

# Dynamical decomposition theorems

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**Abstract.** In this article, we study some dynamical decomposition theorems of spaces related to given homeomorphisms. First, we introduce new notions of 'bright spaces' and 'dark spaces' of homeomorphisms except  $n$  times, and by use of the notions we show some dynamical decomposition theorems of spaces related to given homeomorphisms. Next, we show that if  $f : X \rightarrow X$  is a homeomorphism of an  $n$ -dimensional separable metric space  $X$  with zero-dimensional set of periodic points, then  $X$  can be decomposed into a zero-dimensional bright space of  $f$  except  $n$  times and an  $(n - 1)$ -dimensional dark space of  $f$  except  $n$  times, and also by use of dark spaces, we can show some decomposition theorems of  $X$  related to dimension theory and dynamical systems. Finally, we study dynamical decompositions of continuum-wise expansive homeomorphisms.

## 1 Introduction

In this article, we assume that all spaces are separable metric spaces and dimension means the topological dimension  $\dim$ . Also, let  $\mathbb{N}$  and  $\mathbb{Z}$  denote the set of natural numbers and the set of integers, respectively. If  $A$  is a subset of a space  $X$ , then  $\text{cl}(A)$ ,  $\text{bd}(A)$  and  $\text{int}(A)$  denote the closure, the boundary and the interior of  $A$  in  $X$ , respectively. For a collection  $\mathcal{G}$  of subsets of  $X$ ,

$$\text{ord}(\mathcal{G}) = \sup\{\text{ord}_x(\mathcal{G}) \mid x \in X\},$$

where  $\text{ord}_x(\mathcal{G})$  is the number of members of  $\mathcal{G}$  which contains  $x$ .

We introduce new notions of 'bright spaces' and 'dark spaces' of homeomorphisms except  $n$  times, and by use of the notions we prove some dynamical decomposition theorems of spaces related to given homeomorphisms. For a homeomorphism  $f : X \rightarrow X$  of a space  $X$  and  $k \in \mathbb{N}$ , let  $P_k(f)$  denote the set of points of period  $\leq k$ . Also,  $P(f)$  denotes the set of all periodic points of  $f$ . A subset  $Z$  of  $X$  is a *bright space* of  $f$  except  $n$  times ( $n \in \{0\} \cup \mathbb{N}$ ) if for any  $x \in X$ ,

$$|\{p \in \mathbb{Z} \mid f^p(x) \notin Z\}| \leq n,$$

where  $|A|$  denotes the cardinality of a set  $A$ . Also we say that  $L = X - Z$  is a *dark space* of  $f$  except  $n$  times. Note that for any  $x \in X$ ,  $|O_f(x) \cap L| \leq n$ , where  $O_f(x) = \{f^p(x) \mid p \in \mathbb{Z}\}$  denotes the orbit of  $x$ , and also note that  $L \cap P(f) = \emptyset$ . For a dark space  $L$  of  $f$  except  $n$  times and  $0 \leq j \leq n$ , we put

$$A_f(L, j) = \{x \in X \mid |\{p \in \mathbb{Z} \mid f^p(x) \in L\}| = j\} (= \{x \in X \mid |O_f(x) \cap L| = j\}).$$

$A_f(L, j)$  denotes the set of all point  $x \in X$  whose orbit  $O_f(x)$  appears in  $L$  just  $j$  times. Note that  $P(f) \subset A_f(L, 0)$  and  $A_f(L, j)$  is  $f$ -invariant, i.e.  $f(A_f(L, j)) = A_f(L, j)$  and  $A_f(L, i) \cap A_f(L, j) = \emptyset$  if  $i \neq j$ . Hence we have the  $f$ -invariant decomposition related to the dark space  $L$  as follows;

$$X = A_f(L, 0) \cup A_f(L, 1) \cup \cdots \cup A_f(L, n).$$

## 2 Dynamical decomposition theorems of homeomorphisms with zero-dimensional sets of periodic points

It is well-known that a space  $X$  has at most dimension  $n$  ( $n \in \{0\} \cup \mathbb{N}$ ) (i.e.  $\dim X \leq n$ ) if and only if  $X$  can be represented as a union of  $(n + 1)$  zero-dimensional subspaces of  $X$  (see [2, 12]). The following proposition may be known.

**Proposition 2.1.** *Suppose that  $X$  is a space with  $\dim X = n$  ( $< \infty$ ) and  $f : X \rightarrow X$  is a homeomorphism. Then there exist  $f$ -invariant zero-dimensional dense  $G_\delta$ -sets  $A_f(j)$  ( $j = 0, 1, 2, \dots, n$ ) of  $X$  such that*

$$X = A_f(0) \cup A_f(1) \cup \dots \cup A_f(n).$$

In [1], Arts, Fokkink and Vermeer proved the following interesting theorem of dynamical systems of homeomorphisms under some dimensional conditions of periodic points.

**Theorem 2.2.** ([1, Theorem 8]) *Suppose that  $f : X \rightarrow X$  is a homeomorphism of a (metric) space  $X$  with  $\dim X \leq n$  ( $< \infty$ ). Then there exists a dense  $G_\delta$ -set  $Z$  of  $X$  such that  $\dim Z = 0$  and*

$$X = Z \cup f(Z) \cup f^2(Z) \cup \dots \cup f^n(Z)$$

*if and only if  $\dim P_k(f) < k$  for each  $1 \leq k \leq n$ .*

In this article, under the condition of  $\dim P(f) \leq 0$ , we prove more chaotic decomposition theorems of dynamical systems of homeomorphisms. In [3, 4, 5, 8, 9], we studied some dynamical properties of homeomorphisms with zero-dimensional set of periodic points. Now, we need the following lemma.

**Lemma 2.3.** (cf. [4, Lemma 3.5] and [3, Lemma 2.2]) *Suppose that  $X$  is a space with  $\dim X = n$  ( $< \infty$ ) and  $f : X \rightarrow X$  is a homeomorphism with  $\dim P(f) \leq 0$ . Let  $F$  be an  $F_\sigma$ -set of  $X$  with  $\dim F \leq 0$ . Then for each  $j \in \mathbb{N}$ , there is a locally finite countable open cover  $\mathcal{C}(j) = \{C(j)_\alpha \mid \alpha \in \mathbb{N}\}$  of  $X$  such that*

- (1)  $\text{mesh}(\mathcal{C}(j)) < 1/j$ ,
- (2)  $\text{ord}(\mathcal{G}) \leq n$ , where  $\mathcal{G} = \{f^p(\text{bd}(C(j)_\alpha)) \mid \alpha \in \mathbb{N}, j \in \mathbb{N} \text{ and } p \in \mathbb{Z}\}$  and
- (3)  $F \cap L = \emptyset$ , where  $L = \bigcup \{\text{bd}(C(j)_\alpha) \mid \alpha \in \mathbb{N}, j \in \mathbb{N}\}$ .

The following theorem is a key result.

**Theorem 2.4.** *Suppose that  $X$  is a space with  $\dim X = n$  ( $< \infty$ ) and  $f : X \rightarrow X$  is a homeomorphism. Then there exists a bright space  $Z$  of  $f$  except  $n$  times such that  $Z$  is a zero-dimensional dense  $G_\delta$ -set of  $X$  and the dark space  $L = X - Z$  of  $f$  is a  $(n - 1)$ -dimensional  $F_\sigma$ -set of  $X$  if and only if  $\dim P(f) \leq 0$ .*

**Corollary 2.5.** *Suppose that  $X$  is a space with  $\dim X = n$  ( $< \infty$ ) and  $f : X \rightarrow X$  is a homeomorphism. Then there exists a zero-dimensional  $G_\delta$ -dense set  $Z$  of  $X$  such that for any  $(n + 1)$  integers  $k_0 < k_1 < \dots < k_n$ ,*

$$X = f^{k_0}(Z) \cup f^{k_1}(Z) \cup \dots \cup f^{k_n}(Z)$$

*if and only if  $\dim P(f) \leq 0$ .*

**Theorem 2.6.** *Suppose that  $X$  is a space with  $\dim X = n$  ( $< \infty$ ) and  $f : X \rightarrow X$  is a homeomorphism with  $\dim P(f) \leq 0$ . If  $L$  is a dark space of  $f$  except  $n$  times such that  $L$  is an  $F_\sigma$ -set of  $X$  and  $\dim (X - L) \leq 0$ , then  $\dim A_f(L, j) = 0$  for each  $j = 0, 1, 2, \dots, n$ . In particular, there is the  $f$ -invariant zero-dimensional decomposition of  $X$  related to the dark space  $L$ :*

$$X = A_f(L, 0) \cup A_f(L, 1) \cup \dots \cup A_f(L, n).$$

Finally, as a special case we consider the case that  $f : X \rightarrow X$  is a continuum-wise expansive homeomorphism of a compact metric space  $X$ . A homeomorphism  $f : X \rightarrow X$  of a compact metric space  $(X, d)$  is *expansive* (see [11]) if there is  $c > 0$  such that for any  $x, y \in X$  with  $x \neq y$ , there is an integer  $k \in \mathbb{Z}$  such that  $d(f^k(x), f^k(y)) \geq c$ . Similarly, a homeomorphism  $f : X \rightarrow X$  of a compact metric space  $(X, d)$  is *continuum-wise expansive* (see [6, 7]) if there is  $c > 0$  such that for any nondegenerate subcontinuum  $A$  of  $X$ , there is an integer  $k \in \mathbb{Z}$  such that  $\text{diam } f^k(A) \geq c$ . Note that every expansive homeomorphism is continuum-wise expansive. Such  $c > 0$  is called an *expansive constant* for  $f$ . It is known that if a compact metric space  $X$  admits a continuum-wise expansive homeomorphism  $f$  on  $X$ , then  $\dim X < \infty$  and every minimal set of  $f$  is zero-dimensional (see [11] and [6]). Moreover,  $\dim I_0(f) \leq 0$ , where

$$I_0(f) = \bigcup \{M \mid M \text{ is a zero-dimensional } f\text{-invariant closed set of } X\}$$

(see [7, Proposition 2.5]). In particular,  $\dim P(f) \leq 0$ . We need the following proposition.

**Proposition 2.7.** ([6, Proposition 5.1]) *Suppose that  $f : X \rightarrow X$  is a homeomorphism of a compact metric space  $X$ . Then the following are equivalent.*

- (1)  *$f$  is continuum-wise expansive.*
- (2) *There is  $\delta > 0$  such that if  $\mathcal{C}$  is any finite open cover of  $X$  with  $\text{mesh}(\mathcal{C}) < \delta$  and any  $\gamma > 0$ , there is a sufficiently large natural number  $N$  such that if  $A, B \in \mathcal{C}$ , each component of  $f^{-n}(\text{cl}(A)) \cap f^n(\text{cl}(B))$  has diameter less than  $\gamma$  for each  $n \geq N$ .*

In the case of continuum-wise expansive homeomorphisms, by use of compact dark spaces we obtain the following decomposition theorem.

**Theorem 2.8.** *Suppose that  $X$  is a compact metric space with  $\dim X = n (< \infty)$  and  $f : X \rightarrow X$  is a continuum-wise expansive homeomorphism. Then there exists a compact  $(n-1)$ -dimensional dark space  $L$  of  $f$  except  $n$  times such that  $\dim A_f(L, j) = 0$  for each  $j = 0, 1, 2, \dots, n$ . In particular, there is the  $f$ -invariant zero-dimensional decomposition of  $X$  related to the compact dark space  $L$ :*

$$X = A_f(L, 0) \cup A_f(L, 1) \cup \dots \cup A_f(L, n).$$

**Remark.** (1) In Theorem 2.8, the bright space  $Z = X - L$  of  $f$  is open in  $X$  and  $n$ -dimensional. (2) In Theorem 2.8, suppose that  $\dim X = 1$ . Then  $L$  is a compact zero-dimensional dark space of  $f$  except 1 time such that  $\dim A_f(L, j) = 0$  for each  $j = 0, 1$  if and only if  $L$  is a zero-dimensional compactum such that  $f^i(L) \cap L = \emptyset$  for any  $i \in \mathbb{N}$  and  $\dim (X - \bigcup_{i \in \mathbb{Z}} f^i(L)) = 0$ .

**Example.** Let  $f : I = [0, 1] \rightarrow I$  be the 'tent' map of the unit interval  $I$  defined by  $f(x) = 2x$  for  $0 \leq x \leq 1/2$  and  $f(x) = 2 - 2x$  for  $1/2 \leq x \leq 1$ . Consider the inverse limit

$$X = \{(x_i)_{i=1}^\infty \in I^\infty \mid f(x_{i+1}) = x_i \text{ for } i \in \mathbb{N}\} \subset I^\infty$$

of  $f$  and the shift map  $\tilde{f} : X \rightarrow X$  defined by  $\tilde{f}((x_i)_{i=1}^\infty) = (f(x_i))_{i=1}^\infty$ . Then  $\tilde{f}$  is a continuum-wise expansive homeomorphism of the Knaster continuum  $X$ . Consider the subset

$$L = \{(x_i)_{i=1}^\infty \in X \mid x_1 = 1\}.$$

Then we can easily see that  $L$  is a zero-dimensional compactum (in fact, a Cantor set) such that  $\tilde{f}^i(L) \cap L = \emptyset$  for any  $i \in \mathbb{N}$  and  $\dim (X - \bigcup_{i \in \mathbb{Z}} \tilde{f}^i(L)) = 0$  and hence  $L$  is a compact zero-dimensional dark space  $L$  of  $\tilde{f}$  except 1 time such that  $\dim A_{\tilde{f}}(L, 0) = 0$ . In fact,  $X = A_{\tilde{f}}(L, 0) \cup A_{\tilde{f}}(L, 1)$  is a zero-dimensional decomposition of the Knaster continuum  $X$ .

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