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Relative invariants and irreducible highest weight modules

熊本電波高専　菅　修一(SUGA Shuichi)

1. Introduction.

It is well known that finite dimensional irreducible sl(2,**C**)-modules are constructed in the following way. Let $\{e, h, f\}$ be a standard basis of sl(2,**C**) so that their Lie brackets are given by

$$[h, e]=2e, \quad [h, f]=-2f, \quad [e, f]=h \tag{1.1}$$

Fix a nonnegative integer n. Let $V=\sum_{i=0}^{n} \mathbf{C}x^i$ be the vector space of polynomials of x whose degrees are at most n. Then the following correspondence defines an irreducible representation of sl(2,**C**) on V:

$$\begin{cases} e & \to & x^2\dfrac{d}{dx}-nx \\ h & \to & 2x\dfrac{d}{dx}-n \\ f & \to & -\dfrac{d}{dx} \end{cases} \tag{1.2}$$

It is easy to see that n is highest weight of V and x^n is a highest weight vector.

If one replace n by any complex number λ and V by $W=\sum_{i=0}^{\infty}\mathbf{C}x^{\lambda-i}$, then W is still an irreducible highest weight sl(2,**C**)-module with highest weight λ and a highest weight vector x^λ. In this note we shall generalize the above construction of irreducible highest weight representations to Lie algebras related to certain Hermitian symmetric pairs. The generalization is done by replacing x^λ by a complex power of relative invariant polynomials of prehomogeneous vector spaces attached to Hermitian symmetric spaces. This also gives a representation theoretic interpretation of the zeros of the b-function.

2. Construction of irreducible highest weight modules.

Let (G_o, K_o) be one of the following pairs.

1. $G_o = SU(n,n)$, $\quad K_o = S(U(n) \times U(n))$
2. $G_o = Sp(2n,\mathbf{R})$, $\quad K_o = U(n)$
3. $G_o = SO(4n)^*$, $\quad K_o = U(2n)$
4. $G_o = SO(n,2)_o$, $\quad K_o = SO(n) \times SO(2)$.

Where we regard K_o as a maximal compact subgroup of the Lie group G_o. The quotient space G_o/K_o is an Hermitian symmetric space of tube type. Let g_o (resp. k_o) be the Lie algebra of G_o (resp. K_o) , and $g_o = k_o \oplus p_o$ the Cartan decomposition of g_o. By convention we delete the subscript o to denote complexified Lie algebras. So we have the decomposition $g = k \oplus p$ of the compexified Lie algebra g.

The Lie algebra k_o has the one dimensional center $Z = \mathbf{R}z$, where the eigenvalues of z under the adjoint action on p are $\pm i$.

Let

$$p^+ = \{x \in p \mid [z , x] = ix\} \text{ and } p^- = \{x \in p \mid [z , x] = -ix\} .$$

Then p^+ (resp. p^-) corresponds to the holomorphic (resp. anti-holomorphic) vector fields on the Hermitian symmetric space G_o/K_o . Let t_o be a Cartan subalgebra of k_o then t is a Cartan subalgebra of g. We set $q = k \oplus p^+$ then q is a maximal parabolic subalgebra of g. Let b be a Borel subalgebra contained in q and contains t. Let G be the connected and simply connected complex Lie group with the Lie algebra g and K (resp. Q) the subgroup of G corresponding to the Lie subalgebra k (resp. q). We denote the universal covering groups of K and Q by $\tilde{K}$ and $\tilde{Q}$ respectively. Let $\pi : \tilde{Q} \to Q$ be the projection homomorphism. We choose an open neighborhood $U \subset Q$ of the identity so that there exists a section $\sigma : U \to \tilde{Q}$.

It is known that the pair (K,p^-) is a regular irreducible prehomogeneous vector spaces via the adjoint K-action. There exists a unique (up to constant multiple) irreducible K-relative invariant polynomial f on p^-. By definition f is an irreducible polynomial on p^- satisfying

$$f(Ad(k)x) = \chi(k)f(x) \quad (k \in K, x \in p^-) \qquad (2.1)$$

for some one dimensional character χ of K.

We fix an arbitrary 1-dimensional character λ of $\widetilde{K}$, then there exists a complex number m such that $\lambda = \mu\chi$. (We denote the group of one dimensional characters of $\widetilde{K}$ additively.) We extend λ to $\widetilde{Q}$ trivially and denote it by the same letter. We also denote the differential of λ by the same letter and consider it as an element of t^*.

Let N^- be the subgroup of G corresponding to $ad(p^-)$. Then the exponential map $\exp: p^- \to N^-$ is a diffeomorphism. We denote its inverse by $\log: N^- \to p^-$ We put $O = N^-U$; an open subset of G.

Let $L(\lambda) = \{ h : O \to \mathbf{C} \mid h(g\,q) = \lambda(\sigma(q))\, h(g)\,,\ g \in O,\ q \in U \}$. By differentiating the left translation of G we get an algebra homomorphism $\varphi : U(g) \to D(O)$. Here U(g) is the universal enveloping algebra of g and D(O) is the algebra of differential operators on O with holomorphic coefficients. The homomorphism φ defines a U(g) - action on $L(\lambda)$.

Using the relative invariant polynomial f on p^-, we define an element v^λ of $L(\lambda)$ by the following formula:

$$v^\lambda(n\,q) = \lambda(\sigma(q))\, f^{-2\mu}(\log(n)) \quad n \in N^-,\ q \in U. \qquad (2.2)$$

Consider the g-module $W(\lambda) = \varphi(U(g))\, v^\lambda$ generated by v^λ.

Theorem. W(λ) is an irreducible highest weight g-module (with respect to the Borel subalgebra b) with highest weight λ and a highest weight vector v^λ.

The proof of Theorem will appear in the forthcoming paper [6].

3. The case $G_0 = SU(n,n)$, $K_0 = S(U(n) \times U(n))$.

We illustrate the representation in Section 2. We realize the group G_0 as

$$SU(n,n) = \{ g \in GL_{2n}(\mathbf{C}) \mid {}^t\bar{g} H_{2n} g = H_{2n} , \ \det g = 1 \} \qquad (3.1)$$

where $H_{2n} = \begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix}$. Then the Lie algebras g , k , $p^{\pm}$ and q defined in Section 2 can be taken as the following form:

$$\begin{aligned} & g = sl(2n,\mathbf{C}) \ , \ k = \{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in g \} \ , \ p^+ = \{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in g \} , \\ & p^- = \{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \in g \} \text{ and } q = \{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in g \}. \end{aligned} \qquad (3.2)$$

Let λ be any complex number we regard it as a one dimensional character of k by

$$\lambda(a) = \lambda \mathrm{tr} A \ \text{ for } \ a = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in k . \qquad (3.3)$$

From the definition we can identify $L(\lambda)$ with the space of $\mathbf{C}$-valued functions on N^-. Since the latter space can be identified with the space $\mathbf{C}(p^-)$ of $\mathbf{C}$-valued functions on p^- via the exponential map, we shall use this space in the follwing . Then the highest weight vector v^{λ} of $W(\lambda)$ is corresponds to f^{λ}.

Let $(e_{jk})_{j,k=1,\cdots,n}$ be the $n \times n$ matrix units and $(x_{jk})_{j,k=1,\cdots,n}$ a standard coordinate system on p^-. Then the set of matrices

$$F_{jk} = \begin{pmatrix} 0 & 0 \\ e_{jk} & 0 \end{pmatrix} \ (\text{resp. } E_{jk} = \begin{pmatrix} 0 & e_{jk} \\ 0 & 0 \end{pmatrix}), \ j,k = 1,2,\cdots,n$$

gives a basis of p^- (resp. p^+).

The elements of g acts on $\mathbf{C}(p^-)$ by the following formulas:

$$\begin{cases} E_{jk} \to & \displaystyle\sum_{l,m=1}^{n} x_{lj}\, x_{km} \frac{\partial}{\partial x_{lm}} - \lambda x_{kj} \\ a \ \to & \displaystyle\sum_{l=1}^{n} (x_{jl}\, a_{lk} - d_{jl}\, x_{lk}) \frac{\partial}{\partial x_{jk}} - \lambda \mathrm{tr}(A) \ . \\ F_{jk} \to & \displaystyle -\frac{\partial}{\partial x_{jk}} \end{cases} \qquad (3.4)$$

Where $a=\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$, $A=(a_{jk})$ and $D=(d_{jk})$. This gives a generalization of (1.2).

By the Poincaré-Birkhoff-Witt theorem we have

$$\varphi(U(g))v^{\lambda}=\varphi(U(p^{-})U(k)U(p^{+}))v^{\lambda}=\varphi(U(p^{-}))v^{\lambda}. \qquad (3.5)$$

Hence in particular we conclude that $W(\lambda)$ consists of all the differential polynomials of v^{λ}.

4. Reducibilities of generalized Verma modules

In this section we discuss reducibilities of generalized Verma modules induced from the maximal parabolic subalgebra q. This gives a representation theoretic interpretation of zeros of the b-function. We retain the notations in the previous sections.

Let f^{*} be the relative invariant polynomial of the prehomogeneous vector space (K, p^{+}) and $f^{*}(D_x)$ the linear differential operator with constant coefficients defined by the equation:

$$f^{*}(D_x)\exp\langle\xi,x\rangle=f^{*}(\xi)\exp\langle\xi,x\rangle \text{ for } \xi\in p^{+} \text{ and } x\in p^{-}. \qquad (4.1)$$

where $\langle\ ,\ \rangle$ is the Killing form on g. Let s be a complex parameter then there is a polynomial b(s) such that the following differential equation holds:

$$f^{*}(D_x)f(x)^{s}=b(s)f(x)^{s-1}. \qquad (4.2)$$

The polynomial b(s) is called the b-function of the relative invariant f.

Let λ be a one dimensional representation of k and extend it trivially to q. Let $\mathbf{C}_{\lambda}$ be the representation space of λ and define generalized Verma module $V(\lambda)$ as follows:

$$V(\lambda)=U(g)\otimes_{U(q)}\mathbf{C}_{\lambda}. \qquad (4.3)$$

Jantzen [2] gave reducibility criterion for $V(\lambda)$ using his formulas on determinants of contravariant forms. But our construction of irreducible highest weight modules $W(\lambda)$ reproduces a part of his result.

Corollary. Let r be the split rank of G_0 and $t=-2i\lambda(z)/r$. Then if t is a positive integer or satisfies $b(t)=0$ then $V(\lambda)$ is reducible.

Proof. Here we consider only the first case but the same argument is true for any other case. As in the previous section we regard a complex number λ as a 1-dimensional character of k. If t is a positive integer then highest weight vector $v^\lambda = f^\lambda$ of $W(\lambda)$ is a polynomial function on p^-. Hence $W(\lambda)$ becomes finite dimensional g-module and $V(\lambda)$ is reducible.

Now we consider when $b(\lambda)=0$. In this case the b-function and its defining equation (4.2) is given by the following Capelli's identity (Weyl [7]):

$$\det\left(\frac{\partial}{\partial x_{jk}}\right)\cdot\det(x_{jk})^s = s(s+1)\cdots(s+n-1)\det(x_{jk})^{s-1}$$
$$= b(s)\det(x_{jk})^{s-1}. \qquad (4.4)$$

Suppose $V(\lambda)$ is irreducible then $W(\lambda)$ and $V(\lambda)$ are isomorphic. But then by the Poincare-Birkhoff-Witt theorem $W(\lambda)$ is isomorphic to $U(p^-)$ as a vector space. Then

$$(-1)^n F_{1\,\sigma(1)} F_{2\,\sigma(2)} \cdots F_{n\,\sigma(n)}\, v^\lambda = \frac{\partial^n}{\partial x_{1\,\sigma(1)}\,\partial x_{2\,\sigma(2)}\cdots\partial x_{n\,\sigma(n)}}\det(x_{jk})^\lambda \quad (4.5)$$

must be linearly independent, where σ runs over the set of all permutations of $\{1,2,\cdots,n\}$. But this contradicts to the Capelli's identity. Hence $V(\lambda)$ is also reducible in this case.

Q.E.D.

Remark. The above Corollary holds for any Hermitian symmetric pair listed in section 2. Also it is known that if λ is a zero of the b-function then $W(\lambda)$ is unitarizable (Enright, Howe and Wallach [1]). Moreover these facts still remains true for the exceptional Hermitian symmetric pair

$$G_0 = \text{real form of } E_7,\ K_0 = \text{compact form of } E_6 \times SO(2).$$

These observations are the original motivation of this research.

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