T, \mathcal{M}_T)$ is r -classical.

Indeed, by the claim (b), pullback functors gives the functor (6.4.9), and Proposition 6.2.26 and the claim (a) imply the equivalences

$$\text{Vect}_{\text{kfl}}^{q\text{-cl}}(T, \mathcal{M}_T) \simeq \varprojlim_{\bullet \in \Delta} \text{Vect}(T'^{(\bullet)}) \simeq \varprojlim_{(V, \mathcal{M}_V) \in \mathcal{C}_{(T, \mathcal{M}_T)}} \text{Vect}(V),$$

where $(\text{Spf}(T'^{(\bullet)}), \mathcal{M}_{T'^{(\bullet)}})$ denotes the saturated Čech nerve of $(T', \mathcal{M}_{T'})$ over $(T, \mathcal{M}_T)$.

Since the claim (b) follows from the claim (a) and Lemma 6.2.23, the proof of the equivalence (6.4.9) is reduced to showing the claim (a). Let $(W, V, \mathcal{M}_V) \in \mathcal{C}_{(T, \mathcal{M}_T)}$, and fix a strict map $(W, \mathcal{M}_W) \rightarrow (Y, \mathcal{M}_Y)$. Write Q as a union $\bigcup_{i \in I} P_i$ where P_i runs over all fs submonoids of Q containing $\mathbb{N}^r$. We set

$$(T'_i, \mathcal{M}_{T'_i}) := (T, \mathcal{M}_T) \otimes_{(\mathbb{Z}[\mathbb{N}^r], \mathbb{N}^r)^a} (\mathbb{Z}[P_i], P_i)^a$$

for each $i \in I$. By [DA25, Lemma 2.7], the PD-structure on T extends to T'_i , and so $(T'_i, \mathcal{M}_{T'_i})$ is a log PD-thickening. We let $\mathcal{M}_{i, W}$ denote the log structure on W associated with the composite

$$P_i \rightarrow Q \rightarrow \Gamma(Y, \mathcal{M}_Y) \rightarrow \Gamma(W, \mathcal{M}_W),$$

and define the log structure $\mathcal{M}_{i, V} := \mathcal{M}_W \times_{\mathcal{M}_V} \mathcal{M}_{i, W}$ on V under the identification $V_{\text{ét}} \simeq W_{\text{ét}}$. If we put

$$(Z_i, \mathcal{M}_{Z_i}) := (X, \mathcal{M}_X) \otimes_{(\mathbb{Z}[\mathbb{N}^r], \mathbb{N}^r)^a} (\mathbb{Z}[P_i], P_i)^a,$$

the log PD-thickenings $(T'_i, \mathcal{M}_{T'_i})$ and $(V, \mathcal{M}_{i, V})$ are regarded as objects of $(Z_i, \mathcal{M}_{Z_i})_{\text{cris}}^{\text{str}}$. We set

$$(V'_i, \mathcal{M}_{V'_i}) := (T'_i, \mathcal{M}_{T'_i}) \times_{(T, \mathcal{M}_T)}^{\text{sat}} (V, \mathcal{M}_{i, V}).$$

By Lemma 6.4.28, $(V'_i, \mathcal{M}_{V'_i})$ is a PD-thickening, and $(V'_i, \mathcal{M}_{V'_i}) \rightarrow (V, \mathcal{M}_{i, V})$ is a strict flat cover. Since $(T', \mathcal{M}_{T'})$ and $(V, \mathcal{M}_V)$ are isomorphic to the inverse limits of $(T'_i, \mathcal{M}_{T'_i})$ and $(V, \mathcal{M}_{i, V})$ for $i \in I$ respectively, if we set $(V', \mathcal{M}_{V'})$ to be the inverse limit of $(V'_i, \mathcal{M}_{V'_i})$, the log scheme $(V', \mathcal{M}_{V'})$ is the saturated fiber product of $(V, \mathcal{M}_V)$ and $(T', \mathcal{M}_{T'})$ over $(T, \mathcal{M}_T)$. Since $(V', \mathcal{M}_{V'})$ is isomorphic to the inverse limit of $(V'_i, \mathcal{M}_{V'_i}) \times_{(V, \mathcal{M}_{i, V})} (V, \mathcal{M}_V)$ (which is a strict flat cover of $(V, \mathcal{M}_V)$), the natural map $(V', \mathcal{M}_{V'}) \rightarrow (V, \mathcal{M}_V)$ is a

strict flat cover, and so the PD structure on V extends to V' , and so $(V', \mathcal{M}_{V'})$ is a log PD-thickening. This proves the claim (a).

Using the equivalence (6.4.9), we get exact functors

$$\begin{aligned} \varprojlim_{\bullet \in \Delta} \text{Vect}((Y^{(\bullet)}, \mathcal{M}_{Y^{(\bullet)}}^{\text{str}}_{\text{crys}})) &\rightarrow \varprojlim_{(V, \mathcal{M}_V) \in \mathcal{C}_{(T, \mathcal{M}_T)}} \text{Vect}(T) \\ &\simeq \text{Vect}_{\text{kff}}^{q\text{-cl}}(T, \mathcal{M}_T) \\ &\rightarrow \text{Vect}_{\text{kff}}(T, \mathcal{M}_T). \end{aligned}$$

Taking a limit with respect to $(T, \mathcal{M}_T) \in (X, \mathcal{M}_X)^{\text{str}}_{\text{crys}}$, we obtain an exact functor

$$\varprojlim_{\bullet \in \Delta} \text{Vect}((Y^{(\bullet)}, \mathcal{M}_{Y^{(\bullet)}}^{\text{str}}_{\text{crys}})) \rightarrow \text{Vect}_{\text{kff}}((X, \mathcal{M}_X)_{\text{crys}}).$$

The essential image of this functor is contained in $\text{Vect}_{\text{kff}}^{f\text{-cl}}((X, \mathcal{M}_X)_{\text{crys}})$ by the claim (b), and the induced functor

$$\varprojlim_{\bullet \in \Delta} \text{Vect}((Y^{(\bullet)}, \mathcal{M}_{Y^{(\bullet)}}^{\text{str}}_{\text{crys}})) \rightarrow \text{Vect}_{\text{kff}}^{f\text{-cl}}((X, \mathcal{M}_X)_{\text{crys}})$$

is a desired quasi-inverse functor. $\square$

Proposition 6.4.31. Let $(X, \mathcal{M}_X)$ be a strongly quasi-syntomic log scheme over $\mathbb{F}_p$. Then there exist natural bi-exact equivalences

$$\begin{aligned} \text{Vect}_{\text{kff}}((X, \mathcal{M}_X)_{\text{crys}}) &\simeq \varprojlim_{(Y, \mathcal{M}_Y) \in (X, \mathcal{M}_X)_{\text{pkqsyn}}} \text{Vect}((Y, \mathcal{M}_Y)^{\text{str}}_{\text{crys}}) \\ &\simeq \varprojlim_{(Y, \mathcal{M}_Y) \in (X, \mathcal{M}_X)_{\text{pkqsyn}}} \text{Vect}(Y_{\text{crys}}) \\ &\simeq \varprojlim_{(S, \mathcal{M}_S) \in (X, \mathcal{M}_X)_{\text{pkqrsp}}} \text{Vect}(S_{\text{crys}}). \end{aligned}$$

Analogous statements hold for $\text{Vect}_{\text{kff}}^{\varphi}((X, \mathcal{M}_X)_{\text{crys}})$ and $\text{DM}_{\text{kff}}((X, \mathcal{M}_X)_{\text{crys}})$.

Proof. We shall prove only the first equivalence because the second equivalence follows from Lemma 6.4.18 and the third equivalence follows from [IKY24, Proposition 2.16].

It remains to prove the first equivalence. This is deduced from Proposition 6.4.30 and Lemma 6.A.2 (3) in the same way as the proof of Proposition 6.4.16. $\square$

Proposition 6.4.32. Let $(X, \mathcal{M}_X)$ be a strongly quasi-syntomic log scheme over $\mathbb{F}_p$. Then there exist natural bi-exact equivalences

$$\begin{aligned} \text{Vect}_{\text{kff}}^{\star}((X, \mathcal{M}_X)_{\Delta}) &\simeq \text{Vect}_{\text{kff}}^{\star}((X, \mathcal{M}_X)_{\text{crys}}), \\ \text{DM}_{\text{kff}}((X, \mathcal{M}_X)_{\Delta}) &\simeq \text{DM}_{\text{kff}}((X, \mathcal{M}_X)_{\text{crys}}), \end{aligned}$$

for $\star \in \{\emptyset, \varphi\}$.

Proof. The assertion follows from Proposition 6.4.16, Proposition 6.4.31, and the equivalence in [IKY25, §1.1.1]. $\square$

Definition 6.4.33 (Crystalline realization). Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a strongly quasi-syntomic log formal scheme and $(X_{p=0}, \mathcal{M}_{X_{p=0}})$ be its mod- p fiber. Suppose that

$(X_{p=0}, \mathcal{M}_{X_{p=0}})$ is also strongly quasi-syntomic. Then, taking the composition of the equivalences in Proposition 6.4.32 with the restriction functors from $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ to $(X_{p=0}, \mathcal{M}_{X_{p=0}})$, we obtain exact functors

$$\begin{aligned} \mathrm{Vect}_{\mathrm{kfl}}^*((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) &\rightarrow \mathrm{Vect}_{\mathrm{kfl}}^*((X_{p=0}, \mathcal{M}_{X_{p=0}})_{\mathrm{crys}}), \\ \mathrm{DM}_{\mathrm{kfl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) &\rightarrow \mathrm{DM}_{\mathrm{kfl}}((X_{p=0}, \mathcal{M}_{X_{p=0}})_{\mathrm{crys}}), \end{aligned}$$

for $\star \in \{\emptyset, \varphi\}$. These functors are also called *crystalline realization functors* and denoted by T_{crys} .

When $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is semi-stable log formal scheme over $\mathcal{O}_K$, for $\star \in \{\emptyset, \varphi\}$, crystalline realization functors

$$\mathrm{Vect}^*((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \mathrm{Vect}^*((X_{p=0}, \mathcal{M}_{X_{p=0}}))$$

are constructed in Construction 5.6.13 following [DLMS24, §3.5] by using Breuil log prisms. In the same way, one can also define a variant of crystalline realization functor

$$\mathrm{DM}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \mathrm{DM}((X_{p=0}, \mathcal{M}_{X_{p=0}})_{\mathrm{crys}}).$$

To distinguish them from the functors defined in Definition 6.4.33, we write $T_{\mathrm{crys}}^{\mathrm{Br}}$ for these functors. The relationship between these two crystalline realization functors is described as follows.

Proposition 6.4.34. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a semi-stable log formal scheme over $\mathcal{O}_K$. Then, for $\star \in \{\emptyset, \varphi\}$, the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Vect}^*((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) & \xrightarrow{T_{\mathrm{crys}}^{\mathrm{Br}}} & \mathrm{Vect}^*((X_{p=0}, \mathcal{M}_{X_{p=0}})_{\mathrm{crys}}) \\ \downarrow & & \downarrow \\ \mathrm{Vect}_{\mathrm{kfl}}^*((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) & \xrightarrow{T_{\mathrm{crys}}} & \mathrm{Vect}_{\mathrm{kfl}}^*((X_{p=0}, \mathcal{M}_{X_{p=0}})_{\mathrm{crys}}). \end{array}$$

An analogous statement holds for $\mathrm{DM}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$.

Proof. We prove only the case that $\star = \emptyset$ because other assertions are proved in the same way. By Proposition 6.4.31, it suffices to prove that there exists a natural isomorphism

$$T_{\mathrm{crys}}(\mathcal{E})_{(A_{\mathrm{crys}}(S) \twoheadrightarrow S, \mathcal{M}_{A_{\mathrm{crys}}(S)})} \cong T_{\mathrm{crys}}^{\mathrm{Br}}(\mathcal{E})_{(A_{\mathrm{crys}}(S) \twoheadrightarrow S, \mathcal{M}_{A_{\mathrm{crys}}(S)})}$$

for $(S, \mathcal{M}_S) \in (X_{p=0}, \mathcal{M}_{X_{p=0}})_{\mathrm{pkqrsp}}$, where $\mathcal{M}_{A_{\mathrm{crys}}(S)}$ is the unique log structure on $\mathrm{Spf}(A_{\mathrm{crys}}(S))$ whose pullback to $\mathrm{Spec}(S)$ is isomorphic to $\mathcal{M}_S$ given by Lemma 5.A.5. Working étale locally on $\mathfrak{X}$, we may assume that $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) = (\mathrm{Spf}(R), \mathcal{M}_R)$ is small affine. Fix a framing on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Let $\alpha: \mathbb{N}^r \rightarrow \mathcal{M}_{\mathfrak{X}}$ denote the chart induced from a fixed chart, and set

$$(\mathfrak{X}_{\infty}, \mathcal{M}_{\mathfrak{X}_{\infty}}) := (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \otimes_{\alpha, (\mathbb{Z}[\mathbb{N}^r], \mathbb{N}^r)^a} (\mathbb{Z}[\mathbb{Q}_{\geq 0}], \mathbb{Q}_{\geq 0}^r)^a.$$

Since the saturated fiber product of $(\mathrm{Spec}(S), \mathcal{M}_S)$ and $(X_{\infty, p=0}, \mathcal{M}_{X_{\infty, p=0}})$ over $(X_{p=0}, \mathcal{M}_{X_{p=0}})$ is a strict affine quasi-syntomic cover by Lemma 6.A.2 (2), passing to a quasi-syntomic cover of S allows us to assume that the structure map $(\mathrm{Spec}(S), \mathcal{M}_S) \rightarrow (X_{p=0}, \mathcal{M}_{X_{p=0}})$ admits a factorization

$$(6.4.10) \quad (\mathrm{Spec}(S), \mathcal{M}_S) \rightarrow (X_{\infty, p=0}, \mathcal{M}_{X_{\infty, p=0}}) \rightarrow (X_{p=0}, \mathcal{M}_{X_{p=0}}).$$

In this case, a map of p -adic log PD-thickenings in $(X_{p=0}, \mathcal{M}_{X_{p=0}})_{\mathrm{crys}}$

$$s: (S_R \twoheadrightarrow R/p, \mathcal{M}_{S_R}) \rightarrow (A_{\mathrm{crys}}(S) \twoheadrightarrow S, \mathcal{M}_{A_{\mathrm{crys}}(S)})$$

that is compatible with $\delta_{\log}$ -structures. Indeed, if we fix a factorization as in (6.4.10), there exists a unique monoid map $\mathbb{Q}_{\geq 0}^r \rightarrow \mathcal{M}_{A_{\text{crys}}(S)}$ fitting into the following commutative diagram:

$$\begin{array}{ccc} & & \Gamma(\text{Spf}(A_{\text{crys}}(S)), \mathcal{M}_{A_{\text{crys}}(S)}) \\ & \nearrow & \downarrow \\ \mathbb{Q}_{\geq 0}^r & \longrightarrow \Gamma(X_{\infty, p=0}, \mathcal{M}_{X_{\infty, p=0}}) & \longrightarrow \Gamma(\text{Spec}(S), \mathcal{M}_S). \end{array}$$

by Lemma 5.A.3. This lift gives a chart of $(\text{Spf}(A_{\text{crys}}(S)), \mathcal{M}_{A_{\text{crys}}(S)})$. Consider a prelog ring map $(\mathfrak{S}_{R^0}, \mathbb{N}^r) \rightarrow (A_{\text{crys}}(S), \mathbb{Q}_{\geq 0}^r)$ given by the ring map $\mathfrak{S}_{R^0} \rightarrow A_{\text{crys}}(S)$ sending x_i to $[x_i^b]$ and the natural inclusion map $\mathbb{N}^r \hookrightarrow \mathbb{Q}_{\geq 0}^r$. This prelog ring map induces a map of p -adic log PD-thickenings in $(\text{Spec}(R^0/p), \mathcal{M}_{R^0/p})_{\text{crys}}$

$$(S_{R^0} \twoheadrightarrow R^0/p, \mathcal{M}_{S_{R^0}}) \rightarrow (A_{\text{crys}}(S) \twoheadrightarrow S, \mathcal{M}_{A_{\text{crys}}(S)})$$

which is compatible with $\delta_{\log}$ -structures. Since $S_{R^0} \rightarrow S_R$ is étale, this map uniquely lifts to a map of p -adic log PD-thickenings in $(X_{p=0}, \mathcal{M}_{X_{p=0}})_{\text{crys}}$

$$s: (S_R \twoheadrightarrow R/p, \mathcal{M}_{S_R}) \rightarrow (A_{\text{crys}}(S) \twoheadrightarrow S, \mathcal{M}_{A_{\text{crys}}(S)})$$

that is compatible with $\delta_{\log}$ -structures.

When we fix s as in the previous paragraph, s naturally induces a log prism map

$$s: (S_R, (p), \phi^* \mathcal{M}_{S_R}) \rightarrow (A_{\text{crys}}(S), (p), \phi^* \mathcal{M}_{A_{\text{crys}}(S)}),$$

and we have isomorphisms

$$\begin{aligned} T_{\text{crys}}(\mathcal{E})_{(A_{\text{crys}}(S) \twoheadrightarrow S, \mathcal{M}_{A_{\text{crys}}(S)})} &\cong \mathcal{E}_{(A_{\text{crys}}(S), (p), \phi^* \mathcal{M}_{A_{\text{crys}}(S)})} \\ &\cong \mathcal{E}_{(S_R, (p), \phi^* \mathcal{M}_{S_R})} \otimes_{S_R, s} A_{\text{crys}}(S) \\ &\cong T_{\text{crys}}^{\text{Br}}(\mathcal{E})_{(S_R \twoheadrightarrow R/p, \mathcal{M}_{S_R})} \otimes_{S_R, s} A_{\text{crys}}(S) \\ &\cong T_{\text{crys}}^{\text{Br}}(\mathcal{E})_{(A_{\text{crys}}(S) \twoheadrightarrow S, \mathcal{M}_{A_{\text{crys}}(S)})}, \end{aligned}$$

where the first (resp. third) isomorphism follows from the definition of T_{crys} (resp. $T_{\text{crys}}^{\text{Br}}$). If we take another choice s' as in the previous paragraph, the maps s and s' induce a map of p -adic log PD-thickenings

$$(S_R^{(1)} \twoheadrightarrow R/p, \mathcal{M}_{S_R^{(1)}}) \rightarrow (A_{\text{crys}}(S) \twoheadrightarrow S, \mathcal{M}_{A_{\text{crys}}(S)})$$

that is compatible with the $\delta_{\log}$ -structures. This induces a log prism map

$$(S_R^{(1)}, (p), \phi^* \mathcal{M}_{S_R^{(1)}}) \rightarrow (A_{\text{crys}}(S), (p), \phi^* \mathcal{M}_{A_{\text{crys}}(S)}).$$

Then it follows from the crystal property of $\mathcal{E}$ and $T_{\text{crys}}^{\text{Br}}(\mathcal{E})$ that the isomorphism defined above is independent of the choice of s . In a similar way, one can check that the resulting isomorphism is also independent of the choice of a framing on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ (cf. [DLMS24, Lemma 3.31]). This completes the proof. $\square$

6.5. Log prismatic Dieudonné theory

6.5.1. Review on prismatic Dieudonné theory. we recall some results on prismatic Dieudonné theory for classical p -divisible groups. Let $\mathfrak{X}$ be a bounded p -adic formal scheme.

Construction 6.5.1. Let G be a finite and locally free group scheme or a p -divisible group over $\mathfrak{X}$. We define a sheaf G_Δ on $\mathfrak{X}_\Delta$ by

$$G_\Delta(A, I) := G(A/I).$$

Here, we consider an $\mathcal{O}_\Delta$ -module

$$\mathcal{M}_\Delta(G) := \mathcal{E}xt_{(\mathfrak{X}_\Delta)}^1(G_\Delta, \mathcal{O}_\Delta).$$

The morphism $\varphi: \mathcal{O}_\Delta \rightarrow \mathcal{O}_\Delta$ induces a $\mathcal{O}_\Delta$ -linear morphism $\varphi_{\mathcal{M}_\Delta(G)}: \varphi^* \mathcal{M}_\Delta(G) \rightarrow \mathcal{M}_\Delta(G)$.

Theorem 6.5.2 (Anschütz–Le Bras). Let $\mathfrak{X}$ be a quasi-syntomic p -adic formal scheme. Then the following statements hold.

- (1) Let G be a truncated Barsotti-Tate group over $\mathfrak{X}$ of level n and height h . Then $\mathcal{M}_\Delta(G)$ is a vector bundle of rank h on the ringed site $(\mathfrak{X}_\Delta, \mathcal{O}_\Delta/p^n)$.
- (2) Let G be a p -divisible group over $\mathfrak{X}$. Then $(\mathcal{M}_\Delta(G), \varphi_{\mathcal{M}_\Delta(G)})$ belongs to $\mathrm{DM}(\mathfrak{X}_\Delta)$, and the contravariant functor

$$\mathcal{M}_\Delta: \mathrm{BT}(\mathfrak{X}) \rightarrow \mathrm{DM}(\mathfrak{X}_\Delta)$$

is fully faithful. Moreover, if $\mathfrak{X}$ has an étale cover consisting of an affine formal scheme $\mathrm{Spf} R$ which admits a quasi-syntomic cover $R \rightarrow S$ with S being an integral perfectoid ring, this functor gives an anti-equivalence.

Proof. (1) See [ALB23, Proposition 4.68].

- (2) These are direct consequences of [ALB23, Theorem 4.74] and the proof of [ALB23, Proposition 5.10].

□

Theorem 6.5.3 (Anschütz–Le Bras). Let X be a quasi-syntomic scheme over $\mathbb{F}_p$ and G be a p -divisible group on X . Then there exists a natural isomorphism

$$T_{\mathrm{crys}}(\mathcal{M}_\Delta(G)) \cong \mathcal{M}_{\mathrm{crys}}(G),$$

where $\mathcal{M}_{\mathrm{crys}}$ is the contravariant crystalline Dieudonné functor of Berthelot-Breen-Messing ([BBM82, Définition 3.3.6]).

Proof. See [ALB23, Theorem 4.44].

□

Theorem 6.5.4 (Anschütz–Le Bras). Let $\mathfrak{X}$ be a quasi-syntomic p -adic formal scheme and G be a p -divisible group on $\mathfrak{X}$. Then there exists a canonical isomorphism

$$T_{\mathrm{ét}}(\mathcal{M}_\Delta(G))^\vee \cong G_\eta.$$

Proof. See [ALB23, Proposition 5.25].

□

6.5.2. Log prismatic Dieudonné functors. Let $(\mathfrak{X}, \mathcal{M}_\mathfrak{X})$ be a strongly quasi-syntomic log formal scheme. The aim of this subsection is to construct a functor

$$\mathcal{M}_\Delta: \mathrm{BT}(\mathfrak{X}, \mathcal{M}_\mathfrak{X}) \rightarrow \mathrm{DM}((\mathfrak{X}, \mathcal{M}_\mathfrak{X})_\Delta)$$

and study its properties. To this end, we work in larger categories and apply saturated descent properties to these larger categories.

Construction 6.5.5. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a strongly quasi-syntomic log formal scheme. Let $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{pkqsyn}}$ and $f: (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be the structure morphism. Since every weak log p -divisible group on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is f -classical by Lemma 6.3.12 and Lemma 6.3.13, we have pullback functors

$$\text{wBT}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \xrightarrow{f^*} \text{BT}(\mathfrak{Y}) \xrightarrow{\mathcal{M}_{\Delta}} \text{DM}(\mathfrak{Y}_{\Delta}),$$

where $\mathcal{M}_{\Delta}$ is the prismatic Dieudonné functor by Anschütz-Le Bras (see Theorem 6.5.2). Taking a limit with respect to $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\text{pkqsyn}}$, we obtain an exact functor

$$\text{wBT}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow \text{DM}_{\text{kfl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$$

by Proposition 6.4.16. The resulting functor is called a *log prismatic Dieudonné functor* and also denoted by $\mathcal{M}_{\Delta}$.

Proposition 6.5.6. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a strongly quasi-syntomic log formal scheme.

- (1) The log prismatic Dieudonné functors are compatible with the (non-log) prismatic Dieudonné functors. To be precise, we have the following commutative diagram:

$$\begin{array}{ccc} \text{BT}(\mathfrak{X}) & \xrightarrow{\mathcal{M}_{\Delta}} & \text{DM}(\mathfrak{X}_{\Delta}) \\ \downarrow & & \downarrow \\ \text{wBT}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) & \xrightarrow{\mathcal{M}_{\Delta}} & \text{DM}_{\text{kfl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}). \end{array}$$

- (2) The log prismatic Dieudonné functor $\mathcal{M}_{\Delta}$ is fully faithful.
- (3) Suppose that there exist an étale cover $\{\mathfrak{U}_i = \text{Spf}(R_i)\}_{i \in I}$ and a finite free chart $\mathbb{N}^{r_i} \rightarrow \mathcal{M}_{\mathfrak{U}_i}$ for each $i \in I$ such that, for each $i \in I$, there is a quasi-syntomic cover $R_i \widehat{\otimes}_{\mathbb{Z}_p \langle \mathbb{N}^r \rangle} \mathbb{Z}_p \langle \mathbb{Q}_{\geq 0}^r \rangle \rightarrow S_i$ for an integral perfectoid ring S_i . Then the log prismatic Dieudonné functor $\mathcal{M}_{\Delta}$ gives an equivalence.

Proof. The assertion (1) follows from the construction. For assertions (2) and (3), we may assume that $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ admits a finite free chart $\mathbb{N}^r \rightarrow \mathcal{M}_{\mathfrak{X}}$ by working étale locally on $\mathfrak{X}$. If we put

$$(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) := (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \otimes_{(\mathbb{Z}[\mathbb{N}^r], \mathbb{N}^r)^a} (\mathbb{Z}[\mathbb{Q}_{\geq 0}^r], \mathbb{Q}_{\geq 0}^r)^a$$

and set $(\mathfrak{Y}^{(\bullet)}, \mathcal{M}_{\mathfrak{Y}^{(\bullet)}})$ to be its saturated Čech nerve over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$, the log prismatic Dieudonné functor $\mathcal{M}_{\Delta}$ coincides with the functor defined as a limit of (non-log) prismatic Dieudonné functors

$$\mathcal{M}_{\Delta}: \text{BT}(\mathfrak{Y}^{(\bullet)}) \rightarrow \text{DM}(\mathfrak{Y}_{\Delta}^{(\bullet)})$$

with respect to $\bullet \in \Delta$ via equivalences in Proposition 6.3.27 and Proposition 6.4.12. Hence, the assertions from Theorem 6.5.2 (2). $\square$

Theorem 6.5.7. Suppose that $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is a semi-stable log formal scheme over $\mathcal{O}_K$. Then the functor

$$\text{wBT}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow \text{DM}_{\text{kfl}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$$

gives an bi-exact equivalence.

Proof. The assertion follows Lemma 5.2.3 and Proposition 6.5.6 (3). $\square$

Proposition 6.5.8 (Pullback compatibility of log prismatic Dieudonné functors). Let $f: (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a morphism of strongly quasi-syntomic log formal schemes. Let G be a weak log p -divisible group on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Then there exists a natural isomorphism

$$\mathcal{M}_{\Delta}(f^*G) \cong f^*\mathcal{M}_{\Delta}(G).$$

Proof. It is enough to construct a natural isomorphism

$$(\mathcal{M}_\Delta(f^*G))|_{(\mathfrak{Y}', \mathcal{M}_{\mathfrak{Y}'})} \cong (f^*\mathcal{M}_\Delta(G))|_{(\mathfrak{Y}', \mathcal{M}_{\mathfrak{Y}'})}$$

for every $(\mathfrak{Y}', \mathcal{M}_{\mathfrak{Y}'}) \in (\mathfrak{Y}, \mathcal{M}_\mathfrak{Y})_{\text{pkqsyn}}$. Consider the full subcategory $\mathcal{C}$ of $(\mathfrak{Y}, \mathcal{M}_\mathfrak{Y})_{\text{pkqsyn}}$ consisting of objects $p: (\mathfrak{Y}', \mathcal{M}_{\mathfrak{Y}'}) \rightarrow (\mathfrak{Y}, \mathcal{M}_\mathfrak{Y})$ such that the morphism fp admits a factorization

$$(\mathfrak{Y}', \mathcal{M}_{\mathfrak{Y}'}) \xrightarrow{f'} (\mathfrak{X}', \mathcal{M}_{\mathfrak{X}'}) \xrightarrow{q} (\mathfrak{X}, \mathcal{M}_\mathfrak{X})$$

for some object $(\mathfrak{X}', \mathcal{M}_{\mathfrak{X}'}) \in (\mathfrak{X}, \mathcal{M}_\mathfrak{X})_{\text{pkqsyn}}$. For $(\mathfrak{Y}', \mathcal{M}_{\mathfrak{Y}'}) \in \mathcal{C}$, if we fix the above factorization, we have isomorphisms

$$\begin{aligned} p^*(\mathcal{M}_\Delta(f^*G)) &\cong \mathcal{M}_\Delta(p^*(f^*G)) \\ &\cong \mathcal{M}_\Delta(f'^*(q^*G)) \\ &\cong f'^*\mathcal{M}_\Delta(q^*G) \\ &\cong f'^*q^*\mathcal{M}_\Delta(G) \\ &\cong p^*f^*\mathcal{M}_\Delta(G), \end{aligned}$$

where the first and forth isomorphisms arises from the definition of log prismatic Dieudonné functors, the second one is given by Lemma 6.3.12, the third one is given by the pullback compatibility of (non-log) prismatic Dieudonné functor, and the fifth isomorphism comes from Lemma 6.4.6. The resulting isomorphism

$$p^*(\mathcal{M}_\Delta(f^*G)) \cong p^*f^*\mathcal{M}_\Delta(G)$$

is independent of a choice of the above factorization because two such factorizations are dominated by a single factorization.

Therefore, to prove the statement, it suffices to show that every $(\mathfrak{Y}', \mathcal{M}_{\mathfrak{Y}'}) \in (\mathfrak{Y}, \mathcal{M}_\mathfrak{Y})_{\text{pfqsyn}}$ has a strict quasi-syntomic cover consisting of objects of $\mathcal{C}$. Working étale locally on $\mathfrak{Y}$, we may assume that there exist a chart $\alpha: \mathbb{N}^r \rightarrow \mathcal{M}_\mathfrak{Y}$ and a factorization

$$(\mathfrak{Y}', \mathcal{M}_{\mathfrak{Y}'}) \rightarrow (\mathfrak{Y}_{\infty, \alpha}, \mathcal{M}_{\mathfrak{Y}_{\infty, \alpha}}) := (\mathfrak{Y}, \mathcal{M}_\mathfrak{Y}) \otimes_{\alpha, (\mathbb{Z}[\mathbb{N}^r], \mathbb{N}^r)^a} (\mathbb{Z}[\mathbb{Q}_{\geq 0}^r], \mathbb{Q}_{\geq 0}^r)^a \rightarrow (\mathfrak{Y}, \mathcal{M}_\mathfrak{Y}).$$

Working further étale locally on $\mathfrak{X}$ and $\mathfrak{Y}$, we may assume that there exist a chart of f

$$\Gamma(\mathfrak{X}, \mathcal{M}_\mathfrak{X}) \xleftarrow{\beta} \mathbb{N}^s \rightarrow \mathbb{N}^r \oplus \mathbb{Z}^s \xrightarrow{\gamma} \Gamma(\mathfrak{Y}, \mathcal{M}_\mathfrak{Y})$$

such that the composite $\mathbb{N}^r \hookrightarrow \mathbb{N}^r \oplus \mathbb{Z}^s \xrightarrow{\gamma} \mathcal{M}_\mathfrak{Y}$ coincides with α by Lemma 6.A.3. We set

$$\begin{aligned} (\mathfrak{X}_{\infty, \beta}, \mathcal{M}_{\mathfrak{X}_{\infty, \beta}}) &:= (\mathfrak{X}, \mathcal{M}_\mathfrak{X}) \otimes_{\beta, (\mathbb{Z}[\mathbb{N}^s], \mathbb{N}^s)^a} (\mathbb{Z}[\mathbb{Q}_{\geq 0}^s], \mathbb{Q}_{\geq 0}^s)^a, \\ (\mathfrak{Y}_{\infty, \gamma}, \mathcal{M}_{\mathfrak{Y}_{\infty, \gamma}}) &:= (\mathfrak{Y}, \mathcal{M}_\mathfrak{Y}) \otimes_{\gamma, (\mathbb{Z}[\mathbb{N}^r \oplus \mathbb{Z}^s], \mathbb{N}^r \oplus \mathbb{Z}^s)^a} (\mathbb{Z}[\mathbb{Q}_{\geq 0}^r \oplus \mathbb{Q}^s], \mathbb{Q}_{\geq 0} \oplus \mathbb{Q})^a. \end{aligned}$$

Then we have the following commutative diagram:

$$\begin{array}{ccccc} (\mathfrak{Y}_{\infty, \gamma}, \mathcal{M}_{\mathfrak{Y}_{\infty, \gamma}}) & \longrightarrow & (\mathfrak{Y}_{\infty, \alpha}, \mathcal{M}_{\mathfrak{Y}_{\infty, \alpha}}) & \longrightarrow & (\mathfrak{Y}, \mathcal{M}_\mathfrak{Y}) \\ \downarrow & & & & \downarrow f \\ (\mathfrak{X}_{\infty, \beta}, \mathcal{M}_{\mathfrak{X}_{\infty, \beta}}) & \longrightarrow & & \longrightarrow & (\mathfrak{X}, \mathcal{M}_\mathfrak{X}). \end{array}$$

The upper left horizontal morphism is a strict affine quasi-syntomic cover because $\mathbb{Z}_p\langle\mathbb{Z}^s\rangle \rightarrow \mathbb{Z}_p\langle\mathbb{Q}^s\rangle$ is a quasi-syntomic cover. Therefore, if we put

$$(\mathfrak{Y}'', \mathcal{M}_{\mathfrak{Y}''}) := (\mathfrak{Y}', \mathcal{M}_{\mathfrak{Y}'}) \times_{(\mathfrak{Y}_{\infty, \alpha}, \mathcal{M}_{\mathfrak{Y}_{\infty, \alpha}})} (\mathfrak{Y}_{\infty, \gamma}, \mathcal{M}_{\mathfrak{Y}_{\infty, \gamma}}),$$

the natural map $(\mathfrak{Y}'', \mathcal{M}_{\mathfrak{Y}''}) \rightarrow (\mathfrak{Y}', \mathcal{M}_{\mathfrak{Y}'})$ is a strict affine quasi-syntomic cover, and $(\mathfrak{Y}'', \mathcal{M}_{\mathfrak{Y}''})$ belongs to $\mathcal{C}$. $\square$

Proposition 6.5.9. Let G be a log p -divisible group over a strongly quasi-syntomic fs log formal scheme $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Then the kfl prismatic Dieudonné crystal $\mathcal{M}_{\Delta}(G)$ belongs to $\mathrm{DM}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$.

Proof. First, we consider the case where $\mathfrak{X} = \mathrm{Spf}(R)$ with R/p being strict local. In this case, applying the functor $\mathcal{M}_{\Delta}$ to the exact sequence in Lemma 6.3.20, we obtain an exact sequence

$$0 \rightarrow \mathcal{M}_{\Delta}(G^0) \rightarrow \mathcal{M}_{\Delta}(G) \rightarrow \mathcal{M}_{\Delta}(G^{\mathrm{ét}}) \rightarrow 0.$$

Since $\mathcal{M}_{\Delta}(G^0)$ and $\mathcal{M}_{\Delta}(G^{\mathrm{ét}})$ belongs to $\mathrm{DM}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ by Proposition 6.5.6, so does $\mathcal{M}_{\Delta}(G)$ by Proposition 6.2.18.

In general, we may assume that $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) = (\mathrm{Spf}(R), \mathcal{M}_R)$ is affine. Let $(A, I, \mathcal{M}_A)$ be a log prism in $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}^{\mathrm{str}}$ with structure map $\pi: (\mathrm{Spf}(A/I), \mathcal{M}_{A/I}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. It is enough to prove that the kfl vector bundle $\mathcal{M}_{\Delta}(G)_{(A, I, \mathcal{M}_A)}$ on $(\mathrm{Spf}(A), \mathcal{M}_A)$ is classical. Take a point $x \in \mathrm{Spf}(A)$. Due to Lemma 6.2.16, it suffices to show that the pullback of $\mathcal{M}_{\Delta}(G)_{(A, I, \mathcal{M}_A)}$ to $(\mathrm{Spec}(k(x)), \mathcal{M}_{k(x)})$ is classical. Let S denote the p -completed strict localization of $\mathfrak{X}$ at a geometric point $\overline{\pi(x)}$ above $\pi(x)$. By [BS22, Lemma 2.18], there exists a unique prism (B, IB) over (A, I) with $B/IB \cong A \widehat{\otimes}_R S$. Now, the compatibility of log prismatic Dieudonné functors with pullback functors gives a natural isomorphism

$$(\mathcal{M}_{\Delta}(G)_{(A, I, \mathcal{M}_A)})|_{(\mathrm{Spf}(B), \mathcal{M}_B)} \cong \mathcal{M}_{\Delta}(G|_{(\mathrm{Spf}(S), \mathcal{M}_S)})_{(B, IB, \mathcal{M}_B)},$$

where $\mathcal{M}_B$ is the log structure on $\mathrm{Spf}(B)$ induced from $\mathcal{M}_A$. Hence, what we proved in the previous paragraph implies that $(\mathcal{M}_{\Delta}(G)_{(A, I, \mathcal{M}_A)})|_{(\mathrm{Spf}(B), \mathcal{M}_B)}$ is classical. Since the image of $\mathrm{Spf}(B) \rightarrow \mathrm{Spf}(A)$ contains x , the pullback of $\mathcal{M}_{\Delta}(G)_{(A, I, \mathcal{M}_A)}$ to $(\mathrm{Spec}(k(x)), \mathcal{M}_{k(x)})$ is also classical. $\square$

Definition 6.5.10. For a strongly quasi-syntomic log formal scheme $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$, the functor $\mathcal{M}_{\Delta}$ induces a fully faithful functor

$$\mathrm{BT}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow \mathrm{DM}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$$

by Proposition 6.5.9. This functor is also denoted by $\mathcal{M}_{\Delta}$.

The following proposition is a log version of the étale comparison theorem for prismatic Dieudonné crystals (Theorem 6.5.4).

Proposition 6.5.11 (Étale comparison isomorphisms for log prismatic Dieudonné functors). Let G be a log p -divisible group over a strongly quasi-syntomic log formal scheme $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Then there exists a natural isomorphism

$$(T_{\mathrm{ét}} \mathcal{M}_{\Delta}(G))^{\vee} \cong G_{\eta}.$$

Proof. Working étale locally on $\mathfrak{X}$, we may assume that there is a finite free chart $\mathbb{N}^r \rightarrow \mathcal{M}_{\mathfrak{X}}$. If we set

$$(\mathfrak{X}_{\infty}, \mathcal{M}_{\mathfrak{X}_{\infty}}) := (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \otimes_{(\mathbb{Z}[\mathbb{N}^r], \mathbb{N}^r)^a} (\mathbb{Z}[\mathbb{Q}_{\geq 0}^r], \mathbb{Q}_{\geq 0}^r)^a,$$

the natural morphism

$$(\mathfrak{X}_{\infty}, \mathcal{M}_{\mathfrak{X}_{\infty}})_{\eta}^{\diamond} \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta}^{\diamond}$$

is a quasi-pro-Kummer étale cover by [KY25, Proposition 7.20], and so there is a natural equivalence

$$(6.5.1) \quad \mathrm{Loc}_{\mathbb{Z}_p}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta, \mathrm{qprokét}}^{\diamond}) \rightarrow \varinjlim_{(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\mathrm{pkqsyn}}} \mathrm{Loc}_{\mathbb{Z}_p}((\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})_{\eta, \mathrm{qprokét}}^{\diamond}).$$

Take an object $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ of $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\mathrm{pkqsyn}}$ with structure map $f: (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Under the identification

$$(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})_{\eta, \mathrm{qprokét}}^{\diamond} \simeq \mathfrak{Y}_{\eta, \mathrm{qproét}}^{\diamond}$$

(proved in Lemma 5.B.1), we have isomorphisms of quasi-pro-étale $\mathbb{Z}_p$ -local systems on $\mathfrak{Y}_{\eta}^{\diamond}$

$$\begin{aligned} f_{\eta}^* T_{\mathrm{ét}} \mathcal{M}_{\Delta}(G)^{\vee} &\cong T_{\mathrm{ét}} f^* \mathcal{M}_{\Delta}(G)^{\vee} \\ &\cong T_{\mathrm{ét}} \mathcal{M}_{\Delta}(f^* G)^{\vee} \\ &\cong (f^* G)_{\eta} \\ &\cong f_{\eta}^*(G_{\eta}). \end{aligned}$$

Here, the third isomorphism comes from étale comparison isomorphisms for (non-log) prismatic Dieudonné functors (Theorem 6.5.4). By the functoriality of the isomorphism in Theorem 6.5.4, the isomorphism $f_{\eta}^* T_{\mathrm{ét}} \mathcal{M}_{\Delta}(G)^{\vee} \cong f_{\eta}^*(G_{\eta})$ constructed above is natural in $(\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\mathrm{pkqsyn}}$. Therefore, via the equivalence (6.5.1), we obtain an isomorphism $T_{\mathrm{ét}} \mathcal{M}_{\Delta}(G)^{\vee} \cong G_{\eta}$. $\square$

Theorem 6.5.12. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a semi-stable log formal scheme over $\mathcal{O}_K$. Then the log prismatic Dieudonné functor

$$\mathcal{M}_{\Delta}: \mathrm{BT}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow \mathrm{DM}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$$

gives an exact anti-equivalence.

6.5.3. Log crystalline Dieudonné functors. Let $(X, \mathcal{M}_X)$ be an fs log scheme over $\mathbb{F}_p$ admitting finite free charts étale locally.

Construction 6.5.13. Let $(Y, \mathcal{M}_Y) \in (X, \mathcal{M}_X)_{\mathrm{pkqsyn}}$ with structure map $f: (Y, \mathcal{M}_Y) \rightarrow (X, \mathcal{M}_X)$. Since every weak log p -divisible group on $(X, \mathcal{M}_X)$ is f -classical by Lemma 6.3.13 and Lemma 6.3.12, we have pullback functors

$$\mathrm{wBT}(X, \mathcal{M}_X) \xrightarrow{f^*} \mathrm{BT}(Y) \xrightarrow{\mathcal{M}_{\mathrm{crys}}} \mathrm{DM}(Y_{\mathrm{crys}}),$$

where $\mathcal{M}_{\mathrm{crys}}$ is the crystalline Dieudonné functor defined by Berthelot-Breen-Messing in [BBM82, Définition 3.3.6]. Taking a limit with respect to $(Y, \mathcal{M}_Y) \in (X, \mathcal{M}_X)_{\mathrm{pkqsyn}}$, we obtain an exact functor

$$\mathrm{wBT}(X, \mathcal{M}_X) \rightarrow \mathrm{DM}_{\mathrm{kfl}}((X, \mathcal{M}_X)_{\mathrm{crys}})$$

by Proposition 6.4.31. The resulting functor is called a log crystalline Dieudonné functor and also denoted by $\mathcal{M}_{\mathrm{crys}}$.

Proposition 6.5.14. The log crystalline Dieudonné functors are compatible with the (non-log) crystalline Dieudonné functors. Precisely, there is the following commutative diagram:

$$\begin{array}{ccc} \mathrm{BT}(X) & \xrightarrow{\mathcal{M}_{\mathrm{crys}}} & \mathrm{DM}(X_{\mathrm{crys}}) \\ \downarrow & & \downarrow \\ \mathrm{wBT}(X, \mathcal{M}_X) & \xrightarrow{\mathcal{M}_{\mathrm{crys}}} & \mathrm{DM}_{\mathrm{kfl}}((X, \mathcal{M}_X)_{\mathrm{crys}}). \end{array}$$

Proof. This follows from the definition. $\square$

Proposition 6.5.15 (Pullback compatibility of log crystalline Dieudonné functors). Let $f: (Y, \mathcal{M}_Y) \rightarrow (X, \mathcal{M}_X)$ be a morphism of log schemes over $\mathbb{F}_p$ admitting finite free charts étale locally. Let G be a weak log p -divisible group on $(X, \mathcal{M}_X)$. Then there exists a natural isomorphism

$$\mathcal{M}_{\text{crys}}(f^*G) \cong f^*\mathcal{M}_{\text{crys}}(G).$$

Proof. This follows from the same argument as the proof of Proposition 6.5.8 because the ring map $\mathbb{F}_p[\mathbb{Z}^s] \rightarrow \mathbb{F}_p[\mathbb{Q}^s]$ is a root map. $\square$

Proposition 6.5.16. Let G be a log p -divisible group over $(X, \mathcal{M}_X)$. Then the kfl crystalline Dieudonné crystal $\mathcal{M}_{\text{crys}}(G)$ belongs to $\text{DM}((X, \mathcal{M}_X)_{\text{crys}})$.

Proof. The assertion is proved in the same way as the proof of Proposition 6.5.9 by passing to a strict localization at a point on X . $\square$

Definition 6.5.17. The functor $\mathcal{M}_{\text{crys}}$ induces a functor

$$\text{BT}(X, \mathcal{M}_X) \rightarrow \text{DM}((X, \mathcal{M}_X)_{\text{crys}})$$

by Proposition 6.5.16. This functor is also denoted by $\mathcal{M}_{\text{crys}}$.

Proposition 6.5.18. Let G be a weak log p -divisible group over $(X, \mathcal{M}_X)$. Then there exists a natural exact sequence of kfl vector bundles on $(X, \mathcal{M}_X)$

$$0 \rightarrow \text{Lie}(G)^\vee \rightarrow \mathcal{M}_{\text{crys}}(G)_{(X, \mathcal{M}_X)} \rightarrow \text{Lie}(G^*) \rightarrow 0.$$

Proof. Let $(Y, \mathcal{M}_Y) \in (X, \mathcal{M}_X)_{\text{pkqsyn}}$ with structure map $f: (Y, \mathcal{M}_Y) \rightarrow (X, \mathcal{M}_X)$. Applying [BBM82, Corollaire 3.3.5] to a p -divisible group f^*G over Y , we get a natural exact sequence of vector bundles on Y

$$0 \rightarrow \text{Lie}(f^*G)^\vee \rightarrow \mathcal{M}_{\text{crys}}(f^*G)_{(Y, \mathcal{M}_Y)} \rightarrow \text{Lie}(f^*G^*) \rightarrow 0.$$

By Lemma 6.3.26, we have natural isomorphisms

$$\begin{aligned} \text{Lie}(f^*G)^\vee &\cong f^*(\text{Lie}(G)^\vee), \\ \text{Lie}(f^*G^*) &\cong f^*\text{Lie}(G^*). \end{aligned}$$

By the construction of $\mathcal{M}_{\text{crys}}$, there are natural isomorphisms

$$\begin{aligned} \mathcal{M}_{\text{crys}}(f^*G)_{(Y, \mathcal{M}_Y)} &\cong (f^*\mathcal{M}_{\text{crys}}(G))_{(Y, \mathcal{M}_Y)} \\ &\cong f^*(\mathcal{M}_{\text{crys}}(G)_{(X, \mathcal{M}_X)}). \end{aligned}$$

As a result, we obtain a natural exact sequence

$$0 \rightarrow f^*(\text{Lie}(G)^\vee) \rightarrow f^*(\mathcal{M}_{\text{crys}}(G)_{(X, \mathcal{M}_X)}) \rightarrow f^*\text{Lie}(G^*) \rightarrow 0.$$

Since strongly pro-kfl descent for kfl vector bundles (Proposition 6.2.26) gives a natural bi-exact equivalence

$$\text{Vect}_{\text{kfl}}(X, \mathcal{M}_X) \simeq \varprojlim_{(Y, \mathcal{M}_Y) \in (X, \mathcal{M}_X)_{\text{pkqsyn}}} \text{Vect}(Y),$$

the above exact sequence descends to the desired exact sequence. $\square$

The relationship between log prismatic Dieudonné functors and log crystalline Dieudonné functors is described as follows.

Proposition 6.5.19 (The crystalline comparison theorem for log prismatic Dieudonné functor). Let G be a weak log p -divisible group over a strongly quasi-syntomic log scheme $(X, \mathcal{M}_X)$ over $\mathbb{F}_p$. Then there is a natural isomorphism

$$T_{\text{crys}}(\mathcal{M}_{\Delta}(G)) \cong \mathcal{M}_{\text{crys}}(G).$$

Proof. Let $(Y, \mathcal{M}_Y) \in (X, \mathcal{M}_X)_{\text{pkqsyn}}$ with structure map $f: (Y, \mathcal{M}_Y) \rightarrow (X, \mathcal{M}_X)$. By Proposition 6.4.31, it is enough to construct a natural isomorphism

$$f^*T_{\text{crys}}(\mathcal{M}_{\Delta}(G)) \cong f^*\mathcal{M}_{\text{crys}}(G).$$

By the definition of the log crystalline Dieudonné functor, the right hand side is isomorphic to $\mathcal{M}_{\text{crys}}(f^*G)$. On the other hand, we have isomorphisms

$$\begin{aligned} f^*T_{\text{crys}}(\mathcal{M}_{\Delta}(G)) &\cong T_{\text{crys}}(f^*\mathcal{M}_{\Delta}(G)) \\ &\cong T_{\text{crys}}(\mathcal{M}_{\Delta}(f^*G)). \end{aligned}$$

Hence, the claim follows from the crystalline comparison for (non-log) prismatic Dieudonné functors (Theorem 6.5.3). $\square$

Proof of Theorem 6.5.12. The fully faithfulness is already proved in Proposition 6.5.6. We shall prove the essential surjectivity. Let $\mathcal{E}$ be an object of $\text{DM}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$. By Theorem 6.5.7, there exists $G \in \text{wBT}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ with $\mathcal{M}_{\Delta}(G) \cong \mathcal{E}$. It suffices to prove that G is a log p -divisible group. Let $(X_{p=0}, \mathcal{M}_{X_{p=0}})$ denote the mod- p fiber of $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$. Since $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is a semi-stable log formal scheme, the kfl crystalline crystal in vector bundles $\mathcal{M}_{\text{crys}}(G|_{(X_{p=0}, \mathcal{M}_{X_{p=0}})}) \cong T_{\text{crys}}(\mathcal{M}_{\Delta}(G))$ belongs to $\text{DM}((X_{p=0}, \mathcal{M}_{X_{p=0}})_{\text{crys}})$ by Proposition 6.4.34. Proposition 6.2.18 and Proposition 6.5.18 imply that both of $\text{Lie}(G)$ and $\text{Lie}(G^*)$ are classical. Therefore, G belongs to $\text{BT}(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ by Proposition 6.3.23. This completes the proof. $\square$

6.6. A Tannakian framework for log prismatic F -crystals

6.6.1. Preliminaries for Tannakian formalism. Throughout this section, we consider a right action when we refer to a pseudo-torsor.

Let $\mathcal{G}$ be a smooth affine group scheme over $\mathbb{Z}_p$ and $\text{Rep}_{\mathbb{Z}_p}(\mathcal{G})$ denote the category of finite free representations of $\mathcal{G}$ over $\mathbb{Z}_p$. Fix a finite free $\mathbb{Z}_p$ -module Λ_0 , a faithful representation $\mathcal{G} \hookrightarrow \text{GL}(\Lambda_0)$, and tensors $\mathbb{T}_0 \subset \Lambda_0^{\otimes}$ with $\mathcal{G} = \text{Fix}(\mathbb{T}_0)$. (This is possible by [Bro13, Theorem 1.1].)

Definition 6.6.1. For an exact $\mathbb{Z}_p$ -category $\mathcal{A}$, the groupoid of $\mathbb{Z}_p$ -linear exact tensor functors $\text{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \rightarrow \mathcal{A}$ is denoted by $\mathcal{G}\text{-}\mathcal{A}$.

Let $\mathcal{C}$ be a site and $\mathcal{O}$ be a sheaf of $\mathbb{Z}_p$ -algebras on $\mathcal{C}$. Let $\text{Vect}(\mathcal{C}, \mathcal{O})$ denote the category of vector bundles on the ringed site $(\mathcal{C}, \mathcal{O})$, which is an exact $\mathbb{Z}_p$ -category. For objects $\omega \in \mathcal{G}\text{-Vect}(\mathcal{C}, \mathcal{O})$ and $T \in \mathcal{C}$, the object of $\mathcal{G}\text{-Vect}(\mathcal{C}_T, \mathcal{O})$ obtained by restricting ω to the site $\mathcal{C}_T$ is denoted by ω_T . We define a group sheaf $\mathcal{G}_{\mathcal{O}}$ on $\mathcal{C}$ by $\mathcal{G}_{\mathcal{O}}(T) := \mathcal{G}(\mathcal{O}(T))$ for $T \in \mathcal{C}$.

Definition 6.6.2. ([IKY24, Appendix])

- (1) We define an object ω_{triv} of $\mathcal{G}\text{-Vect}(\mathcal{C}, \mathcal{O})$ by $\omega_{\text{triv}}(\Lambda) := \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}$ for $\Lambda \in \text{Rep}_{\mathbb{Z}_p}(\mathcal{G})$. We say that $\mathcal{G}$ is *reconstructible* in $(\mathcal{C}, \mathcal{O})$ if the natural map $\mathcal{G}_{\mathcal{O}} \rightarrow \underline{\text{Aut}}(\omega_{\text{triv}})$ is an isomorphism, where $\underline{\text{Aut}}(\omega_{\text{triv}})$ is a group sheaf on $\mathcal{C}$ given by

$$\underline{\text{Aut}}(\omega_{\text{triv}})(T) := \text{Aut}(\omega_{\text{triv}, T})$$

for $T \in \mathcal{C}$.

- (2) For an object ω of $\mathcal{G}\text{-Vect}(\mathcal{C}, \mathcal{O})$, we define an $\underline{\text{Aut}}(\omega_{\text{triv}})$ -pseudo-torsor $\underline{\text{Isom}}(\omega_{\text{triv}}, \omega)$ by

$$\underline{\text{Isom}}(\omega_{\text{triv}}, \omega)(T) := \text{Isom}_{\mathcal{G}\text{-Vect}(\mathcal{C}/T, \mathcal{O})}(\omega_{\text{triv}, T}, \omega_T)$$

for $T \in \mathcal{C}$. The object ω is called *locally trivial* if the pseudo-torsor $\underline{\text{Isom}}(\omega_{\text{triv}}, \omega)$ is a torsor. Let $\mathcal{G}\text{-Vect}^{\text{lt}}(\mathcal{C}, \mathcal{O})$ denote the full subcategory of $\mathcal{G}\text{-Vect}(\mathcal{C}, \mathcal{O})$ consisting of locally trivial objects.

We let $\text{Twist}_{\mathcal{O}}(\Lambda_0, \mathbb{T}_0)$ denote the groupoid of a pair $(\mathcal{E}, \mathbb{T})$ consisting of $\mathcal{E} \in \text{Vect}(\mathcal{C}, \mathcal{O})$ and $\mathbb{T} \in \mathcal{E}^{\otimes}$ such that $\mathcal{G}_{\mathcal{O}}$ -pseudo-torsor $\underline{\text{Isom}}((\Lambda_0 \otimes_{\mathbb{Z}_p} \mathcal{O}, \mathbb{T}_0 \otimes 1), (\mathcal{E}, \mathbb{T}))$ given by

$$T \mapsto \{f: \Lambda_0 \otimes_{\mathbb{Z}_p} \mathcal{O}_T \xrightarrow{\sim} \mathcal{E}_T : f(\mathbb{T}_0 \otimes 1) = \mathbb{T}\}$$

is a $\mathcal{G}_{\mathcal{O}}$ -torsor. We have a natural functor $\text{Twist}_{\mathcal{O}}(\Lambda_0, \mathbb{T}_0) \rightarrow \text{Tors}_{\mathcal{G}_{\mathcal{O}}}(\mathcal{C})$ sending $(\mathcal{E}, \mathbb{T})$ to $\underline{\text{Isom}}((\Lambda_0 \otimes_{\mathbb{Z}_p} \mathcal{O}, \mathbb{T}_0 \otimes 1), (\mathcal{E}, \mathbb{T}))$.

For an object $\mathcal{P} \in \text{Tors}_{\mathcal{G}_{\mathcal{O}}}(\mathcal{C})$, we define $\omega_{\mathcal{P}} \in \mathcal{G}\text{-Vect}^{\text{lt}}(\mathcal{C}, \mathcal{O})$ by $\omega_{\mathcal{P}}(\Lambda) := \mathcal{P} \wedge^{\mathcal{G}_{\mathcal{O}}} (\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O})$ for $\Lambda \in \text{Rep}_{\mathbb{Z}_p}(\mathcal{G})$.

Proposition 6.6.3. ([IKY24, Proposition A.17]) Assume that $\mathcal{G}$ is reconstructible in $(\mathcal{C}, \mathcal{O})$. Then we have the following triangle of equivalences.

$$\begin{array}{ccc} & & \text{Tors}_{\mathcal{G}_{\mathcal{O}}}(\mathcal{C}) \\ & \nearrow & \downarrow \mathcal{P} \mapsto \omega_{\mathcal{P}} \\ \text{Twist}_{\mathcal{O}}(\Lambda_0, \mathbb{T}_0) & \xleftarrow[\omega \mapsto (\omega(\Lambda_0), \omega(\mathbb{T}_0))]{\omega \mapsto (\omega(\Lambda_0), \omega(\mathbb{T}_0))} & \mathcal{G}\text{-Vect}^{\text{lt}}(\mathcal{C}, \mathcal{O}) \end{array}$$

6.6.2. The Tannakian theory of Kummer étale local systems. For convenience, we write $\mathbb{Z}_p/p^\infty$ for $\mathbb{Z}_p$.

Definition 6.6.4. Let $n \in \mathbb{N}_{\geq 1} \cup \{\infty\}$. Let $(X, \mathcal{M}_X)$ be a locally noetherian fs log adic space. We let $\text{Loc}_{\mathbb{Z}_p/p^n}(X, \mathcal{M}_X)$ (resp. $\text{Loc}_{\mathbb{Z}_p/p^n}(X)$) denote the category of $\mathbb{Z}_p/p^n$ -local system on $(X, \mathcal{M}_X)_{\text{prokét}}$ (resp. $X_{\text{prokét}}$). When $n \in \mathbb{N}_{\geq 1}$, the category $\text{Loc}_{\mathbb{Z}_p/p^n}(X, \mathcal{M}_X)$ (resp. $\text{Loc}_{\mathbb{Z}_p/p^n}(X)$) is equivalent to the category of $\mathbb{Z}/p^n$ -local systems on $(X, \mathcal{M}_X)_{\text{két}}$ (resp. $X_{\text{két}}$). We regard $\text{Loc}_{\mathbb{Z}_p/p^n}(X)$ as the full subcategory of $\text{Loc}_{\mathbb{Z}_p/p^n}(X, \mathcal{M}_X)$.

An object of $\mathcal{G}\text{-Loc}_{\mathbb{Z}_p/p^n}(X, \mathcal{M}_X)$ (resp. $\mathcal{G}\text{-Loc}_{\mathbb{Z}_p/p^n}(X)$) is called a *két $\mathbb{Z}_p/p^n$ -local system with G -structure* on $(X, \mathcal{M}_X)$ (resp. *étale $\mathbb{Z}_p/p^n$ -local system with G -structure* on X).

Proposition 6.6.5 ([DLLZ23b]). Let $(X, \mathcal{M}_X)$ be a connected and locally noetherian fs log adic space and $\bar{x}$ be a log geometric point on $(X, \mathcal{M}_X)$ (for the definition, see [DLLZ23b, Definition 4.4.2]). The geometric point on X defined by $\bar{x}$ is also denoted by $\bar{x}$.

- (1) The category of finite Kummer étale fs log adic spaces over $(X, \mathcal{M}_X)$ (resp. finite étale log adic spaces over X) together with the functor taking a fiber at $\bar{x}$ forms a Galois category. The corresponding fundamental group is denoted by $\pi_1^{\text{két}}((X, \mathcal{M}_X), \bar{x})$ (resp. $\pi_1^{\text{két}}(X, \bar{x})$). When there exists no possibility of confusion, $\bar{x}$ is omitted.
- (2) For an integer $n \geq 1$, the functor taking a stalk at $\bar{x}$ gives bi-exact equivalences

$$\text{Loc}_{\mathbb{Z}_p/p^n}(X, \mathcal{M}_X) \xrightarrow{\sim} \pi_1^{\text{két}}(X, \mathcal{M}_X)\text{-FMod}_{\mathbb{Z}/p^n}$$

$$\mathrm{Loc}_{\mathbb{Z}/p^n}(X) \xrightarrow{\sim} \pi_1^{\mathrm{\acute{e}t}}(X)\text{-}\mathrm{FMod}_{\mathbb{Z}/p^n}.$$

Here, $G\text{-}\mathrm{FMod}_{\mathbb{Z}/p^n}$ denote the category of finite and free $\mathbb{Z}/p^n$ -modules (equipped with discrete topology) with continuous G -action.

(3) there exists a natural surjection

$$\pi_1^{\mathrm{k\acute{e}t}}(X, \mathcal{M}_X) \rightarrow \pi_1^{\mathrm{\acute{e}t}}(X).$$

Proof. (1) See [DLLZ23b, Lemma 4.4.16].

(2) See [DLLZ23b, Theorem 4.4.15].

(3) Since the natural functor from the category of finite étale adic spaces over X to the category of finite Kummer étale fs log adic spaces over $(X, \mathcal{M}_X)$ is fully faithful, the assertion follows from the well-known property of Galois categories. $\square$

Lemma 6.6.6. Let $(X, \mathcal{M}_X)$ be a locally noetherian fs log adic space and $n \in \mathbb{N}_{\geq 1}$. Let

$$0 \rightarrow \mathbb{L}_1 \rightarrow \mathbb{L}_2 \rightarrow \mathbb{L}_3 \rightarrow 0$$

be an exact sequence in $\mathrm{Loc}_{\mathbb{Z}/p^n}(X, \mathcal{M}_X)$. Suppose that $\mathbb{L}_2$ belongs to $\mathrm{Loc}_{\mathbb{Z}/p^n}(X)$. Then $\mathbb{L}_1$ and $\mathbb{L}_3$ also belong to $\mathrm{Loc}_{\mathbb{Z}/p^n}(X)$.

Proof. We may assume that X is connected. The exact sequence in the statement corresponds to an exact sequence

$$0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$$

of finite free $\mathbb{Z}/p^n$ -modules M_i with continuous $\pi_1^{\mathrm{k\acute{e}t}}(X, \mathcal{M}_X)$ -actions for $i = 1, 2, 3$. By the assumption, $\mathrm{Ker}(\pi_1^{\mathrm{k\acute{e}t}}(X, \mathcal{M}_X) \twoheadrightarrow \pi_1^{\mathrm{\acute{e}t}}(X))$ acts trivially on M_2 . Hence $\pi_1^{\mathrm{k\acute{e}t}}(X, \mathcal{M}_X)$ -action on M_1 and M_3 factors $\pi_1^{\mathrm{\acute{e}t}}(X)$ -action, which implies that $\mathbb{L}_1$ and $\mathbb{L}_3$ belong to $\mathrm{Loc}(X)$. $\square$

Lemma 6.6.7. Let $(X, \mathcal{M}_X)$ be a locally noetherian fs log adic space and $n \in \mathbb{N}_{\geq 1}$. Then, for $\omega \in \mathcal{G}\text{-}\mathrm{Loc}_{\mathbb{Z}/p^n}(X, \mathcal{M}_X)$, there exists a finite Kummer étale cover $(Y, \mathcal{M}_Y) \rightarrow (X, \mathcal{M}_X)$ such that $\omega|_{(Y, \mathcal{M}_Y)}$ belongs to $\mathcal{G}\text{-}\mathrm{Loc}_{\mathbb{Z}/p^n}(Y)$.

Proof. Take a finite Kummer étale cover $(Y, \mathcal{M}_Y) \rightarrow (X, \mathcal{M}_X)$ such that $\omega(\Lambda_0)|_{(Y, \mathcal{M}_Y)}$ belongs to $\mathrm{Loc}_{\mathbb{Z}/p^n}(Y)$. Then it follows from Lemma 6.6.6 and [DS09, Proposition 12] that $\omega(\Lambda)|_{(Y, \mathcal{M}_Y)}$ belongs to $\mathrm{Loc}_{\mathbb{Z}/p^n}(Y)$ for every $\Lambda \in \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G})$. $\square$

Proposition 6.6.8. Let $(X, \mathcal{M}_X)$ be a locally noetherian fs log adic space. Then $\mathcal{G}$ is reconstructible in $((X, \mathcal{M}_X)_{\mathrm{prok\acute{e}t}}, \widehat{\mathbb{Z}}_p)$ and every object of $\mathcal{G}\text{-}\mathrm{Loc}_{\mathbb{Z}_p}(X, \mathcal{M}_X)$ is locally trivial on $(X, \mathcal{M}_X)_{\mathrm{prok\acute{e}t}}$.

Proof. Since $\mathcal{G}\text{-}\mathrm{Loc}_{\mathbb{Z}_p}(X) \rightarrow \mathcal{G}\text{-}\mathrm{Loc}_{\mathbb{Z}_p}(X, \mathcal{M}_X)$ is fully faithful, it follows from [IKY24, Proposition 2.8] that $\mathcal{G}$ is reconstructible in $((S, \mathcal{M}_X)_{\mathrm{prok\acute{e}t}}, \widehat{\mathbb{Z}}_p)$.

Next, let $\omega \in \mathcal{G}\text{-}\mathrm{Loc}_{\mathbb{Z}_p}(X, \mathcal{M}_X)$. To prove ω is locally trivial, it suffices to show that ω_n is trivial after finite Kummer étale base change for every $n \geq 1$. Here, $\omega_n \in \mathcal{G}\text{-}\mathrm{Loc}_{\mathbb{Z}/p^n}(X, \mathcal{M}_X)$ is the mod- p^n reduction of ω . Fix $n \geq 1$. By Lemma 6.6.7, we may assume that ω_n belongs to $\mathcal{G}\text{-}\mathrm{Loc}_{\mathbb{Z}/p^n}(X)$. Then the claim follows from the argument in [IKY24, Proposition 2.3]. $\square$

Proposition 6.6.9. We have the following commutative triangle of equivalences.

$$\begin{array}{ccc} \mathrm{Twist}_{\widehat{\mathbb{Z}_p}}(\Lambda_0, \mathbb{T}_0) & \longrightarrow & \mathrm{Tors}_{\mathcal{G}_{\widehat{\mathbb{Z}_p}}}((X, \mathcal{M}_X)_{\mathrm{prokét}}) \\ & \nwarrow & \downarrow \\ & & \mathcal{G}\text{-}\mathrm{Loc}_{\mathbb{Z}_p}(X, \mathcal{M}_X) \end{array}$$

Proof. The assertion follows from Proposition 6.6.3 and Proposition 6.6.8. $\square$

6.6.3. The Tannakian theory of prismatic F -crystals. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a bounded p -adic log formal scheme.

Definition 6.6.10. An object of $\mathcal{G}\text{-}\mathrm{Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ is called a *log prismatic F -crystal with G -structure* on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$.

We define a group sheaf $\mathcal{G}_{\Delta}$ on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}$ by

$$\mathcal{G}_{\Delta}(A, I, \mathcal{M}_A) := \mathcal{G}_{\mathcal{O}_{\Delta}}(A, I, \mathcal{M}_A) = \mathcal{G}(A).$$

The ring map $\phi_A: A \rightarrow A$ for each $(A, I, \mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}$ induces a group map $\phi: \mathcal{G}_{\Delta} \rightarrow \mathcal{G}_{\Delta}$. Let $\mathcal{G}_{\Delta}[1/\mathcal{I}_{\Delta}]$ denote the group sheaf on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}$ sending $(A, I, \mathcal{M}_A)$ to $\mathcal{G}(A[1/I])$. We have a natural injection $\iota: \mathcal{G}_{\Delta} \rightarrow \mathcal{G}_{\Delta}[1/\mathcal{I}_{\Delta}]$. For a $\mathcal{G}_{\Delta}$ -torsor $\mathcal{P}$, we set $\mathcal{P}[1/\mathcal{I}_{\Delta}]$ to be the $\mathcal{G}_{\Delta}[1/\mathcal{I}_{\Delta}]$ -torsor $\iota_*\mathcal{P}$.

Definition 6.6.11. (1) For a bounded prism (A, I) , we let $\mathrm{Twist}_{(A, I)}^{\varphi}(\Lambda_0, \mathbb{T}_0)$ denote the groupoid of a pair $((\mathcal{E}, \varphi_{\mathcal{E}}), \mathbb{T}_{\Delta})$ consisting of an object $(\mathcal{E}, \varphi_{\mathcal{E}}) \in \mathrm{Vect}^{\varphi}(A, I)$ and tensors $\mathbb{T}_{\Delta} \subset \mathcal{E}^{\otimes}$ fixed by $\varphi_{\mathcal{E}}$ such that $\underline{\mathrm{Isom}}((\Lambda_0 \otimes_{\mathbb{Z}_p} A, \mathbb{T}_0 \otimes 1), (\mathcal{E}, \mathbb{T}_{\Delta}))$ is a $\mathcal{G}$ -torsor on $\mathrm{Spf}(A)_{\mathrm{fpqc}}$.
 (2) We let $\mathrm{Twist}_{\mathcal{O}_{\Delta}}^{\varphi}(\Lambda_0, \mathbb{T}_0)$ denote the groupoid of a pair $((\mathcal{E}, \varphi_{\mathcal{E}}), \mathbb{T}_{\Delta})$ consisting of an object $(\mathcal{E}, \varphi_{\mathcal{E}}) \in \mathrm{Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ and tensors $\mathbb{T}_{\Delta} \subset \mathcal{E}^{\otimes}$ fixed by $\varphi_{\mathcal{E}}$ such that $\underline{\mathrm{Isom}}((\Lambda_0 \otimes_{\mathbb{Z}_p} \mathcal{O}_{\Delta}, \mathbb{T}_0 \otimes 1), (\mathcal{E}, \mathbb{T}_{\Delta}))$ is a $\mathcal{G}_{\Delta}$ -torsor on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}$.

Lemma 6.6.12. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a bounded p -adic log formal scheme. Then there exists a natural equivalence

$$\mathrm{Twist}_{\mathcal{O}_{\Delta}}^{\varphi}(\Lambda_0, \mathbb{T}_0) \simeq \varprojlim_{(A, I, \mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}} \mathrm{Twist}_{(A, I)}^{\varphi}(\Lambda_0, \mathbb{T}_0).$$

Proof. The equivalence in Remark 5.5.10 gives a fully faithful functor

$$\mathrm{Twist}_{\mathcal{O}_{\Delta}}^{\varphi}(\Lambda_0, \mathbb{T}_0) \rightarrow \varprojlim_{(A, I, \mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}} \mathrm{Twist}_{(A, I)}^{\varphi}(\Lambda_0, \mathbb{T}_0).$$

To check essential surjectivity, it is enough to show that, for a pair $((\mathcal{E}, \varphi_{\mathcal{E}}), \mathbb{T}_{\Delta})$ consisting of $(\mathcal{E}, \varphi_{\mathcal{E}}) \in \mathrm{Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ and tensors $\mathbb{T}_{\Delta} \subset \mathcal{E}^{\otimes}$ fixed by $\varphi_{\mathcal{E}}$ such that

$$\mathcal{P}_{(A, I, \mathcal{M}_A)} := \underline{\mathrm{Isom}}((\Lambda_0 \otimes_{\mathbb{Z}_p} A, \mathbb{T}_0 \otimes 1), (\mathcal{E}_{(A, I, \mathcal{M}_A)}, \mathbb{T}_{\Delta, (A, I, \mathcal{M}_A)}))$$

is a $\mathcal{G}$ -torsor on $\mathrm{Spf}(A)_{\mathrm{fpqc}}$ for each log prism $(A, I, \mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}$, the object

$$\mathcal{P} := \underline{\mathrm{Isom}}((\Lambda_0 \otimes_{\mathbb{Z}_p} \mathcal{O}_{\Delta}, \mathbb{T}_0 \otimes 1), (\mathcal{E}, \mathbb{T}_{\Delta}))$$

is a $\mathcal{G}_{\Delta}$ -torsor. The smoothness of $\mathcal{G}$ implies that $\mathcal{P}_{(A, I, \mathcal{M}_A)}$ is a $\mathcal{G}$ -torsor on $\mathrm{Spf}(A)_{\mathrm{ét}}$. Since every étale covering $\mathrm{Spf}(B) \rightarrow \mathrm{Spf}(A)$ extends to an étale covering $(A, I, \mathcal{M}_A) \rightarrow (B, IB, \mathcal{M}_B)$ in $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}$ by [Kos22, Lemma 2.13], $\mathcal{P}$ is a $\mathcal{G}_{\Delta}$ -torsor. $\square$

Construction 6.6.13. For a bounded p -adic log formal scheme $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$, we define a functor

$$\mathrm{Twist}_{\mathcal{O}_{\Delta}}^{\varphi}(\Lambda_0, \mathbb{T}_0) \rightarrow \mathcal{G}\text{-Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$$

as follows. Let $((\mathcal{E}, \varphi_{\mathcal{E}}), \mathbb{T}_{\Delta}) \in \mathrm{Twist}_{\mathcal{O}_{\Delta}}^{\varphi}(\Lambda_0, \mathbb{T}_0)$. We set a $\mathcal{G}_{\Delta}$ -torsor

$$\mathcal{P} := \underline{\mathrm{Isom}}((\Lambda_0 \otimes_{\mathbb{Z}_p} \mathcal{O}_{\Delta}, \mathbb{T}_0 \otimes 1), (\mathcal{E}, \mathbb{T}_{\Delta})).$$

Let $\omega_{\mathcal{P}}$ denote the object of $\mathcal{G}\text{-Vect}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ defined by $\omega_{\mathcal{P}}(\Lambda) := \mathcal{P} \wedge^{\mathcal{G}_{\Delta}} (\Lambda \otimes \mathcal{O}_{\Delta})$. The isomorphism $\varphi_{\mathcal{E}}: (\phi^* \mathcal{E})[1/\mathcal{I}_{\Delta}] \xrightarrow{\sim} \mathcal{E}[1/\mathcal{I}_{\Delta}]$ induces isomorphisms of $\mathcal{G}_{\Delta}[1/\mathcal{I}_{\Delta}]$ -torsors

$$\begin{aligned} (\phi_* \mathcal{P})[1/\mathcal{I}_{\Delta}] &\cong \underline{\mathrm{Isom}}((\Lambda_0 \otimes \mathcal{O}_{\Delta}, \mathbb{T}_0 \otimes 1), ((\phi^* \mathcal{E})[1/I], \phi^* \mathbb{T}_{\Delta})) \\ &\xrightarrow{\sim} \underline{\mathrm{Isom}}((\Lambda_0 \otimes \mathcal{O}_{\Delta}, \mathbb{T}_0 \otimes 1), (\mathcal{E}[1/I], \mathbb{T}_{\Delta})) \\ &\cong \mathcal{P}[1/\mathcal{I}_{\Delta}], \end{aligned}$$

which gives isomorphisms of $\mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]$ -modules

$$\begin{aligned} (\phi^* \omega_{\mathcal{P}}(\Lambda))[1/\mathcal{I}_{\Delta}] &\cong (\phi_* \mathcal{P})[1/\mathcal{I}_{\Delta}] \wedge^{\mathcal{G}_{\Delta}[1/\mathcal{I}_{\Delta}]} (\Lambda \otimes \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]) \\ &\xrightarrow{\sim} \mathcal{P}[1/\mathcal{I}_{\Delta}] \wedge^{\mathcal{G}_{\Delta}[1/\mathcal{I}_{\Delta}]} (\Lambda \otimes \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]) \\ &\cong \omega_{\mathcal{P}}(\Lambda)[1/\mathcal{I}_{\Delta}]. \end{aligned}$$

Definition 6.6.14. Let $\mathrm{Loc}_{\mathbb{Z}_p}^{\Delta\text{-gr}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta})$ be the essential image of the étale realization functor $T_{\text{ét}}: \mathrm{Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \mathrm{Loc}_{\mathbb{Z}_p}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta})$ (which is fully faithful by Corollary 5.6.28). Let $\mathrm{Loc}_{\mathbb{Z}_p}^{\Delta\text{-gr}, \text{an}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta})$ be the essential image of the étale realization functor $T_{\text{ét}}: \mathrm{Vect}^{\text{an}, \varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \mathrm{Loc}_{\mathbb{Z}_p}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta})$.

Proposition 6.6.15. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a horizontally semi-stable log formal scheme over $\mathcal{O}_K$. Then the functor induced the étale realization

$$\mathcal{G}\text{-Vect}^{\text{an}, \varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \mathcal{G}\text{-Loc}_{\mathbb{Z}_p}^{\Delta\text{-gr}, \text{an}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta})$$

gives an equivalence.

Proof. The assertion follows from the fact that the étale realization functor

$$T_{\text{ét}}: \mathrm{Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \mathrm{Loc}_{\mathbb{Z}_p}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta})$$

gives a bi-exact equivalence to the essential image (Theorem 5.6.27 and Theorem 5.6.32). $\square$

Theorem 6.6.16. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be a horizontally semi-stable log formal scheme over $\mathcal{O}_K$. Suppose that $\mathcal{G}$ is reductive. Then the functor induced the étale realization

$$\mathcal{G}\text{-Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \mathcal{G}\text{-Loc}_{\mathbb{Z}_p}^{\Delta\text{-gr}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta})$$

gives an equivalence.

proof of Theorem 6.6.16. We follow the argument of the proof of [IKY24, Theorem 2.28]. By working étale locally on $\mathfrak{X}$, we may assume that $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is small affine log formal scheme with a fixed framing. Since $T_{\text{ét}}: \mathrm{Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}) \rightarrow \mathrm{Loc}_{\mathbb{Z}_p}^{\Delta\text{-gr}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta})$ is fully faithful Corollary 5.6.28, the functor in the statement is also fully faithful. We shall prove the essential surjectivity. Let $\omega_{\text{ét}} \in \mathcal{G}\text{-Loc}_{\mathbb{Z}_p}^{\Delta\text{-gr}}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta})$. By Proposition 6.6.8, we have the corresponding object $(\omega_{\text{ét}}(\Lambda_0), \mathbb{T}_{\text{ét}}) \in \mathrm{Twist}_{\mathbb{Z}_p}(\Lambda_0, \mathbb{T}_0)$ where $\omega_{\text{ét}}(\Lambda_0)$ is a $\mathbb{Z}_p$ -local system on

$(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\eta}$ and tensors $\mathbb{T}_{\text{ét}} \subset \omega_{\text{ét}}(\Lambda_0)^{\otimes}$ is induced from $\mathbb{T}_0 \subset \Lambda_0^{\otimes}$. Let $(\mathcal{E}, \varphi_{\mathcal{E}})$ be a prismatic F -crystal on $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ with $T_{\text{ét}}(\mathcal{E}, \varphi_{\mathcal{E}}) \cong \omega_{\text{ét}}(\Lambda_0)$. Since $T_{\text{ét}}$ is fully faithful, the tensors $\mathbb{T}_{\text{ét}} \subset \omega_{\text{ét}}(\Lambda_0)^{\otimes}$ come from unique tensors $\mathbb{T}_{\Delta} \subset \mathcal{E}^{\otimes}$ which is fixed by $\varphi_{\mathcal{E}}$.

We shall show that $((\mathcal{E}, \varphi_{\mathcal{E}}), \mathbb{T}_{\Delta})$ belongs to $\text{Twist}_{\mathcal{O}_{\Delta}}^{\varphi}(\Lambda_0, \mathbb{T}_0)$. To achieve this, it suffices to prove that

$$\mathcal{P} := \underline{\text{Isom}}((\Lambda_0 \otimes_{\mathbb{Z}_p} \mathcal{O}_{\Delta}, \mathbb{T}_0 \otimes 1), (\mathcal{E}, \mathbb{T}_{\Delta}))$$

is a $\mathcal{G}_{\Delta}$ -torsor. Proposition 6.6.15 implies that

$$\mathcal{P}_{(A, I, \mathcal{M}_A)} := \underline{\text{Isom}}((\Lambda_0 \otimes_{\mathbb{Z}_p} A, \mathbb{T}_0 \otimes 1), (\mathcal{E}_{(A, I, \mathcal{M}_A)}, \mathbb{T}_{\Delta}))$$

is a $\mathcal{G}$ -torsor on $U(A, I)_{\text{ét}}$ for every $(A, I, \mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}$. Since $\mathfrak{S}_R$ is normal and $U(\mathfrak{S}_R, (E))$ contains all points of codimension 1 in $\text{Spec}(\mathfrak{S}_R)_{\text{ét}}$, we see that $\mathcal{P}_{(\mathfrak{S}_R, (E), \mathcal{M}_{\mathfrak{S}_R})}$ is a $\mathcal{G}$ -torsor on $\text{Spec}(\mathfrak{S}_R)$ by [IKY24, Proposition A.26 or Remark A.27]. Hence, $\mathcal{P}_{(A, I, \mathcal{M}_A)}$ is a $\mathcal{G}$ -torsor on $\text{Spf}(A)_{\text{fpqc}}$ for every log prism $(A, I, \mathcal{M}_A) \in (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta}$ by Lemma 5.5.16. Therefore, $\mathcal{P}$ is a $\mathcal{G}_{\Delta}$ -torsor.

Since the étale realization functor sends $((\mathcal{E}, \varphi_{\mathcal{E}}), \mathbb{T}_{\Delta})$ to $(\omega_{\text{ét}}(\Lambda_0), \mathbb{T}_{\text{ét}})$, the object $\omega_{\Delta} \in \mathcal{G}\text{-Vect}^{\varphi}((\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})_{\Delta})$ which comes from $((\mathcal{E}, \varphi_{\mathcal{E}}), \mathbb{T}_{\Delta}) \in \text{Twist}_{\mathcal{O}_{\Delta}}^{\varphi}(\Lambda_0, \mathbb{T}_0)$ via the functor defined in Construction 6.6.13 is sent to $\omega_{\text{ét}}$ by Proposition 6.6.9. This completes the proof. $\square$

6.7. Application to Shimura varieties

6.7.1. Log prismatic realization on the toroidal compactifications of Shimura varieties. Throughout this section, let $p > 2$ be a prime number.

Setting 6.7.1. Define the following notation:

- (G, X) is a Shimura datum of Hodge type with reflex field E ,
- v is a place of E above p , and E_v denotes the completion of E at v ,
- $\mathcal{O}_{E_v}$ is the integer ring of E_v with residue field k_v ,
- $\mathcal{G}$ is a reductive model of $G_{\mathbb{Q}_p}$ over $\mathbb{Z}_p$,
- $K_p := \mathcal{G}(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$.

Let $\text{Sh}_K(G, X)$ denote the associated Shimura variety over E . By [Kim18, 3.3.1], we can choose an embedding of Shimura datum $(G, X) \hookrightarrow (\text{GSp}(V), S^{\pm})$ and a self-dual $\mathbb{Z}_{(p)}$ -lattice $\Lambda_{\mathbb{Z}_{(p)}} \subset V$ such that $\Lambda := \Lambda_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p \subset V_{\mathbb{Q}_p}$ is K_p -stable. Let $K'_p := \text{GSp}(\Lambda)$. Let $K'^p \subset \text{GSp}(V)(\mathbb{A}_f^p)$ be a neat open compact subgroup containing K^p and let $K' := K'_p K'^p$. Then the Shimura variety $\text{Sh}_{K'}(\text{GSp}(V), S^{\pm})_{\mathbb{Q}_p}$ has an integral model $\mathcal{S}_{K'}(\text{GSp}(V), S^{\pm})$ over $\mathbb{Z}_p$ defined as the moduli space of polarized abelian varieties with K' -level structures. Let K^p be a open compact subgroup of $G(\mathbb{A}_f^p)$ contained in K'^p . Let $\mathcal{S}_{K_p K^p}(G, X)$ denote the normalization of the scheme theoretic image of

$$\text{Sh}_{K_p K^p}(G, X)_{E_v} \rightarrow \text{Sh}_{K'}(\text{GSp}(V), S^{\pm})_{E_v} \hookrightarrow \mathcal{S}_{K'}(\text{GSp}(V), S^{\pm})_{\mathcal{O}_{E_v}}.$$

Then the tower $\{\mathcal{S}_{K_p K^p}(G, X)\}_{K^p}$ is an integral canonical model of $\text{Sh}_K(G, X)_{E_v}$, which is smooth over $\mathcal{O}_{E_v}$ ([Kis10, Theorem 2.3.8]). Fix such a K^p and set $K := K_p K^p$.

Fix a smooth projective admissible rational cone decomposition Σ' for $((\text{GSp}(V), S^{\pm}), K')$ ([Mad19, 2.1.23]). Let Σ'' denote the induced admissible rational cone decomposition for $((G, X), K)$ from Σ' , and fix a smooth projective admissible rational cone decomposition Σ for $((G, X), K)$ refining Σ'' ([Mad19, 2.1.28]). Let $\mathcal{S}_{K'}^{\Sigma'}(\text{GSp}(V), S^{\pm})$ be the toroidal compactification of the Siegel modular variety with level K' constructed in [FC90]. Let $\mathcal{S}_K^{\Sigma''}(G, X)$ denote the normalization of $\mathcal{S}_{K'}^{\Sigma'}(\text{GSp}(V), S^{\pm})$

in $\mathrm{Sh}_K(G, X)$, which is the toroidal compactification of $\mathcal{S}_K(G, X)$ constructed in [Mad19]. By the technique of toroidal embeddings ([KKMSD, Chapter II]), we have an integral model $\mathcal{S}_K^\Sigma(G, X)$ of $\mathrm{Sh}_K^\Sigma(G, X)$ over $\mathcal{O}_{E_v}$ and a birational morphism $\mathcal{S}_K^\Sigma(G, X) \rightarrow \mathcal{S}_K^{\Sigma'}(G, X)$ (see also [Mad19, 4.1.4 and Remark 4.1.6]). As a result, we obtain the following diagram:

$$\begin{array}{ccc} \mathrm{Sh}_K(G, X)_{E_v} & \hookrightarrow & \mathrm{Sh}_{K'}(\mathrm{GSp}(V), S^\pm)_{E_v} \\ \downarrow & & \downarrow \\ \mathcal{S}_K(G, X) & \longrightarrow & \mathcal{S}_{K'}(\mathrm{GSp}(V), S^\pm)_{\mathcal{O}_{E_v}} \\ \downarrow & & \downarrow \\ \mathcal{S}_K^\Sigma(G, X) & \longrightarrow & \mathcal{S}_{K'}^{\Sigma'}(\mathrm{GSp}(V), S^\pm)_{\mathcal{O}_{E_v}}. \end{array}$$

Here, the vertical morphisms are open immersions.

We write $\mathcal{S}_K^\Sigma, \mathcal{S}_{K'}^{\Sigma'}$ for $\mathcal{S}_K^\Sigma(G, X), \mathcal{S}_{K'}^{\Sigma'}(\mathrm{GSp}(V), S^\pm)$ respectively. We equip $\mathcal{S}_K^\Sigma$ (resp. $\mathcal{S}_{K'}^{\Sigma'}$) with log structures $\mathcal{M}_{\mathcal{S}_K^\Sigma}$ (resp. $\mathcal{M}_{\mathcal{S}_{K'}^{\Sigma'}}$) defined from relatively normal crossings boundary divisors. Note that the p -completions of them are horizontally semi-stable log formal schemes by the smoothness and projectivity of Σ and Σ' .

In [KKN21] and [KKN22, Theorem 2.2.2, Proposition 4.3.4], the toroidal compactification $\mathcal{S}_{K'}^{\Sigma'}$ is realized as the moduli space of principally polarized log abelian varieties with K' -level structure. In particular, the universal abelian scheme over the interior extends to the universal log abelian varieties $\mathcal{A}_{\mathrm{univ}}^{\log}$ over $(\mathcal{S}_{K'}^{\Sigma'}, \mathcal{M}_{\mathcal{S}_{K'}^{\Sigma'}})$. Taking p -power torsion points, we obtain a log p -divisible group $\mathcal{A}_{\mathrm{univ}}^{\log}[p^\infty]$ by [KKN15, Proposition 18.1] and [Kat23, Proposition 4.5]. The log p -divisible group $\mathcal{A}_{\mathrm{univ}}^{\log}[p^\infty]$ is pulled back to $\mathcal{S}_K^\Sigma$ along

$$(\mathcal{S}_K^\Sigma, \mathcal{M}_{\mathcal{S}_K^\Sigma}) \rightarrow (\mathcal{S}_{K'}^{\Sigma'}, \mathcal{M}_{\mathcal{S}_{K'}^{\Sigma'}}).$$

6.7.2. Log prismatic realization. First, we recall the étale realization on Shimura varieties (cf. [IKY23, §2.3]). By varying levels at p , we obtain a tower of Shimura varieties

$$\varprojlim_{K'_p \subset K_p} \mathrm{Sh}_{K'_p K_p}(G, X) \rightarrow \mathrm{Sh}_K(G, X),$$

which is a pro-étale K_p -torsor after taking the analytification. Taking the pushforward along $K_p \rightarrow \mathcal{G}_{\widehat{\mathbb{Z}}_p}$, we obtain a $\mathbb{Z}_p$ -local system on $\mathrm{Sh}_K(G, X)^{\mathrm{an}}$ with $\mathcal{G}$ -structure $\omega_{\mathrm{ét}} \in \mathcal{G}\text{-Loc}_{\mathbb{Z}_p}(\mathrm{Sh}_K(G, X)^{\mathrm{an}})$ via Proposition 6.6.9.

By purity for Kummer étale local systems ([DLLZ23b, Theorem 4.6.1]), there exists a bi-exact equivalence

$$\mathrm{Loc}_{\mathbb{Z}_p}(\mathrm{Sh}_K(G, X)^{\mathrm{an}}) \simeq \mathrm{Loc}_{\mathbb{Z}_p}(\mathrm{Sh}_K^\Sigma(G, X)^{\mathrm{an}}, \mathcal{M}_{\mathrm{Sh}_K^\Sigma(G, X)^{\mathrm{an}}}).$$

Hence, $\omega_{\mathrm{ét}}$ gives a unique Kummer étale $\mathbb{Z}_p$ -local system with $\mathcal{G}$ -structure

$$\omega_{\mathrm{két}}: \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \rightarrow \mathrm{Loc}_{\mathbb{Z}_p}(\mathrm{Sh}_K^\Sigma(G, X)^{\mathrm{an}}, \mathcal{M}_{\mathrm{Sh}_K^\Sigma(G, X)^{\mathrm{an}}}).$$

Theorem 6.7.2. There exists a unique log prismatic F -crystal with G -structure

$$\omega_{\Delta, \log}: \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \rightarrow \mathrm{Vect}^\varphi((\widehat{\mathcal{S}}_K^\Sigma(G, X), \mathcal{M}_{\widehat{\mathcal{S}}_K^\Sigma(G, X)})_{\Delta})$$

with an isomorphism $T_{\mathrm{ét}} \circ \omega_{\Delta, \log} \cong \omega_{\mathrm{két}}$.

Proof. By Theorem 6.6.16 and [DS09, Proposition 12], it suffices to prove that there exists a log prismatic F -crystal $\mathcal{E}$ on $(\widehat{\mathcal{S}}_K^\Sigma(G, X), \mathcal{M}_{\widehat{\mathcal{S}}_K^\Sigma(G, X)})$ with $T_{\text{ét}}(\mathcal{E}) \cong \omega_{\text{két}}(\Lambda)$. Since $\omega_{\text{két}}(\Lambda)$ is characterized as a unique extension of $\omega_{\text{ét}}(\Lambda) \cong \mathcal{A}_{\text{univ}}[p^\infty]$, we have an isomorphism $\omega_{\text{két}}(\Lambda) \cong \mathcal{A}_{\text{univ}}^{\text{log}}[p^\infty]$. Therefore, Proposition 6.5.11 implies that the log prismatic Dieudonné crystal $\mathcal{M}_\Delta(\mathcal{A}_{\text{univ}}^{\text{log}}[p^\infty])$ is the desired one. $\square$

Remark 6.7.3. In [IKY23, Theorem 2.10], a prismatic F -crystal with $\mathcal{G}$ -structure $\omega_\Delta \in \mathcal{G}\text{-Vect}^\varphi(\widehat{\mathcal{S}}_K(G, X)_\Delta)$ with $T_{\text{ét}} \circ \omega_\Delta \cong \omega_{\text{ét}}|_{\widehat{\mathcal{S}}_K(G, X)_\eta}$ is constructed. We see that $\omega_{\Delta, \text{log}}$ constructed in Theorem 6.7.2 restricts to ω_Δ by fully faithfulness of étale realization functor.

Remark 6.7.4. In [Lov17], a strongly divisible filtered F -crystal with $\mathcal{G}$ -structure

$$\omega_{\text{crys}} : \text{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \rightarrow \text{FilCrys}^\varphi(\widehat{\mathcal{S}}_K(G, X))$$

is constructed, and it is proved that $\omega_{\text{ét}}$ and ω_{crys} are associated in the sense of Fontaine-Laffaille. In relation to the prismatic theory, a full subcategory $\text{Vect}^{\varphi, \text{lf}}(\widehat{\mathcal{S}}_K(G, X)_\Delta)$ of $\text{Vect}^\varphi(\widehat{\mathcal{S}}_K(G, X)_\Delta)$ consisting of locally filtered free prismatic F -crystals is introduced in [IKY25, §2.3], an integral crystalline realization functor

$$\mathbb{D}_{\text{crys}} : \text{Vect}^{\varphi, \text{lf}}(\widehat{\mathcal{S}}_K(G, X)_\Delta) \rightarrow \text{FilCrys}^\varphi(\widehat{\mathcal{S}}_K(G, X))$$

is constructed in [IKY25, Proposition 2.13], and it is proved that the functor ω_Δ takes values in $\text{Vect}^{\varphi, \text{lf}}(\widehat{\mathcal{S}}_K(G, X)_\Delta)$ and that there exists an isomorphism $\mathbb{D}_{\text{crys}} \circ \omega_\Delta \cong \omega_{\text{crys}}$ in [IKY23, Theorem 2.18 and Corollary 3.7].

The generalization of this picture to our setting remains as a future work. It seems that a key step is to show the boundedness of $\omega_{\Delta, \text{log}}$.

6.7.3. Lovering's Conjecture. Our work on toroidal compactifications enables us to generalize the comparison isomorphism for $\mathbb{Q}_p$ -local systems coming from $\omega_{\text{ét}}$, which is proved in the case that the Shimura variety is proper in [Lov17, Theorem 3.6.1] or [IKY23, Proposition 2.21].

Theorem 6.7.5. Let $(\overline{\mathcal{S}}_K^\Sigma(G, X), \mathcal{M}_{\overline{\mathcal{S}}_K^\Sigma(G, X)}) := (\mathcal{S}_K^\Sigma(G, X), \mathcal{M}_{\mathcal{S}_K^\Sigma(G, X)}) \otimes_{\mathcal{O}_{E_v}} k$, and consider a filtered F -isocrystal $\omega_{\text{fisoc}} := T_{\text{fisoc}} \circ \omega_{\Delta, \text{log}}$ on $(\widehat{\mathcal{S}}_K^\Sigma(G, X), \mathcal{M}_{\widehat{\mathcal{S}}_K^\Sigma(G, X)})$. Then the $\text{Gal}(\overline{E}_v/E_v)$ -representation $H_{\text{ét}}^i(\text{Sh}_K(G, X)_{\overline{E}_v}, \omega_{\text{ét}}(\xi)[1/p])$ is crystalline, and we have an isomorphism

$$\begin{aligned} & H_{\text{ét}}^i(\text{Sh}_K(G, X)_{\overline{E}_v}, \omega_{\text{ét}}(\xi)[1/p]) \otimes_{\mathbb{Q}_p} B_{\text{crys}} \\ & \cong H_{\text{logcrys}}^i((\overline{\mathcal{S}}_K^\Sigma(G, X), \mathcal{M}_{\overline{\mathcal{S}}_K^\Sigma(G, X)})_{\text{crys}}, \omega_{\text{fisoc}}(\xi)) \otimes_{W(k_v)} B_{\text{crys}} \end{aligned}$$

that is compatible with Galois actions and Frobenius isomorphisms and a filtered isomorphism after taking the base change along $B_{\text{crys}} \rightarrow B_{\text{dR}}$, for any $\xi \in \text{Rep}_{\mathbb{Z}_p}(\mathcal{G})$.

Proof. By Theorem 5.6.18, the $\mathbb{Q}_p$ -local system $\omega_{\text{két}}(\xi)[1/p]$ is associated with $\omega_{\text{fisoc}}(\xi)$ in the sense of Definition 5.4.18. Then the assertion follows from Faltings' p -adic Hodge theory [Fal89, Theorem 6.3]. $\square$

Remark 6.7.6. This comparison isomorphism is conjectured in [Lov17, §3.6.2]. Furthermore, in *loc. cit.*, it is also conjectured that there exists an integral version of ω_{fisoc} on the toroidal compactification, which is still open.

Remark 6.7.7. The fact that the $\mathrm{Gal}(\overline{E}_v/E_v)$ -representation

$$H_{\text{ét}}^i(\mathrm{Sh}_K(G, X)_{\overline{E}_v}, \omega_{\text{ét}}(\xi)[1/p])$$

is crystalline can be deduced more easily as follows. We may assume that $\xi = \Lambda^{\otimes n}$, and so it is enough to show that the $\mathrm{Gal}(\overline{E}_v/E_v)$ -representation

$$H_{\text{ét}}^i(\mathrm{Sh}_K(G, X)_{\overline{E}_v}, R^n f_{\text{ét},*} \mathbb{Q}_p)$$

is crystalline, where f is the structure morphism of the n -th fold self-product of the universal abelian scheme. We have a spectral sequence

$$E_2^{p,q} = H_{\text{ét}}^p(\mathrm{Sh}_K(G, X)_{\overline{E}_v}, R^q f_{\text{ét},*} \mathbb{Q}_p) \Rightarrow H_{\text{ét}}^{p+q}(\mathcal{A}_{\text{univ}, \overline{E}_v}^n, \mathbb{Q}_p).$$

Since the multiplication by m on $\mathcal{A}_{\text{univ}, \overline{E}_v}$ induces the multiplication by m^{nq} on $E_2^{p,q}$, this spectral sequence degenerates at the E_2 -page. Hence, it suffices to prove that the $\mathrm{Gal}(\overline{E}_v/E_v)$ -representation $H_{\text{ét}}^i(\mathcal{A}_{\text{univ}, \overline{E}_v}^n, \mathbb{Q}_p)$ is crystalline, which follows from Faltings' p -adic Hodge theory and the existence of a smooth compactification of $\mathcal{A}_{\text{univ}, \mathcal{O}_{E_v}}^n$ with a relatively normal crossings boundary divisor over $\mathcal{O}_{E_v}$ ([Lan12, Theorem 2.15]). We learned this argument from Kai-Wen Lan.

Clearly, this method gives no explicit description of the filtered F -isocrystal

$$D_{\text{crys}}(H_{\text{ét}}^i(\mathrm{Sh}_K(G, X)_{\overline{E}_v}, \omega_{\text{ét}}(\xi)[1/p])).$$

6.A. Some lemmas on log schemes

Lemma 6.A.1. Let P and Q be a finitely generated free monoids (i.e. $P \simeq \mathbb{N}^r$ and $Q \simeq \mathbb{N}^s$ for some $r, s \geq 0$). Let $\alpha : P \rightarrow Q$ be a homomorphism inducing $P/\alpha^{-1}(0) \xrightarrow{\sim} Q$. Then, there exist a basis $p_1, \dots, p_r$ (resp. $q_1, \dots, q_s$) of P (resp. Q) such that

$$\alpha(p_i) = \begin{cases} q_i & (1 \leq i \leq s) \\ 0 & (s+1 \leq i \leq r). \end{cases}$$

Proof. Choose a basis $p_1, \dots, p_r$ (resp. $q_1, \dots, q_s$) of P (resp. Q). Since $\alpha^{-1}(0)$ is a face of P , the submonoid $\alpha^{-1}(0)$ is generated by $\{p_i | i \in I\}$ for some $I \in \{1, \dots, r\}$. For each $1 \leq j \leq s$, there exist an integer $1 \leq i_j \leq r$ such that $\alpha(p_{i_j}) = q_j$ by the surjectivity of α and the irreducibility of q_j .

By renumbering, we may assume that there exists an integer t with $1 \leq s \leq t \leq r$ such that $\alpha(p_i) = q_i$ for $1 \leq i \leq s$ and $\alpha^{-1}(0)$ is generated by $p_{s+1}, \dots, p_t$. By $P^{\text{gp}}/\alpha^{-1}(0)^{\text{gp}} \xrightarrow{\sim} Q^{\text{gp}}$ and the computation of ranks, we conclude that $s = t$. $\square$

Lemma 6.A.2. Let $n \geq 2$ be an integer. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ be an fs log formal scheme with fs charts $\alpha_i : P_i \rightarrow \mathcal{M}_{\mathfrak{X}}$ for integers $1 \leq i \leq n$. We set

$$(\mathfrak{X}_{\infty, i}, \mathcal{M}_{\mathfrak{X}_{\infty, i}}) := (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \otimes_{(\mathbb{Z}[P_i], P_i)^a} (\mathbb{Z}[P_i, \mathbb{Q}_{\geq 0}], P_{i, \mathbb{Q}_{\geq 0}})^a$$

for each $1 \leq i \leq n$. For a non-empty subset $I \subset \{1, \dots, n\}$, we let $(\mathfrak{X}_{\infty, I}, \mathcal{M}_{\mathfrak{X}_{\infty, I}})$ denote the fiber product of $(\mathfrak{X}_{\infty, i}, \mathcal{M}_{\mathfrak{X}_{\infty, i}})$ for $i \in I$ over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ in the category of saturated log formal schemes. Let $\emptyset \neq J \subset I \subset \{1, \dots, n\}$.

- (1) The projection map $(\mathfrak{X}_{\infty, I}, \mathcal{M}_{\mathfrak{X}_{\infty, I}}) \rightarrow (\mathfrak{X}_{\infty, J}, \mathcal{M}_{\mathfrak{X}_{\infty, J}})$ is a strict affine flat cover.
- (2) Suppose that $\mathfrak{X}$ is a bounded p -adic formal scheme and $P_i = \mathbb{N}^{r_i}$ for each $1 \leq i \leq n$. Then the projection map $(\mathfrak{X}_{\infty, I}, \mathcal{M}_{\mathfrak{X}_{\infty, I}}) \rightarrow (\mathfrak{X}_{\infty, J}, \mathcal{M}_{\mathfrak{X}_{\infty, J}})$ is a strict affine quasi-syntomic cover.
- (3) Suppose that $\mathfrak{X}$ is a scheme and $P_i = \mathbb{N}^{r_i}$ for each $1 \leq i \leq n$. Then the projection map $(\mathfrak{X}_{\infty, I}, \mathcal{M}_{\mathfrak{X}_{\infty, I}}) \rightarrow (\mathfrak{X}_{\infty, J}, \mathcal{M}_{\mathfrak{X}_{\infty, J}})$ is a strict affine root covering.

Proof. We may assume $n = 2$ because every projection map is written as the composition of maps which are the base change of projection maps in the $n = 2$ case. (For example, in the $n = 3$ case, a projection map $(\mathfrak{X}_{\infty, \{1,2,3\}}, \mathcal{M}_{\mathfrak{X}_{\infty, \{1,2,3\}}}) \rightarrow (\mathfrak{X}_{\infty, \{3\}}, \mathcal{M}_{\mathfrak{X}_{\infty, \{3\}}})$ factors as

$$(\mathfrak{X}_{\infty, \{1,2,3\}}, \mathcal{M}_{\mathfrak{X}_{\infty, \{1,2,3\}}}) \rightarrow (\mathfrak{X}_{\infty, \{2,3\}}, \mathcal{M}_{\mathfrak{X}_{\infty, \{2,3\}}}) \rightarrow (\mathfrak{X}_{\infty, \{3\}}, \mathcal{M}_{\mathfrak{X}_{\infty, \{3\}}}),$$

where the projection map $(\mathfrak{X}_{\infty, \{1,2,3\}}, \mathcal{M}_{\mathfrak{X}_{\infty, \{1,2,3\}}}) \rightarrow (\mathfrak{X}_{\infty, \{2,3\}}, \mathcal{M}_{\mathfrak{X}_{\infty, \{2,3\}}})$ is written as the base change of a projection map $(\mathfrak{X}_{\infty, \{1,2\}}, \mathcal{M}_{\mathfrak{X}_{\infty, \{1,2\}}}) \rightarrow (\mathfrak{X}_{\infty, \{2\}}, \mathcal{M}_{\mathfrak{X}_{\infty, \{2\}}})$. By the symmetry, it suffice to prove the assertion for the projection map

$$(\mathfrak{X}_{\infty, \{1,2\}}, \mathcal{M}_{\mathfrak{X}_{\infty, \{1,2\}}}) \rightarrow (\mathfrak{X}_{\infty, \{2\}}, \mathcal{M}_{\mathfrak{X}_{\infty, \{2\}}}).$$

First, we shall prove (1). For any $n \geq 1$, there exists an integer $m \geq 1$ such that the projection map from the saturated fiber product

$$(\mathfrak{X}_{n, \alpha_1}, \mathcal{M}_{\mathfrak{X}_{n, \alpha_1}}) \times_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})} (\mathfrak{X}_{m, \alpha_2}, \mathcal{M}_{\mathfrak{X}_{m, \alpha_2}}) \rightarrow (\mathfrak{X}_{m, \alpha_2}, \mathcal{M}_{\mathfrak{X}_{m, \alpha_2}})$$

is a strict affine flat cover by Lemma 6.2.2 and the fact that a saturation map for a quasi-coherent log formal scheme is affine. Therefore, the projection from the saturated fiber product

$$(\mathfrak{X}_{n, \alpha_1}, \mathcal{M}_{\mathfrak{X}_{n, \alpha_1}}) \times_{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})} (\mathfrak{X}_{\infty, \alpha_2}, \mathcal{M}_{\mathfrak{X}_{\infty, \alpha_2}}) \rightarrow (\mathfrak{X}_{\infty, \alpha_2}, \mathcal{M}_{\mathfrak{X}_{\infty, \alpha_2}})$$

is a strict affine flat cover for any $n \geq 1$. Taking a limit with respect to $n \geq 1$, we obtain the claim.

Next, we shall prove (2) and (3) simultaneously. Take $x \in \mathfrak{X}$. Applying Lemma 6.A.1 to maps $\alpha_{1,x} : \mathbb{N}^{r_1} \rightarrow \overline{\mathcal{M}_{\mathfrak{X}, x}}$ and $\alpha_{2,x} : \mathbb{N}^{r_2} \rightarrow \overline{\mathcal{M}_{\mathfrak{X}, x}}$ induced by α_1 and α_2 , we see that, after changing orders of bases of $\mathbb{N}^{r_1}$ and $\mathbb{N}^{r_2}$ appropriately, there exists an integer s with $1 \leq s \leq \min\{r_1, r_2\}$ such that $\alpha_{1,x}(e_i) = \alpha_{2,x}(e_i)$ for every $1 \leq i \leq s$ and $\alpha_{1,x}(e_j) = \alpha_{2,x}(e_k) = 1$ for every $s < j \leq r_1$ and $s < k \leq r_2$. By shrinking $\mathfrak{X}$ to an étale neighborhood of x , we may assume that we can write

$$\alpha_1(e_i) = u_i \alpha_2(e_i) \quad (1 \leq i \leq s)$$

$$\alpha_1(e_j) = v_j \quad (s < j \leq r_1)$$

$$\alpha_2(e_k) = w_k \quad (s < k \leq r_2)$$

for $u_i, v_j, w_k \in \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})^\times$. Then $\mathfrak{X}_{\infty, \{1,2\}}$ is obtained by adding n -th power roots of u_i and v_j to $\mathfrak{X}_{\infty, \alpha_2}$. We shall give a more precise explanation. Consider a map $\mathfrak{X}_{\infty, \alpha_2} \rightarrow \mathrm{Spf}(\mathbb{Z}_p\langle x_1, \dots, x_{r_1} \rangle)$ corresponding to a ring map $\mathbb{Z}_p\langle x_1, \dots, x_{r_1} \rangle \rightarrow \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$ mapping x_i to u_i or v_i depending on whether $i \leq s$ or not. Let

$$\mathfrak{X}_{\infty, \alpha_1, \alpha_2} := \mathfrak{X}_{\alpha_2} \times_{\mathrm{Spf}(\mathbb{Z}_p\langle x_1, \dots, x_{r_1} \rangle)} \mathrm{Spf}(\mathbb{Z}_p\langle x_1^{\mathbb{Q}_{\geq 0}}, \dots, x_{r_1}^{\mathbb{Q}_{\geq 0}} \rangle)$$

and the log structure $\mathcal{M}_{\mathfrak{X}_{\infty, \alpha_1, \alpha_2}}$ on $\mathfrak{X}_{\infty, \alpha_1, \alpha_2}$ be the inverse image log structure of $\mathcal{M}_{\mathfrak{X}_{\infty, \alpha_2}}$. We have a strict morphism $(\mathfrak{X}_{\infty, \alpha_1, \alpha_2}, \mathcal{M}_{\mathfrak{X}_{\infty, \alpha_1, \alpha_2}}) \rightarrow (\mathfrak{X}_{\infty, \alpha_1}, \mathcal{M}_{\mathfrak{X}_{\infty, \alpha_1}})$ whose composition with $(\mathfrak{X}_{\infty, \alpha_1}, \mathcal{M}_{\mathfrak{X}_{\infty, \alpha_1}}) \rightarrow (\mathrm{Spf}(\mathbb{Z}_p\langle \mathbb{Q}_{\geq 0}^{r_1} \rangle), \mathbb{Q}_{\geq 0}^{r_1})$ corresponds to a monoid map $\mathbb{Q}_{\geq 0}^{r_1} \rightarrow \Gamma(\mathfrak{X}_{\infty, \alpha_1, \alpha_2}, \mathcal{M}_{\mathfrak{X}_{\infty, \alpha_1, \alpha_2}})$ mapping $(1/n)e_i$ to $x_i^{1/n} \alpha_2((1/n)e_i)$ or $x_i^{1/n}$ for every $n \geq 1$ and i depending on whether $i \leq s$ or not. Here, $\alpha_2((1/n)e_i)$ is the image of $(1/n)e_i$ by the map $\mathbb{Q}_{\geq 0}^{r_2} \rightarrow \Gamma(\mathfrak{X}_{\infty, \alpha_1, \alpha_2}, \mathcal{M}_{\mathfrak{X}_{\infty, \alpha_1, \alpha_2}})$ corresponding to the morphism

$$(\mathfrak{X}_{\infty, \alpha_1, \alpha_2}, \mathcal{M}_{\mathfrak{X}_{\infty, \alpha_1, \alpha_2}}) \rightarrow (\mathfrak{X}_{\infty, \alpha_2}, \mathcal{M}_{\mathfrak{X}_{\infty, \alpha_2}}) \rightarrow (\mathrm{Spf}(\mathbb{Z}_p\langle \mathbb{Q}_{\geq 0}^{r_2} \rangle), \mathbb{Q}_{\geq 0}^{r_2})^a.$$

Then we can check directly that $(\mathfrak{X}_{\infty, \alpha_1, \alpha_2}, \mathcal{M}_{\mathfrak{X}_{\infty, \alpha_1, \alpha_2}})$ is the fiber product of $(\mathfrak{X}_{\infty, \alpha_1}, \mathcal{M}_{\mathfrak{X}_{\infty, \alpha_1}})$ and $(\mathfrak{X}_{\infty, \alpha_2}, \mathcal{M}_{\mathfrak{X}_{\infty, \alpha_2}})$ over $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$, and this description of

$(\mathfrak{X}_{\infty, \{1,2\}}, \mathcal{M}_{\mathfrak{X}_{\infty, \{1,2\}}})$ implies the assertion (2) and (3) for the projection map

$$(\mathfrak{X}_{\infty, \{1,2\}}, \mathcal{M}_{\mathfrak{X}_{\infty, \{1,2\}}}) \rightarrow (\mathfrak{X}_{\infty, \alpha_2}, \mathcal{M}_{\mathfrak{X}_{\infty, \alpha_2}}).$$

□

Lemma 6.A.3. Let $f: (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \rightarrow (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ be a morphism of fs log formal schemes. Suppose that we are given a chart $\alpha: \mathbb{N}^r \rightarrow \mathcal{M}_{\mathfrak{X}}$ and a monoid map $\beta: \mathbb{N}^s \rightarrow \mathcal{M}_{\mathfrak{Y}}$. Let $x \in \mathfrak{X}$. Then, after shrinking $\mathfrak{X}$ to an étale neighborhood of x , there exist a chart $\alpha': \mathbb{N}^r \oplus \mathbb{Z}^s \rightarrow \mathcal{M}_{\mathfrak{X}}$ which restricts to α and a monoid map $\gamma: \mathbb{N}^s \rightarrow \mathbb{N}^r \oplus \mathbb{Z}^s$ such that g gives a chart of f .

Proof. For each $1 \leq i \leq s$, we can write the image of $\beta(e_i)$ in $\mathcal{M}_{\mathfrak{X}, \bar{x}}$ as $f(m_i)u_i$ for some $m_i \in \mathbb{N}^r$ and $u_i \in \mathcal{O}_{\mathfrak{X}, \bar{x}}^\times$ because α is a chart. By shrinking $\mathfrak{X}$ to an étale neighborhood of x , we may assume that $u_i \in \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^\times)$ for every $1 \leq i \leq s$. Let $\{e'_i\}_{1 \leq i \leq s}$ denote the standard basis of $\mathbb{Z}^s$. We set $\alpha': \mathbb{N}^r \oplus \mathbb{Z}^s \rightarrow \mathcal{M}_{\mathfrak{X}}$ to be the monoid map induced by $\alpha: \mathbb{N}^r \rightarrow \mathcal{M}_{\mathfrak{X}}$ and $\mathbb{Z}^s \rightarrow \mathcal{M}_{\mathfrak{X}}$ mapping e'_i to u_i , and we define a monoid map $\gamma: \mathbb{N}^s \rightarrow \mathbb{N}^r \oplus \mathbb{Z}^s$ by $\gamma(e_i) := m_i + e'_i$. Then the desired condition is satisfied. □

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