

Eisenstein cocycles for arithmetic groups and values of zeta functions

Robert Sczech (九州大学)

Let F be a totally real number field of degree n over $\mathbb{Q}$, and f a conductor of a ray class group in F . By definition, $f = f_\infty f_{\text{fin}}$ is the product of the finite part f_{fin} which is an integral ideal of $\mathbb{Z}_F$, and the infinite part $f_\infty = \prod \mathfrak{P}_i$, where $\mathfrak{P}_i$ runs through a set of embeddings of F into $\mathbb{R}$, indexed by a subset $S \subseteq \{1, 2, \dots, n\}$. Let $I(f)$ be the multiplicative group of fractional ideals in F generated by all prime ideals in $\mathbb{Z}_F$ which do not divide f_{fin} . Two ideals $a, b \in I(f)$ belong to the same class mod f iff ab^{-1} is a principal ideal (α) generated by an element $\alpha \in 1 + f_{\text{fin}} b^{-1}$ such that $\mathfrak{P}_i(\alpha) > 0$ for all $i \in S$. Modulo this relation, $I(f)$ decomposes into finitely many classes $C \bmod f$. To every class C there is associated the partial zeta function

$$\zeta(C, s) = \sum_{a \in C} N(a)^{-s}, \quad \text{Re}(s) > 1.$$

According to Hecke, this function has an analytic continuation to the whole complex s -plane except for a simple pole at $s=1$, and, by results of Klingen and Siegel, the special values of $\zeta(C, s)$ at non-positive integral $s=0, -1, -2, \dots$ are all rational numbers which can be calculated explicitly using a well known formula of Shintani. In the simplest case, this is the classical formula of Euler,

$$\zeta(1-k) = -\frac{B_k}{k}, \quad k=1, 2, 3, \dots,$$

for the special values of the Riemann zeta function $\zeta(s)$. Since the Bernoulli numbers B_k of an odd index $k > 1$ are all zero, it follows that $\zeta(-2k) = 0$ for $k=1, 2, 3, \dots$. This is in fact a general phenomenon. Because of Gamma factors in Hecke's functional equation, $\zeta(C, s)$ vanishes at $s=-2k$ of order

$$\text{ord}_{s=-2k} \zeta(C, s) \geq r = n - |S|, \quad k=0, 1, 2, \dots$$

In particular, $\zeta(C, -2k)=0$ if $r > 0$. It is therefore of interest to investigate the coefficients

$$\zeta^{(r)}(C, -2k) = \frac{d^r}{ds^r} \zeta(C, s) \Big|_{s=-2k}.$$

For instance, these numbers are the subject of the well known conjectures of Stark ($k=0$) and Beilinson-Gross ($k > 0$). In this report, we are interested in the cohomological interpretation of these values in terms of the group cohomology of the unit group

$$U = \{\eta \in \mathbb{Z}_F \mid \eta \in 1 + f_{\text{fin}}, \mathfrak{P}_i(\eta) > 0 \text{ for all } i \in S\}.$$

It is convenient to assume that U is torsionfree. Then, according to Dirichlet, U is a free abelian group of rank $n-1$, and therefore, the homology as well as the cohomology groups of U are isomorphic to the (co)homology of the torus T^{n-1} , $T=\mathbb{R}/\mathbb{Z}$. In particular, the homology group $H_{n-1}(U, \mathbb{Z})$ is free abelian of rank one, so we can talk about a fundamental class Z of U , which is a generator of $H_{n-1}(U, \mathbb{Z})$. (In the case $n=2$, Z corresponds to a fundamental unit of U).

Theorem 1. There is a cohomology class $\varepsilon_p(C, k) \in H^{n-1}(U, \mathbb{R})$ such that the evaluation on Z gives

$$\zeta^{(p)}(C, -k) = \varepsilon_p(C, k)(Z)$$

provided that either $p = 0$ and $k=1, 3, 5, \dots$ or $p = n-|S|$ and $k=0, 2, 4, \dots$. Moreover, ε_p is the restriction of a universal Eisenstein cohomology class in $H^{n-1}(GL_n \mathbb{Z})$ which depends only on n and p , but not on the particular field F or ray class C .

This is a generalization of a previous result [1] which deals with the special case $p=0$. In that case, it can be shown that the cohomology class $\varepsilon_0(C, k)$ is in fact rational, $\varepsilon_0(C, k) \in H^{n-1}(U, \mathbb{Q})$. Moreover, a finite formula exists for $\varepsilon_0(C, k)$ which generalizes the classical Dedekind sum. In general, our method does not lead to any conclusion about the arithmetic nature of the cohomology classes $\varepsilon_p(C, k)$ for $p > 0$. The proof of the above theorem will be published elsewhere. In this report, we wish to illustrate the construction of the Eisenstein cocycle in the simplest non-trivial case: $n=2, p=1, k$ even.

Let $G = GL_2 \mathbb{R}$ and H be the subspace of homogenous polynomials in $\mathbb{R}[x_1, x_2]$. The set $M = \{f: H \times \mathbb{R}^2 \rightarrow \mathbb{C}\}$ is then a G -module under the action

$$(Af)(P, x) = \det(A) f(A^t P, xA), \quad A \in G, \quad f \in M.$$

Here, $A^t P$ denotes the polynomial defined by $(A^t P)(y) = P(yA^t)$. We first construct a homogenous 1-cocycle ψ for G with values in M . By definition, ψ is a map $\psi: G \times G \rightarrow M$ satisfying the properties

$$\psi(A_1, A_2) + \psi(A_2, A_3) = \psi(A_1, A_3), \quad (1)$$

$$\psi(AA_1, AA_2) = A\psi(A_1, A_2); \quad A, A_j \in G. \quad (2)$$

For $A_i \in G$, we denote the j th column of the matrix A_i by A_{ij} . Then the cocycle ψ is defined for $x \neq 0$ by

$$\psi(A_1, A_2)(P, x) = P(\partial_{x_1}, \partial_{x_2}) \left(\frac{\det(A_{11}, A_{21})}{\langle x, A_{11} \rangle \langle x, A_{21} \rangle} \right) \quad (3)$$

where $P(\partial_{x_1}, \partial_{x_2})$ denotes the differential operator formed with the partial derivatives with respect to x_1 and x_2 . The definition needs a modification if one of the scalar products $\langle x, y \rangle = x_1 y_1 + x_2 y_2$ in the denominator vanishes. For instance, if $\langle x, A_{11} \rangle = 0$, then $\langle x, A_{12} \rangle \neq 0$ since $x \neq 0$; assuming that the second scalar product $\langle x, A_{21} \rangle$ in (3) does not vanish, the right side of (3) must be replaced in that case by

$$P(\partial_{x_1}, \partial_{x_2}) \left(\frac{\det(A_{12}, A_{21})}{\langle x, A_{12} \rangle \langle x, A_{21} \rangle} \right).$$

A similar modification applies in all other cases except when $x=0$ in which case we set $\psi=0$. For details of this construction and the proof that the so defined map ψ does indeed represent a cohomology class in $H^1(G, M)$, we refer the reader to [1].

The basic idea behind the construction of the Eisenstein cocycle $\varepsilon = \varepsilon_1$ is to average the values of ψ with respect to the variable x over the lattice $\mathbb{Z}^2$. Let $\Gamma = GL_2 \mathbb{Z}$ and let N be the set of complex valued functions $f(P, Q, u, v)$ on $H \times H \times \mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z})^2$. N is a left Γ -module under the action

$$(Af)(P, Q, u, v) = \det(A)f(A^tP, A^{-1}Q, A^{-1}u, A^{-1}v).$$

For $A_i \in \Gamma$, the Eisenstein cocycle ε is the map $\varepsilon : \Gamma \times \Gamma \rightarrow N$ defined by ($\mathbf{e}(z) = \exp(2\pi iz)$)

$$\varepsilon(A_1, A_2)(P, Q, u, v) \stackrel{\text{def}}{=} \sum_{x \in \mathbb{Z}^2} \text{sign}(xu) \mathbf{e}(-xv) \psi(A_1, A_2)(P, x) \Big|_Q,$$

where the "Q-limit" notation on the right has to be understood as

$$\sum_x h(x) \Big|_Q \stackrel{\text{def}}{=} \lim_{t \rightarrow \infty} \left(\sum_{|Q(x)| < t} h(x) \right).$$

Theorem 2. The map $\varepsilon : \Gamma \times \Gamma \rightarrow N$ is well defined and has the properties

$$\begin{aligned} \varepsilon(A_1, A_2) + \varepsilon(A_2, A_3) &= \varepsilon(A_1, A_3), \quad A_i \in \Gamma \\ \varepsilon(AA_1, AA_2) &= A\varepsilon(A_1, A_2), \quad A \in \Gamma. \end{aligned}$$

Moreover, ε represents a non trivial cohomology class in $H^1(\Gamma, N)$.

For the proof, see [2]. We return now to the partial zeta function of the introduction and consider the case of a real quadratic field F with one distinguished real embedding $\mathfrak{P} : F \rightarrow \mathbb{R}$ such that $f_\infty = \mathfrak{P}$. Let $b \in C$ be a fixed representative of the ray class C and choose a $\mathbb{Z}$ -basis W for $f_{\text{fin}} b^{-1} = \mathbb{Z}W_1 + \mathbb{Z}W_2$. The trace form in F determines the dual basis V by $\text{tr}(V_i W_j) = \delta_{ij}$. Define P, Q, v by

$$P(x) = N(\sum x_i W_i), \quad Q(x) = N(\sum x_i V_i), \quad v_j = \text{tr}(V_j), \quad j=1, 2.$$

P and Q are normforms determined by the bases W resp. V . Finally, let $A \in \Gamma$ be the hyperbolic matrix corresponding to a generator of U under the regular representation of U with respect to the basis V . Then, as a special case of Theorem 1, we have the explicit relation

$$\zeta'(C, -2k) = \pm (2\pi i)^{-1-4k} \varepsilon(1, A)(P^{2k}, Q, \mathfrak{P}(V), v).$$

The sign ambiguity is due to the fact that the right side changes its sign when A is replaced by A^{-1} .

Acknowledgement. This report is based on work partly supported by the NSF.

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Kyushu University 33
Fukuoka 812, Japan

Rutgers University
Newark NJ 07102, USA

email: sczech@math.kyushu-u.ac.jp