

Residual vanishing of concentration arising in the mean field equations

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Abstract

In this short report, we study the Sawada-Suzuki equation. In the positive case, we prove the property called *Residual vanishing* which means that a blow-up solution sequence (more precisely, its subsequence) converges to a finite sum of Dirac's measures in the sense of measure.

1 Introduction

In this report, we consider the Sawada-Suzuki equation ([6]):

$$\begin{cases} -\Delta v_n = \lambda_n \int_I \alpha \left(\frac{e^{\alpha v_n}}{\int_{\Omega} e^{\alpha v_n}} - \frac{1}{|\Omega|} \right) \mathcal{P}(d\alpha) & \text{in } \Omega \\ \int_{\Omega} v_n = 0, \end{cases} \quad (1.1)$$

where (λ_n, v_n) is a solution sequence to (1.1), λ_n a non-negative number sequence tending to some non-negative number λ_0 , $I = [-1, 1]$, $\Omega = (\Omega, g)$ a two dimensional orientable compact Riemannian manifold, and $\mathcal{P}(d\alpha)$ a Borel probability measure on I . According to the result of [4], the following alternative holds:

(i) (*Compactness*) $\limsup_{n \rightarrow \infty} \|v_n\|_{\infty} < +\infty$, namely, there exist $v \in \mathcal{E}$ and a subsequence $\{v_{n_k}\} \subset \{v_n\}$ such that $v_{n_k} \rightarrow v$ in $\mathcal{E}$ as $k \rightarrow \infty$, where

$$\mathcal{E} = \left\{ v \in H^1(\Omega) \mid \int_{\Omega} v = 0 \right\}.$$

(ii) (*Concentration*) $\limsup_{n \rightarrow \infty} \|v_n\|_{\infty} = +\infty$, namely, the set $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_-$ is a non-empty and finite set, and there exists $0 \leq s_{\pm} \in L^1(\Omega)$ such that

$$\nu_{\pm, n} := \lambda_n \int_{I_{\pm}} \frac{\alpha e^{\alpha v_n}}{\int_{\Omega} e^{\alpha v_n}} \mathcal{P}(d\alpha) dx \xrightarrow{*} \nu_{\pm} = s_{\pm} dx + \sum_{x_0 \in \mathcal{S}_{\pm}} m(x_0) \delta_{x_0}(dx) \quad (1.2)$$

in $\mathcal{M}(\Omega)$ with $m(x_0) \geq 4\pi$ for all $x_0 \in \mathcal{S}_{\pm}$, where $I_+ = (0, 1]$, $I_- = [-1, 0)$, δ_x is the Dirac measure supported at x , $\mathcal{M}(\Omega) = C(\Omega)^*$ and

$$\mathcal{S}_{\pm} = \{x_0 \in \Omega \mid \text{there exists } \{x_n\} \subset \Omega \text{ such that } x_n \rightarrow x_0 \text{ and } v_n(x_n) \rightarrow \pm\infty\}. \quad (1.3)$$

It is natural to ask whether $s_{\pm}$ is zero or not in (1.2). If this is the case, we call this property *residual vanishing* in this report. In the positive case, we obtain

Proposition 1. *If (ii) above holds and $I = I_+$, then $s = s_+ = 0$.*

Remark 1. *We note that $\mathcal{S} = \mathcal{S}_+$ in the case $I = I_+$, see [4] for details. The proof of this fact is based on the boundedness from below of the Green function associated to $-\Delta$ on Ω , i.e.,*

$$\begin{cases} -\Delta_x G(x, y) = \delta_y - \frac{1}{|\Omega|} & \text{in } \Omega \\ \int_{\Omega} G(x, y) dx = 0, & \forall y \in \Omega, \end{cases}$$

see [1].

Remark 2. *Residual vanishing also holds in the case $I = I_-$.*

Remark 3. *It is open whether residual vanishing is true or not in the general case. On the contrary, the problem is not solved even in the simple case $\mathcal{P}(d\alpha) = \frac{1}{2}(\delta_{-1} + \delta_1)$ treated in [5].*

It is not difficult to show *residual vanishing* in the case $\mathcal{P}(d\alpha) = \delta_p$ for $p \in I$ by a direct application of the result (Theorem 3) of [2]. Just to be safe, we show it here, assuming $p = 1$ for simplicity, i.e.,

$$-\Delta v_n = \lambda_n \left(\frac{e^{v_n}}{\int_{\Omega} e^{v_n}} - \frac{1}{|\Omega|} \right).$$

Fix $x_0 \in \mathcal{S}$. If it fails then it holds that

$$\liminf_{n \rightarrow \infty} \int_{\Omega} e^{v_n} < +\infty.$$

We introduce

$$z_n = v_n - \log \int_{\Omega} e^{v_n}$$

and obtain

$$-\Delta z_n = \lambda_n e^{z_n} - \frac{\lambda_n}{|\Omega|} \quad \text{in } \Omega.$$

It follows from the assumption of contradiction that $z_n \rightarrow +\infty$ (for some subsequence still denoted by the same notation). Since λ_n is uniformly bounded and $-\lambda_n/|\Omega|$ can be regarded as a simple perturbed term, we can safely apply the result of [2] to the equation of z_n to find that $z_n \rightarrow -\infty$ in $B(x_0, r_0) \setminus \{x_0\}$ for $0 < r_0 \ll 1$, where $B(x, r)$ denotes a disk centered at x with radius r for $x \in \mathbf{R}^2$ and $r > 0$, in particular, B_r in the case $x = 0$. On the other hand, z_n is bounded below in $B(x_0, r_0) \setminus \{x_0\}$ since $\mathcal{S} = \mathcal{S}_+ \neq \emptyset$, a contradiction.

Still, it seems to be difficult to directly apply the result of [2] to the general positive case. To overcome this difficulty, we introduce the key transformation, see (2.3) below, and then develop a blowup analysis.

This report consists of three sections. We prove Proposition 1 in Section 2, and several lemmas stated there are shown in Section 3.

2 Proof of Proposition 1

In this section, we write I and $\mathcal{S}$ by I_+ and $\mathcal{S}_+$, respectively, in order to stress that we treat the positive case.

To prove the proposition, we have only to show

$$\mathcal{P}(\{\alpha \in I_+ \mid \liminf_{n \rightarrow \infty} \int_{\Omega} e^{\alpha v_n} = +\infty\}) = \mathcal{P}(I_+). \quad (2.1)$$

To confirm this, we fix $\omega \subset\subset \Omega \setminus \mathcal{S}_+$. Then, it holds that

$$\begin{aligned} 0 &\leq \int_{\omega} s_+ dx = \lim_{n \rightarrow \infty} \int_{\omega} \nu_{+,n} = \lim_{n \rightarrow \infty} \lambda_n \int_{\omega} \int_{I_+} \left(\frac{\alpha e^{\alpha v_n}}{\int_{\Omega} e^{\alpha v_n}} - \frac{1}{|\Omega|} \mathcal{P}(d\alpha) \right) \\ &\leq (\lambda_0 + 1) C(\omega) \lim_{n \rightarrow \infty} \int_{I_+} \frac{\mathcal{P}(d\alpha)}{\int_{\Omega} e^{\alpha v_n}} = 0 \end{aligned}$$

because $\lambda_n \rightarrow \lambda_0$ and v_n is uniformly bounded in ω . Hence, we obtain $s = 0$ in ω by $0 \leq s_{+,n} \in L^1(\Omega)$. Since $\omega \subset\subset \Omega \setminus \mathcal{S}_+$ is arbitrary, the proposition holds if (2.1) is true.

Now, we suppose that (2.1) is false. Then, there exists a number α_* such that

$$0 < \alpha_* := \sup\{\alpha \in I_+ \mid \liminf_{n \rightarrow \infty} \int_{\Omega} e^{\alpha v_n} < +\infty\} \quad \text{and} \quad \mathcal{P}((0, \alpha_*]) > 0. \quad (2.2)$$

Fix $x_0 \in \mathcal{S}_+$ and take $r_0 > 0$ satisfying $\overline{B(x_0, r_0)} \cap \mathcal{S}_+ = \{x_0\}$. It is possible to take such an r_0 because $\mathcal{S}$ is a finite set. We may assume $x_0 = 0$ by a translation. Then, there exist $x_n \in B_{r_0}$ and $\alpha_n \in \mathbf{R}$ such that

$$\begin{aligned} x_n &\rightarrow 0 \quad v_n(x_n) = \max_{\overline{B_{3r_0}}} v_n \rightarrow +\infty, \\ e^{\alpha_n v_n(x_n)} &= \int_{I_+} \frac{\alpha e^{\alpha v_n(x_n)}}{\int_{\Omega} e^{\alpha v_n}} \mathcal{P}(d\alpha). \end{aligned} \quad (2.3)$$

For this α_n , we obtain the following lemmas shown in next section.

Lemma 1. *There exists $C_1 > 0$, independent of n , such that*

$$\int_{I_+} \frac{\alpha e^{(\alpha - \alpha_n) v_n(x)}}{\int_{\Omega} e^{\alpha v_n}} \mathcal{P}(d\alpha) \leq C_1$$

for all $x \in \overline{B_{2r_0}}$.

Lemma 2. *We have*

$$\alpha_n \rightarrow \alpha_0 \in [\alpha_*, 1],$$

passing to a subsequence.

Here, we develop a blow-up argument. Set

$$\begin{cases} w_n(x) = \alpha_n v_n(x_n) - L, \\ \tilde{w}_n(x) = w_n(\sigma_n x + x_n) + 2 \log \sigma_n, \\ \sigma_n = e^{-w_n(x_n)/2} \rightarrow 0 \text{ by Lemma 2),} \end{cases}$$

where $L \gg 1$ will be determined later on. The function $\tilde{w}_n = \tilde{w}_n(x)$ is a solution to

$$\begin{cases} -\Delta \tilde{w}_n = \alpha_n \tilde{V}_n(x) e^{\tilde{w}_n} - \sigma_n^2 \frac{\alpha_n \lambda_n}{|\Omega|} \int_{I_+} \alpha \mathcal{P}(d\alpha) & \text{in } B_{r_0/\sigma_n} \\ \tilde{w}_n \leq \tilde{w}_n(0) = 0 & \text{in } B_{r_0/\sigma_n} \\ \int_{B_{r_0/\sigma_n}} \tilde{V}_n e^{\tilde{w}_n} \leq m(0), \end{cases} \quad (2.4)$$

where

$$\tilde{V}_n(x) = e^L \cdot \lambda_n \int_{I_+} \frac{\alpha e^{(\alpha - \alpha_n)v_n(\sigma_n x + x_n)}}{\int_{\Omega} e^{\alpha v_n}} \mathcal{P}(d\alpha).$$

Lemma 3. *There exist $\tilde{w} \in C^2(\mathbf{R}^2)$ and $0 < \tilde{V} \in C^2(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2)$ such that*

$$\tilde{w}_n \rightarrow \tilde{w}, \quad \tilde{V}_n \rightarrow \tilde{V} \quad \text{in } \mathbf{R}^2$$

and

$$\begin{cases} -\Delta \tilde{w} = \alpha_0 \tilde{V}(x) e^{\tilde{w}} & \text{in } \mathbf{R}^2 \\ \tilde{w} \leq \tilde{w}(0) = 0 & \text{in } \mathbf{R}^2 \\ \int_{\mathbf{R}^2} \tilde{V} e^{\tilde{w}} \leq m(0). \end{cases} \quad (2.5)$$

Lemma 3 is also shown in next section.

For a solution $\tilde{w}$ to (2.5), we set

$$\tilde{\phi}(x) = \frac{\alpha_0}{2\pi} \int_{\mathbf{R}^2} \tilde{V}(y) e^{\tilde{w}(y)} \log \frac{|x - y|}{1 + |y|} dy, \quad (2.6)$$

complying [3]. Noting that

$$\tilde{V} e^{\tilde{w}} \in L^1 \cap L^\infty(\mathbf{R}^2), \quad (2.7)$$

we find that the function $\tilde{\phi}$ set by (2.6) is well-defined in $\mathbf{R}^2$, and can show the following lemma because the proof of Lemma 1.1 of [3] is applicable to our case, see also Remark below.

Lemma 4. *There exists $C_2 > 0$, independent of L , such that*

$$\tilde{w}(x) \geq -\beta \log(1 + |x|) - C_2 \quad (2.8)$$

for $x \in \mathbf{R}^2$, where

$$\beta = \frac{\alpha_0}{2\pi} \int_{\mathbf{R}^2} \tilde{V} e^{\tilde{w}}. \quad (2.9)$$

Remark 4. *In Lemma 1.1 of [3], the integrability condition $\int_{\mathbf{R}^2} e^{\tilde{w}} dx < +\infty$ is assumed to show the estimates from above and below for solutions and the estimate from below for β . However, it is not required if one only needs the estimate from below (2.8).*

Proof of Proposition 1: Fix $R \gg 1$. It follows from Lemmas 3-4 that

$$v_n(x) \geq v_n(x_n) - \frac{\beta}{\alpha_n} \log \left(1 + \left| \frac{x - x_n}{\sigma_n} \right| \right) - \frac{C_2}{\alpha_n} + \varepsilon_n$$

for all $x \in B(x_n, \sigma_n R)$, where ε_n is a quantity converging to 0 as $n \rightarrow \infty$. This ε_n may be changed in the following but keeps the property that $\varepsilon_n \rightarrow 0$. We obtain

$$\begin{aligned} \int_{B(x_n, \sigma_n)} e^{\alpha v_n} &\geq e^{\alpha v_n(x_n) - \alpha C_2 / \alpha_n - 1} \int_{B(x_n, \sigma_n R)} \left(1 + \left| \frac{x - x_n}{\sigma_n} \right| \right)^{-\alpha \beta / \alpha_n} dx \\ &= e^{(\alpha - \alpha_n) v_n(x_n)} \cdot e^{L - \alpha C_2 / \alpha_n - 1} \int_{B_R} (1 + |x|)^{-\alpha \beta / \alpha_n} dx \end{aligned} \quad (2.10)$$

for all $\alpha \in I_+$. Thus, (2.3) and (2.10) yield

$$\begin{aligned} 1 &= \int_{I_+} \frac{\alpha e^{(\alpha - \alpha_n) v_n(x_n)}}{\int_{\Omega} e^{\alpha v_n}} \mathcal{P}(d\alpha) \\ &\leq \varepsilon_n + \int_{[\alpha_n, 1]} \frac{\int_{B(x_n, \sigma_n)} e^{\alpha v_n}}{\int_{\Omega} e^{\alpha v_n}} \cdot \frac{\alpha}{e^{L - \alpha C_2 / \alpha_n - 1} \int_{B_R} (1 + |x|)^{-\alpha \beta / \alpha_n} dx} \mathcal{P}(d\alpha) \\ &\leq \varepsilon_n + \frac{1}{e^{L - C_2 / \alpha_n - 1} \int_{B_R} (1 + |x|)^{-\beta / \alpha_n} dx}. \end{aligned} \quad (2.11)$$

Since $\beta / \alpha_n \leq (\alpha_0 / \alpha_n) \cdot (m(0) / 2\pi)$ by (2.9) and Lemma 3, inequality (2.11) implies

$$1 \leq \varepsilon_n + \frac{1}{e^{L - C_2 / \alpha_n - 1} \int_{B_R} (1 + |x|)^{-\frac{\alpha_0}{\alpha_n} \cdot \frac{m(0)}{2\pi}} dx},$$

or

$$1 \leq \frac{e^{1 + C_2 / \alpha_0} - L}{\int_{B_R} (1 + |x|)^{-\frac{m(0)}{2\pi}} dx},$$

which is a contradiction if L is sufficiently large. The proof is complete. $\square$

3 Proof of Lemmas 1-3

As having announced in the previous sections, we show Lemmas 1-3 in this section. We again consider the positive case (i.e., $\mathcal{S} = \mathcal{S}_+$ and $I = I_+$) in what follows.

Proof of Lemma 1: Since $\mathcal{S} = \mathcal{S}_+$, there exists $C_3 > 0$, independent of n , such that $v_n > -C_3$ in Ω . We use (2.3) and Jensen's inequality to calculate

$$\begin{aligned} & \int_{I_+} \frac{\alpha e^{(\alpha-\alpha_n)v_n(x)}}{\int_{\Omega} e^{\alpha v_n}} \mathcal{P}(d\alpha) \\ & \leq \int_{I'_{+,n}} \frac{\alpha e^{-(\alpha_n-\alpha)v_n(x)}}{\int_{\Omega} e^{\alpha v_n}} + \int_{I_+} \frac{\alpha e^{(\alpha-\alpha_n)v_n(x)}}{\int_{\Omega} e^{\alpha v_n}} \mathcal{P}(d\alpha) \\ & \leq \frac{\alpha_n \mathcal{P}(I'_{+,n}) e^{\alpha_n C_3}}{|\Omega|} + 1 \leq \frac{e^{C_3}}{|\Omega|} + 1 \end{aligned}$$

for all $x \in \overline{B_{2r_0}}$ and n , where

$$I'_{+,n} = \begin{cases} (0, \alpha_n) & \text{if } \alpha_n > 0 \\ \emptyset & \text{if } \alpha_n \leq 0. \end{cases}$$

The lemma is completely shown. $\square$

Proof of Lemma 2: Put $\alpha_0 = \lim_{n \rightarrow \infty} \alpha_n$.

Assume that $\alpha_0 > 1$. Then, there exists $\delta > 0$ such that

$$e^{(1+\delta)v_n(x_n)} \leq e^{\alpha_n v_n(x_n)},$$

that is, by Jensen's inequality,

$$e^{\frac{\delta}{2}v_n(x_n)} \leq \int_{I_+} \frac{\alpha e^{(\alpha-1-\delta/2)v_n(x_n)}}{\int_{\Omega} e^{\alpha v_n}} \mathcal{P}(d\alpha) \leq e^{-\frac{\delta}{2}v_n(x_n)} |\Omega|^{-1}$$

for $n \gg 1$, which is a contradiction because $v_n(x_n) \rightarrow +\infty$.

Next, assume that $\alpha_0 \leq 0$. In the case that $\mathcal{P}((0, \alpha_*)) > 0$, there exists $0 < \varepsilon \ll 1$ such that $\mathcal{P}([\varepsilon, \alpha_* - \varepsilon]) > 0$, and therefore

$$\begin{aligned} 1 &= \int_{I_+} \frac{\alpha e^{(\alpha-\alpha_n)v_n(x_n)}}{\int_{\Omega} e^{\alpha v_n}} \mathcal{P}(d\alpha) \\ &\geq \int_{[\varepsilon, \alpha_* - \varepsilon]} \frac{\alpha e^{(\alpha-\varepsilon/2)v_n(x_n)}}{\int_{\Omega} e^{\alpha v_n}} \mathcal{P}(d\alpha) \\ &\geq c(\varepsilon) e^{\frac{\varepsilon}{2}v_n(x_n)} \mathcal{P}([\varepsilon, \alpha_* - \varepsilon]) \rightarrow +\infty \end{aligned}$$

as $n \rightarrow \infty$, a contradiction. In the case that $\mathcal{P}(\{\alpha_*\}) = \mathcal{P}((0, \alpha_*]) > 0$, it holds that $\liminf_{n \rightarrow \infty} \int_{\Omega} e^{\alpha_* v_n} < +\infty$, and hence

$$\begin{aligned} 1 &= \int_{I_+} \frac{\alpha e^{(\alpha - \alpha_n)v_n(x_n)}}{\int_{\Omega} e^{\alpha v_n}} \mathcal{P}(d\alpha) \\ &\geq \alpha_* e^{(\alpha_* - \alpha_n)v_n(x_n)} \left(\int_{\Omega} e^{\alpha_* v_n} \right)^{-1} \mathcal{P}(\{\alpha_*\}) \rightarrow +\infty \end{aligned}$$

as $n \rightarrow \infty$, a contradiction.

We have shown that $\alpha_0 \in (0, 1]$. It is left to show that $\alpha_0 \geq \alpha_*$. To prove this, we finally assume that $\alpha_0 \in (0, \alpha_*)$. Consider

$$\varphi_n = \alpha_n v_n - \log \int_{\Omega} e^{\alpha_n v_n}.$$

Passing to a subsequence, we have

$$\varphi_n(x_n) \rightarrow +\infty. \quad (3.1)$$

The function $\varphi_n = \varphi_n(x)$ satisfies

$$\begin{cases} -\Delta \varphi_n = K_n(x) e^{\varphi_n} - \frac{\alpha_n \lambda_n}{|\Omega|} \int_{I_+} \alpha \mathcal{P}(d\alpha) & \text{in } B_{2r_0} \\ \int_{\Omega} e^{\varphi_n} = 1, \end{cases} \quad (3.2)$$

where

$$K_n(x) = \alpha_n \lambda_n \left(\int_{\Omega} e^{\alpha_n v_n} \right) \int_{I_+} \frac{\alpha e^{(\alpha - \alpha_n)v_n(x)}}{\int_{\Omega} e^{\alpha v_n}} \mathcal{P}(d\alpha).$$

Lemma 1 and the boundedness $\liminf_{n \rightarrow \infty} \int_{\Omega} e^{\alpha_n v_n} < +\infty$ show that there exists $C_4 > 0$, independent of n , such that

$$0 \leq K_n \leq C_4 \quad \text{in } B_{2r_0}. \quad (3.3)$$

Consequently, (3.1)-(3.3) assure that

$$\varphi_n \rightarrow -\infty \quad \text{locally uniformly in } B_{2r_0} \setminus \{0\} \quad (3.4)$$

by virtue of the result of [2]. However, (3.4) is false since $\mathcal{S} = \mathcal{S}_+$ and $\liminf_{n \rightarrow \infty} \int_{\Omega} e^{\alpha_n v_n} < +\infty$. $\square$

Proof of Lemma 3: It follows from Lemma 2 that

$$0 \leq \tilde{V}_n \leq e^{L(\lambda_0+1)} C_1 \quad \text{in } B_{r_0/\sigma_n}$$

for $n \gg 1$. We also have

$$0 \leq e^{\tilde{w}_n} \leq 1 \quad \text{in } B_{r_0/\sigma_n}$$

for all n , and

$$\sigma_n^2 \frac{\alpha_n \lambda_n}{|\Omega|} \int_{I_+} \alpha \mathcal{P}(d\alpha) \rightarrow 0$$

as $n \rightarrow \infty$. Combining these properties with $\tilde{w}_n(0) = 0$, we can safely apply the result of [2] to find that, for every $R > 0$, there exists $C_5(R) > 0$ such that

$$\tilde{w}_n \geq -C_5(R) \quad \text{in } B_R \quad (3.5)$$

for $n \gg 1$. Thus, the elliptic regularity and a diagonal argument show that there exists $\tilde{w} \in C^{1+\alpha}(\mathbf{R}^2)$, $\alpha \in (0, 1)$, such that

$$\tilde{w}_n \rightarrow \tilde{w} \quad \text{in } C_{loc}^{1+\alpha}(\mathbf{R}^2). \quad (3.6)$$

Noting the definitions of $\tilde{V}_n$ and $\tilde{w}_n$, we see that there exists $\tilde{V} \in C^{1+\alpha}(\mathbf{R}^2)$, $\alpha \in (0, 1)$, such that

$$\tilde{V}_n \rightarrow \tilde{V} \quad \text{in } C_{loc}^{1+\alpha}(\mathbf{R}^2). \quad (3.7)$$

We again use the elliptic regularity, together with (3.6)-(3.7), and conclude the relation (2.5) and $\tilde{w}, \tilde{V} \in C^2(\mathbf{R}^2)$.

It is clear that $\tilde{V} \in L^\infty(\mathbf{R}^2)$ by Lemma 1, and therefore, we must show that $\int_{\mathbf{R}^2} \tilde{V} e^{\tilde{w}} \leq m(0)$ and that $\tilde{V} > 0$ in $\mathbf{R}^2$.

For every $R > 0$ and $0 < r \ll 1$,

$$\begin{aligned} \int_{B_R} \tilde{V} e^{\tilde{w}} &\leq \liminf_{n \rightarrow \infty} \int_{B_R} \tilde{V}_n e^{\tilde{w}_n} \leq \liminf_{n \rightarrow \infty} \int_{B_{r/\sigma_n}} \tilde{V}_n e^{\tilde{w}_n} \\ &= \liminf_{n \rightarrow \infty} \int_{B(x_n, r)} \nu_{+,n} \leq m(0) + \int_{B_{2r}} \nu_+ \end{aligned}$$

by the Fatou lemma, the definitions of $w_n, \tilde{w}_n, \sigma_n$ and $\tilde{V}_n$, and (1.2). Letting $R \uparrow +\infty$ and $r \downarrow 0$, we obtain $\int_{\mathbf{R}^2} \tilde{V} e^{\tilde{w}} \leq m(0)$.

Finally, we use the definitions of $w_n, \tilde{w}_n, \sigma_n$ and $\tilde{V}_n$, (3.5), $\tilde{w}_n \leq 0$ and (1.2) to obtain $C_6(R) > 0$, independent of $n \gg 1$, such that

$$\begin{aligned} \tilde{V}_n(x) &= e^L \lambda_n \int_{I_+} \frac{\alpha e^{\frac{\alpha - \alpha_n}{\alpha_n} (\tilde{w}_n(x) + \alpha_n v_n(x_n))}}{\int_{\Omega} e^{\alpha v_n}} \mathcal{P}(d\alpha) \\ &\geq e^{L - C_6(R)} \lambda_n \int_{I_+} \frac{\alpha e^{(\alpha - \alpha_n) v_n(x_n)}}{\int_{\Omega} e^{\alpha v_n}} \mathcal{P}(d\alpha) = e^{L - C_6(R)} \lambda_n \end{aligned}$$

for all $x \in B_R$ and $n \gg 1$, and for every $R > 0$, which means $\tilde{V} > 0$ in $\mathbf{R}^2$ because $\lambda_n \rightarrow \lambda_0 > 0$ by $\mathcal{S} = \mathcal{S}_+ \neq \emptyset$. $\square$

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