

Supplement to free L-spaces

Ehime Univ., Faculty Sci. K. Nagami

When I introduced the notion of free L-spaces in [2] the concept of canonical neighborhoods were used in the definition. However if canonical neighborhoods are weakened to semi-canonical neighborhoods, yet the theory developed in [2] seems to be almost effective for the weakened case. To note this fact will be convenient for the reader of [2].

Let X be a space, F a closed set of X , and $\mathcal{U}_F$ an anti-cover of F . An open neighborhood U of F is said to be semi-canonical with respect to $\mathcal{U}_F$ if $\text{Cl } \mathcal{U}_F(X-U) \cap F = \emptyset$. Let $\mathcal{F}$ be a closed cover of X . Then $\{\mathcal{F}, \mathcal{U}_F(F \in \mathcal{F})\}$ is said to be a weak L-structure of X if

a) $\mathcal{F}$ is σ -discrete,

b) for each $x \in X$ and each open neighborhood U of x there exist finite elements $F_1, \dots, F_n$ of $\mathcal{F}$ and their corresponding semi-canonical neighborhoods $U_1, \dots, U_n$ with respect to $\mathcal{U}_{F_1}, \dots, \mathcal{U}_{F_n}$ respectively, such that

$$x \in \bigcap_{i=1}^n F_i \subset \bigcap_{i=1}^n U_i \subset U.$$

Let (WL) denote the class of paracompact Hausdorff spaces with weak L-structures. Then (WL) is hereditary and countably productive.

THEOREM. If $X \in (WL)$, then the following conditions are equivalent.

- i) $\dim X \leq n$.
- ii) X is the image of a free L-space Z with $\dim Z \leq 0$ under a perfect map f with order $f \leq n + 1$.
- iii) $\text{Ind } X \leq n$.
- iv) X is the sum of subsets $X_1, \dots, X_{n+1}$ with $\dim X_i \leq 0$ for $i=1, \dots, n+1$.
- v) X has a σ -closure-preserving base $\mathcal{U}$ such that $\dim \partial U \leq n-1$ for each $U \in \mathcal{U}$.
- vi) X has a stratification $\{U_i\}$ such that $\dim \partial U_i \leq n-1$ for each open set U and each i.

The implication iii) $\rightarrow$ v) is verified by a similar way as in [2, Theorem 2.7]. The implication v) $\rightarrow$ iii) needs the following which is just a trivial modification of [1, Lemma 15.5].

LEMMA. Let X be a normal semi-stratifiable space and $\{F_\alpha: \alpha \in A\}$ an order-closure-preserving closed cover of X. Then $\text{Ind } X = \sup \text{Ind } F_\alpha$.

Proof. Assume that $\dim F_\alpha \leq n$ for each $\alpha \in A$. Set

$$\mathcal{H}_i = \{ H_{\alpha i} = F_\alpha \cap (X - \bigcup_{\beta < \alpha} F_\beta) : \alpha \in A \}, i \in \mathbb{N},$$

$$H_i = \mathcal{H}_i^\#.$$

Then $\bigcup \mathcal{H}_i$ covers X. To see that $\mathcal{H}_i$ is discrete let x be an arbitrary point of X and α the minimal with $x \in F_\alpha$. Set

$$V = (X - (X - \bigcup_{\beta \leq \alpha} F_\beta)_i) - \bigcup_{\beta < \alpha} F_\beta.$$

Then V is an open neighborhood of x with $V \cap H_{\beta i} = \emptyset$ for any $\beta \neq \alpha$. Thus $\text{Ind } H_i \leq n$ which implies $\dim X \leq n$ by the sum theorem.

T. Mizokami wrote to me that the equivalence of iii) and v) holds for some kind of spaces which seem to be free L-spaces. As is expected from the condition ii) we can prove that each member of (WL) is the perfect image of a free L-space. If each perfect image of a free L-space has to be a free L-space, we can conclude that (WL) coincides with the class of free L-spaces. I don't know whether this is the case or not. I don't know whether the theorem is valid for M_1 or M_3 spaces.

References

- [1] K. Nagami, Dimension theory, 1970, Academic Press, New York.
- [2] ———, Dimension of free L-spaces, forthcoming in Fund. Math..