

An extension of AKTH-theory to locally compact groups

by Nobuhiko TATSUUMA

1. Let $\{\mathcal{A}, G, \alpha\}$ be a C^* -system. That is, $\mathcal{A}$ is a C^* -algebra, G is a locally compact group, and $G \ni g \mapsto \alpha_g \in \text{Aut}(\mathcal{A})$ is a continuous homomorphism. Consider an α -invariant state ω on $\mathcal{A}$, and the unitary representation $\{\pi, U_g, \mathcal{H}, \Omega\}$ of G deduced by GNS-construction.

For any $A, B \in \mathcal{A}$, put $f_{AB}(g) = \omega(B \alpha_g(A)) - \omega(A)\omega(B) = \langle U_g \pi(A) \Omega, \pi(B^*) \Omega \rangle - \langle \pi(A) \Omega, \Omega \rangle \langle \pi(B) \Omega, \Omega \rangle$ and $g_{AB}(g) = \omega(\alpha_g(A)B) - \omega(A)\omega(B)$. Then evidently,

$$(1) \quad g_{AB}(g) = f_{A^*B^*}(g) = f_{BA}(g^{-1}).$$

Now we assume the existence of a norm dense α -invariant $*$ -subalgebra $\mathcal{A}_0$ of $\mathcal{A}$, for which the followings are valid.

Put

$$\mathcal{F}_0 = (\text{function algebra on } G \text{ algebraically generated by } \{f_{AB}\}_{A, B \in \mathcal{A}_0}),$$

$$\mathcal{F} = (\text{the uniform closure of } \mathcal{F}_0),$$

and construct $\mathcal{G}_0$ and $\mathcal{G}$ as same way from $\{g_{AB}; A, B \in \mathcal{A}_0\}$.

[Assumption 1] $\mathcal{F}$ is closed with respect to complex conjugation.

[Assumption 2] For any $n \geq 1$ and $A_j, B_j \in \mathcal{A}_0$ ($j=1, 2, \dots$),

$$(2) \quad \int_G \left(\prod_{j=1}^n f_{A_j B_j}(g) - \prod_{j=1}^n g_{A_j B_j}(g) \right) dg = 0.$$

[Assumption 3] There exist $1 \leq p, q < +\infty$ and a non-zero

element $f_0 \equiv \sum_k \prod_j f_{A_{j,k}, B_{j,k}} \quad (A_{j,k}, B_{j,k} \in \mathcal{A}_0) \text{ in } \mathcal{F}_0$

such that (i) $f_0 \in L^p(G)$,

(ii) $g_0 \equiv \sum_k \prod_j g_{A_{j,k}, B_{j,k}} \in L^q(G)$.

(We use a right Haar measure dg on G).

The purpose of this paper is to show the KMS-property for C^* -systems which satisfy the above assumptions.

From (1) and [Assumption 1] the following lemma is direct.

Lemma 1 $\mathcal{F} = \mathcal{G}$, and $\mathcal{F}$ is closed under the operation

$$f(g) \mapsto f(g^{-1}) .$$

2. We shall give the formulation of our KMS-property on C^* -systems based on Araki-Kastler-Takesaki-Haag's theory.

When G is the additive group $\mathbb{R}$ of real numbers, the ordinary KMS-property is stated as follows.

[KMS] The function $\psi_{AB}(t) = \omega(B \alpha_t(A))$ can be extended analytically on some strip domain $\{t; 0 \leq \Im(t) \leq \beta\}$ and

$$\psi_{AB}(t+i\beta) = \omega(\alpha_t(A)B) \text{ for any } t \text{ in } \mathbb{R} \text{ and any } A, B \text{ in } \mathcal{A}.$$

In the other hand for any one-parameter subgroup $g(t)$ of G , using the Stone's theorem, we can determine its infinitesimal generator iH , as H is a self-adjoint operator on $\mathcal{H}$ and

$$e^{iHt} = U_{g(t)} .$$

Now in our case, denote by K the kernel in G of the homomorphism $g \rightarrow \alpha_g$, then our main result is given as follows.

MAIN THEOREM. Under the assumptions 1~3, there exists an one-parameter subgroup $g(t)$ of G/K , such that

$$\langle U_g e^{H/2} \pi(A) \Omega, e^{H/2} \pi(B^*) \Omega \rangle = \langle \pi(B) \Omega, U_g \pi(A) \Omega \rangle,$$

for any A, B in $\mathcal{A}_0$.

If the Main Theorem is proved, the function

$$\psi(t) = \omega(B \alpha_{g(t)}(A)) = \langle U_{g(t)} \pi(A) \Omega, \pi(B^*) \Omega \rangle$$

has the analytical extension

$$\psi(t+is) = \langle U_{g(t)} e^{sH/2} \pi(A) \Omega, e^{sH/2} \pi(B^*) \Omega \rangle$$

and $\psi(t+i) = \langle \pi(B^*) \Omega, U_{g(t)} \pi(A) \Omega \rangle = \omega(\alpha_{g(t)}(A)B)$.

This shows that the subsystem $\{\mathcal{A}, R, \alpha_{g(t)}\}$ is just a KMS-C*-system as originally defined.

3. At first we discuss under slightly more general situation and prove a useful Proposition 1.

Let F_0 be a set of bounded uniformly continuous functions on G , and F be the uniform closure of F_0 . For any $f \in F$, put $G_f = \{g \in G; f(gg_1) = f(g_1), \forall g_1 \in G\}$ and $G_{F_0} = \bigcap_{f \in F_0} G_f$, $G_F = \bigcap_{f \in F} G_f$.

Lemma 2. $G_F = G_{F_0}$, and G_F is a closed subgroup of G .

Proof. Because f is continuous, G_f is closed. Hence G_F, G_{F_0} are closed.

For any $k_1, k_2 \in G_f$, $g \in G$, $f(k_1^{-1}k_2g) = f(k_1(k_1^{-1}k_2g)) = f(k_2g) = f(g)$. Thus $k_1^{-1}k_2 \in G_f$, therefore G_f and G_F, G_{F_0} are subgroups.

Obviously $G_F \subset G_{F_0}$. If $G_F \neq G_{F_0}$ there exists $g_1 \in G_{F_0}$ and $\notin G_F$. That is, $\exists g_2, \exists f \in F$ and $f(g_1g_2) \neq f(g_2)$ and for $\forall \varphi \in F_0$, $\varphi(g_1g) =$

$\varphi(g)$. On the other hand, $\forall \varepsilon > 0, \exists \varphi_1 \in F_0$ such that $\|f - \varphi_1\|_\infty < \varepsilon/2$.
 Put $\varepsilon = |f(g_1 g_2) - f(g_2)|$, then $|f(g_1 g_2) - f(g_2)| \leq |f(g_1 g_2) - \varphi(g_1 g_2)| + |\varphi(g_1 g_2) - \varphi(g_2)| + |\varphi(g_2) - f(g_2)| < \varepsilon/2 + \varepsilon/2 = \varepsilon = |f(g_1 g_2) - f(g_2)|$.
 That is contradiction.

Lemma 3. If there is a non-trivial function f_0 of zero at ∞ in F_0 , then the subgroup G_{f_0} and $G_{F_0} = G_F$ are compact.

Proof. If G_{f_0} is not compact, there exists a sequence $\{k_j\} \subset G_{f_0}$ such that $k_j \rightarrow \infty$. Therefore for some $g_0 \in G$, and for all j
 $0 \neq f_0(g_0) = f_0(k_j g_0)$. This contradicts to the assumption for f_0 .

Corollary 1. In such a case, $L^p(G_F \backslash G)$ is imbedded into $L^p(G)$ as a space of functions which are constant on G_F -left cosets.

Hereafter we write $H = G_F$.

Lemma 4. If a uniformly continuous function f on G belongs to $L^p(G)$ for some $p < +\infty$, f is zero at ∞ .

Proof. If f is not zero at ∞ , there exists a sequence $\{k_j\} \subset G$ and $a > 0$ such that $k_j \rightarrow \infty, |f(k_j)| > a$ for any j .
 Uniform continuity of f asserts the existence of a compact neighborhood V of e , such that $|f(g_1) - f(g_2)| < a/2$ for any $g_1, g_2 \in V$ such that $g_1 g_2^{-1} \in V$. Since $k_j \rightarrow \infty$, if it is necessary, taking a subsequence, we can assume $V k_j \cap V k_\ell = \emptyset$ ($j \neq \ell$). Thus,

$$\int_G |f(g)|^p d g \geq \sum_j \int_{V k_j} |f(g)|^p d g \geq \sum_j \int_{V k_j} [|f(k_j)| - (a/2)]^p d g \geq (a/2)^p \sum_j \mu(V) = \infty. \text{ That is contradiction.}$$

Corollary 2. Any $f \in \mathcal{B}(G) \cap L^p(G)$ ($p < +\infty$) is zero at ∞ .

Here $\mathcal{B}(G)$ is the ring of functions generated by $\{ \langle U_g^{\omega} v, u \rangle \}$ (ω runs unitary representations of G , and v, u run vectors of spaces of representation ω).

Proposition 1. Assume that the above F_0 satisfies the followings.

(i) F_0 is a function algebra, that is, closed under the operations $+$, $\times$ and scalar multiplication.

(ii) F_0 is invariant under right translations, that is, for any f in F_0 and any g_1 , the function $(R_{g_1} f)(g) = f(gg_1)$ of g is in F_0 .

(iii) The uniform closure F of F_0 is closed with respect to complex conjugation.

(iv) There exist an $f_0 (\neq 0)$ in F_0 and $p < +\infty$, such that $f_0 \in L^p(G)$.

Then there exists a natural number n and the set

$$F_1 = \left\{ \sum_j^N \varphi_j \cdot R_{g_j} (f_0)^n ; N=1,2,\dots, g_j \in G, \varphi_j \in F_0 \oplus \mathbb{C}1 \right\}$$

in $F_0 \cap L^1(H \setminus G)$, is dense in $L^q(H \setminus G)$ for $1 \leq q < +\infty$, and is dense in $L_c^\infty(H \setminus G) \equiv \{ \text{continuous function of zero at } \infty \text{ on } H \setminus G \}$ with uniform norm.

Proof. If we put $n = [p] + 1$, $(f_0)^n \in L^1(G) \cap L^\infty(G)$, therefore $F_1 \subset F_0 \cap L^1(H \setminus G) \cap L^\infty(H \setminus G)$. Thus replacing f_0^n to f_0 , we can consider $f_0 \in L^1(H \setminus G)$ and $F_1 \subset L^1(H \setminus G) \cap L^\infty(H \setminus G) \subset L^q(H \setminus G)$ for $1 \leq q < +\infty$. And by Lemma 4, $F_1 \subset L_c^\infty(H \setminus G)$. Moreover we consider

$$F_2 = \left\{ \sum_j^N \varphi_j \cdot R_{g_j} |f_0|^2 ; N=1,2,\dots, g_j \in G, \varphi_j \in F_0 \oplus \mathbb{C}1 \right\}.$$

In general $F_2 \not\subset F_0$, but by the assumption (iii) $F_2 \subset F$, since $\overline{R_{g_j} f_0}$ and therefore $R_{g_j} |f_0|^2 = (\overline{R_{g_j} f_0})(R_{g_j} f_0)$ are in F .

Lemma 5. For $\forall \varphi \in F_2, \forall \varepsilon > 0, 1 \leq p \leq +\infty$, there exists $f \in F_1$ such that $\|\varphi - f\|_p < \varepsilon$.

Proof. Let $\varphi = \sum_j^N (\varphi_j \cdot \overline{R_{g_j} f_0}) R_{g_j} f_0 \in F_2$. Here $\varphi_j \overline{R_{g_j} f_0} \in F$, so there exist $f_j' \in F_0$ such that $\|\varphi_j \overline{R_{g_j} f_0} - f_j'\|_\infty < (\varepsilon/N \|f_0\|_p)$. Thus $\|\varphi - \sum_j f_j' R_{g_j} f_0\|_p < \sum_j \|\varphi_j \overline{R_{g_j} f_0} - f_j'\|_\infty \|R_{g_j} f_0\|_p = \sum_j \|\varphi_j \overline{R_{g_j} f_0} - f_j'\|_\infty \|f_0\|_p < \varepsilon$.

By the reason of Lemma 5, it is enough to show that F_2 is dense in $L^q(H \setminus G)$ and $L_c^\infty(H \setminus G)$.

Lemma 6. F_2 is (i) a subring of F , (ii) closed with respect to complex conjugation, (iii) invariant to right translations, (iv) $F_2 \subset L^1(G) \cap L^\infty(G)$, so its elements are zero at ∞ , (v) separates any two points $\tilde{g}_1 \neq \tilde{g}_2$ in $H \setminus G$.

Proof. F_2 is the ideal of F generated by $\Lambda = \{R_g |f_0|^2; g \in G\}$, thus (i) is evident. The fact that $R_g |f_0|^2$ are real-valued, and the assumption (iii) in Proposition 1, give (ii). (iii) is direct result of right invariant properties of F_0 , F and Λ . $R_g |f_0|^2$ are in $L^1(G)$ and F is in $L^\infty(G)$, hence (iv) is true. At last, if $f_0(g_1) \neq f_0(g_2)$ then (v) is true for such g_1, g_2 . And if $0 \neq f_0(g_1 g_0) = f_0(g_2 g_0)$ for some g_0 in G , by the definition of $H = G_F$, there exists a $\varphi \in F$ such that $\varphi(g_1) \neq \varphi(g_2)$, thus $\varphi \cdot R_{g_0} |f_0|^2$ separates these $\tilde{g}_1, \tilde{g}_2$.

Corollary 3. For $\forall \varphi \in L_c^\infty(H \setminus G), \forall \varepsilon > 0$, there exists $f \in F_2$ such that $\|\varphi - f\|_\infty < \varepsilon$, that is, F_2 is dense in $L_c^\infty(H \setminus G)$.

Proof. Consider the one point compactification space X of $H \setminus G$. We apply the Stone-Weierstrass's theorem to $F_2 \oplus \mathbb{C}1$ on $C(X)$. Thus we get $f_1 = f + a1 \in F_2 \oplus \mathbb{C}1$ and $\|\varphi - f_1\|_\infty < \varepsilon/2$. But φ is zero

at ∞ and $f \in F_2$ is too. Hence $|a| < (\varepsilon/2)$, and $\|\varphi - f\|_\infty \leq$

$$\|\varphi - f_1\|_\infty + (\varepsilon/2) < \varepsilon.$$

Since $C_0(H \setminus G) = \{\text{continuous functions on } H \setminus G \text{ with compact supports}\}$ is dense in $L^p(H \setminus G)$ ($p < \infty$), the following Lemma 7 gives directly a proof of Proposition 1.

Lemma 7. For $\forall \varphi \in C_0(H \setminus G), \forall \varepsilon > 0, \forall p < +\infty$, there exists $f \in F_2$ such that $\|\varphi - f\|_p < \varepsilon$.

Proof. Put $C = [\varphi]$ (support of φ), $a = \mu(C)$ (measure of C) and $M = \|\varphi\|_\infty$. Using the regularity of Haar measure, there exists a relative compact open set U containing C such that $\mu(U) < 2a$. Moreover we can take a $\psi \in C_0(H \setminus G)$ such that $\psi(g) = 1$ for $g \in C$, and $= 0$ for $g \notin U$, $0 \leq \psi(g) \leq 1 \quad \forall g \in G$.

By Corollary 3, take $f_1 \in F_2$ such that

$$\|\varphi - f_1\|_\infty < \rho < \min(1, \varepsilon(2^{p+1}a + 1)^{-1/p}).$$

Put $m = \int_{G-U} |f_1(g)|^p dg$, and $0 < \delta < \min(1, \rho/(M+\rho), \rho m^{-1/p})$.

Again by Corollary 3, take $f_2 \in F_2$ such that $\|\psi - f_2\|_\infty < \delta$ and put $f = f_1 \cdot f_2$. Then $|\varphi(g) - f(g)| = |\varphi(g) - f_1(g)f_2(g)|$ is less than $|\varphi(g) - f_1(g)| + |1 - f_2(g)| |f_1(g)| < \rho + \delta(M + \rho) < 2\rho$ for $g \in C$,

$$|f_1(g)| |f_2(g)| < \rho(|\psi(g)| + \delta) < \rho(1 + \delta) < 2\rho \quad \text{for } g \in U - C,$$

$$|f_1(g)| |f_2(g)| < |f_1(g)| \delta < \rho m^{-1/p} |f_1(g)| \quad \text{for } g \notin U.$$

Thus $\|\varphi - f\|_p^p = \int_G |\varphi(g) - f(g)|^p dg = \int_C + \int_{U-C} + \int_{G-U} <$
 $< 2^p \rho^p \mu(C) + 2^p \rho^p \mu(U-C) + \rho^{p_m-1} \int_{G-U} |f_1(g)|^p dg <$
 $< (2^{p+1}a + 1) \rho^p \leq \varepsilon^p.$

4. Now we return to our problem concerning to the C^* -system $\{\alpha, G, \alpha\}$. We apply Proposition 1 twice, at first to the case

$F_0 = \mathcal{F}_0$ and second to the case $F_0 = \mathcal{G}_0$.

Lemma 8. In both cases, G_{F_0} ($= K$) are same one and compact normal subgroup of G .

Proof. If $F_0 = \mathcal{F}_0$, $K = G_{\mathcal{F}_0} = G_{\mathcal{F}}$, and if $F_0 = \mathcal{G}_0$, $K = G_{\mathcal{G}_0} = G_{\mathcal{G}}$. But by Lemma 1, $\mathcal{G} = \mathcal{F}$, thus $G_{\mathcal{F}} = G_{\mathcal{G}}$.

For $\forall k \in K$, $f_{AB}(kg) = f_{AB}(g)$, that is, for $\forall g \in G$ and $\forall A, B \in \mathcal{A}_0$,

$$\langle U_k U_g \pi(A) \Omega, \pi(B^*) \Omega \rangle = \langle U_g \pi(A) \Omega, \pi(B^*) \Omega \rangle$$

Thus $U_k v = v$, for $\forall v \in \mathcal{H}$. This shows $U_k = I$, therefore K is the kernel of this representation, hence normal. [Assumption 3] and Lemma 3, Corollary 2 assure the compactness of K .

Based on Lemma 8, replacing the factor group $K \backslash G$ to G , hereafter we can assume $K = \{e\}$. Moreover we take $p_0 = [\max(p, q)] + 1$ and replace $f_0^{p_0}, g_0^{p_0}$ to f_0, g_0 in Assumption 3. Thus we can assume that $f_0, g_0 \in L^1(G) \cap L^\infty(G)$.

Lemma 9. $\mathcal{G}_0 = \{ \overline{f_1(g)} ; f_1 \in \mathcal{F}_0 \} = \{ f_1(g^{-1}) ; f_1 \in \mathcal{F}_0 \}$.

Proof. Since $\mathcal{A}_0$ is $*$ -invariant, by (1) we obtain the result.

Proposition 1 leads us to the following lemma.

Lemma 10. The following spaces are dense in $L^p(G)$ ($1 \leq p < \infty$) and in $L_c^\infty(G)$.

$$\mathcal{F}_1 = \left\{ \sum_{j=1}^N f_j(R_{g_j'} f_0) ; N=1,2,\dots, g_j' \in G, f_j \in \mathcal{F}_0 \otimes \mathbb{C}1 \right\},$$

$$\mathcal{G}_1 = \left\{ \sum_{j=1}^N g_j(R_{g_j'} g_0) ; N=1,2,\dots, g_j' \in G, g_j \in \mathcal{G}_0 \otimes \mathbb{C}1 \right\}.$$

Now define a map S from $\mathcal{F}_1$ onto $\mathcal{G}_1$ by

$$S : \mathcal{F}_1 \ni \sum_k \prod_j f_{A_{j,k} B_{j,k}} \mapsto \sum_k \prod_j g_{A_{j,k} B_{j,k}} \in \mathcal{G}_1.$$

Lemma 11. (i) The map S is welldefined. That is, for

$\forall f_1 \in \mathcal{F}_1$, $S f_1$ does not depend on the form $f_1 = \sum \prod f_{A_{j,k} B_{j,k}}$.

(ii) As a map defined on dense space in $L^p(G)$ (resp. $L_c^\infty(G)$), S is closable.

Proof. Summing up the relations (2) in [Assumption 2], we obtain for any f_1, f_2 in $\mathcal{K}_1$,

$$(3) \quad \int_G f_1(g) f_2(g) \, dg = \int_G (Sf_1)(g) (Sf_2)(g) \, dg.$$

If f_2 runs over $\mathcal{K}_1$, Sf_2 runs over $\mathcal{G}_1$. Thus if $f_1 \equiv 0$,

$\int_G (Sf_1)(g) k(g) \, dg = 0$ for $\forall k \in \mathcal{G}_1$. Because $\mathcal{G}_1$ is dense in $L^1(G)$, $Sf_1 \equiv 0$. This shows, S is welldefined.

Next if $f_1 \rightarrow 0$ and $Sf_1 \rightarrow f_3$ in $L^p(G)$ (resp. $L_c^\infty(G)$), since $\mathcal{K}_1 \subset L^q(G)$ ($((1/p)+(1/q)=1)$) (resp. $L^1(G)$), the left hand side of (3) tends to zero, and the right hand side tends to

$\int_G f_3(g) Sf_2(g) \, dg$ for any f_2 in $\mathcal{K}_1$. Again by the denseness of $\mathcal{G}_1 = \{Sf_2 ; f_2 \in \mathcal{K}_1\}$ in $L^q(G)$ (resp. in $L^1(G)$), f_3 must be zero.

Corollary 4. For any $f_1, f_2 \in \mathcal{K}_1$,

$$(4) \quad \langle Sf_1, \overline{Sf_2} \rangle = \langle f_1, \overline{f_2} \rangle.$$

Proof. A direct result of (3).

Let T_2 (resp. T_∞) be the closure of S as an operator on $L^2(G)$ (resp. $L_c^\infty(G)$), and $D_2 \equiv D(T_2)$ (resp. $D_\infty = D(T_\infty)$) be the domains of T_2 (resp. T_∞).

Lemma 12. For $\forall \varphi \in D_2, \forall \psi \in D_\infty, \psi \cdot \varphi \in D_2$ and

$$T_2(\psi \cdot \varphi) = T_\infty(\psi) \cdot T_2(\varphi).$$

Proof. Let $\mathcal{K}_1 \ni f_j \rightarrow \varphi, Sf_j \rightarrow T_2(\varphi)$ in $L^2(G)$, and

$\mathcal{K}_1 \ni k_j \rightarrow \psi, Sk_j \rightarrow T_\infty(\psi)$ in $L_c^\infty(G)$, then $\mathcal{K}_1 \ni (k_j f_j) \rightarrow \psi \cdot \varphi$, $(Sk_j)(Sf_j) \rightarrow T_\infty(\psi) T_2(\varphi)$ in $L^2(G)$. By the definition of S , $(Sk_j)(Sf_j) = S(k_j f_j)$ for $\forall k_j, f_j \in \mathcal{K}_1$. Thus we get the result.

Lemma 13. S commutes with right and left translations R_g, L_g .

(We use the notations, $R_g f(g_1) = f(g_1 g)$ and $L_g f(g_1) = f(g^{-1} g_1)$.)

Proof. It is enough to show that S commutes with R_g, L_g on generators $\{f_{AB}\}$ of $\mathcal{F}_1$. And

$$\begin{aligned} (L_{g_1} R_{g_2} f_{AB})(g) &= \omega(B \alpha_{g_1}^{-1} g g_2(A)) - \omega(A) \omega(B) \\ &= \omega(\alpha_{g_1}(B) \alpha_g(\alpha_{g_2}(A))) - \omega(\alpha_{g_2}(A)) \omega(\alpha_{g_1}(B)) \\ &= f_{\alpha_{g_2}(A), \alpha_{g_1}(B)}(g), \end{aligned}$$

in just same way

$$(L_{g_1} R_{g_2} g_{AB})(g) = g \alpha_{g_2}(A) \alpha_{g_1}(B)(g). \text{ Therefore}$$

$$\begin{aligned} S(L_{g_1} R_{g_2} f_{AB})(g) &= S(f_{\alpha_{g_2}(A) \alpha_{g_1}(B)})(g) = g \alpha_{g_2}(A) \alpha_{g_1}(B)(g) \\ &= (L_{g_1} R_{g_2} g_{AB})(g) = (L_{g_1} R_{g_2} S f_{AB})(g). \end{aligned}$$

Lemma 14. For $\forall \varphi \in D_2$, $\forall \psi \in L^1(G) \cap L_c^\infty(G)$, the function

$\langle R_g \varphi, \psi \rangle$ is in D_∞ and

$$(5) \quad T_\infty(\langle R_g \varphi, \psi \rangle) = \langle R_g T_2 \varphi, \psi \rangle.$$

Proof. For $\forall f \in \mathcal{F}_1$,

$$\begin{aligned} \langle R_g f, \psi \rangle &= \int_G f(g_1 g) \overline{\psi(g_1)} d g_1 = \lim \sum_j^N f(g_j g) \overline{\psi(g_j)} |\Delta_j| \\ &= \lim \sum_j^N (L_{g_j}^{-1} f)(g) \overline{\psi(g_j)} |\Delta_j|. \end{aligned}$$

Because of uniform continuity of f, ψ and integrability in our case,

this integral converges uniformly in $g \in G$. Moreover

$$S(\sum_j (L_{g_j}^{-1} f)(g) \overline{\psi(g_j)} |\Delta_j|) = \sum_j (L_{g_j}^{-1} (Sf))(g) \overline{\psi(g_j)} |\Delta_j|.$$

Thus $\sum_j f(g_j g) \overline{\psi(g_j)} |\Delta_j|$ and $S(\sum_j f(g_j g) \overline{\psi(g_j)} |\Delta_j|)$ converge to $\langle R_g f, \psi \rangle$ and $\langle R_g Sf, \psi \rangle$ in $L_c^\infty(G)$ respectively. This shows the results for such a f .

Next for $\forall \varphi \in D_2$, let $\mathcal{F}_1 \ni f_j \rightarrow \varphi$, $Sf_j \rightarrow T_2 \varphi$ in $L^2(G)$,

then $\langle R_g f_j, \psi \rangle$ and $\langle R_g Sf_j, \psi \rangle$ converge to $\langle R_g \varphi, \psi \rangle$ and

$\langle R_g T_2 \varphi, \psi \rangle$ in $L_c^\infty(G)$ respectively. That is, the proof is obtained.

Corollary 5. For $\forall \varphi \in \mathcal{F}_1, \forall \psi \in L^1(G) \cap L_c^\infty(G)$,

$$\langle R_g^{-1} \psi, \bar{\varphi} \rangle \in D_\infty, \text{ and } T_\infty(\langle R_g^{-1} \psi, \bar{\varphi} \rangle) = \langle R_g \varphi, \bar{\psi} \rangle$$

(Here $\bar{\varphi}, \bar{\psi}$ show the complex conjugations of φ, ψ respectively.)

Proof. Indeed, $\langle R_g^{-1} \psi, \bar{\varphi} \rangle = \overline{\langle R_g \bar{\varphi}, \psi \rangle} = \langle R_g \varphi, \bar{\psi} \rangle$.

From assumptions, $\varphi \in \mathcal{F}_1$ and $\bar{\psi} \in L^1(G) \cap L_c^\infty(G)$, so lemma 14 leads us to the results.

5. Now we have to discuss the Katz-Takesaki operator on G , and the relation to the above operator T_2 . We define a unitary operator on $L^2(G) \otimes L^2(G)$ (called the Katz-Takesaki operator) by

$$(4) \quad W(f_1 \otimes f_2)(g_1, g_2) = f_1(g_1 g_2) f_2(g_2).$$

This operator is closely related with duality theorem as follows.

Proposition 2. The operators $U \equiv R_g$ of the right regular representation of G , satisfy

$$(5) \quad W(U \otimes U) = (I \otimes U)W.$$

And conversely, for any non-zero bounded operator U satisfying (5), there exists unique element g in G such that $R_g = U$.

For the proof of Proposition 2, we refer [].

However for our discussion, we don't need this proposition directly, but the following which is deduced from it.

Proposition 3. For any closed operator T on $L^2(G)$ such that

$$(6) \quad W(T \otimes T) = (I \otimes T)W,$$

there exist an element g_0 in G and an one parameter subgroup $g(t)$ of G with infinitesimal generator iH , such that

$$(7) \quad T = g_0 e^H,$$

(Here we denote the closure of algebraic tensor product of two closed operators A and B on $L^2(G)$ by $A \otimes B$.)

Proof. Put $T^*T = A$, then A is a self-adjoint positive definite operator satisfying

$$(8) \quad W(A \otimes A) = (I \otimes A)W.$$

Consider the projection P onto the space $\mathcal{H} = (A^{-1}(0))^\perp = \overline{\text{Range}(A)}$, then by (8) $P \neq 0$, and

$$(9) \quad W(P \otimes P) = (I \otimes P)W.$$

Proposition 2 assures that P is unitary, therefore $P = I$. That is $\mathcal{H} = L^2(G)$, and we can define the self-adjoint operator $H = (1/2)\log A$ satisfying

$$(10) \quad W(H \otimes I + I \otimes H) = (I \otimes H)W.$$

Direct calculations show that for $\forall t \in \mathbb{R}$, $U(t) = e^{iHt}$ is a bounded operator in Proposition 2. Hence we obtain an one-parameter subgroup $g(t)$ in G and

$$(11) \quad U(t) = R_{g(t)} \quad \text{for } \forall t \in \mathbb{R}.$$

On the other hand, the bounded operator $Te^{-H} = U$ satisfies

(5) too. Again Proposition 2 gives an element g_0 in G such that $R_{g_0} = U$. This completes the proof.

We shall call that these operators given in Proposition 3 admissible. In after propositions, we show that the our operator T_2 is admissible.

At first we must remark the following.

Lemma 15. Using any fixed complete orthonormal system $\{e_\alpha\}$ in $L^2(G)$, the Katz-Takesaki operator is expanded as follows.

$$(12) \quad W(f_1 \otimes f_2)(g_1, g_2) = \sum_{\alpha} \varphi_{\alpha}(g_1) \langle R_{g_2} f_1, \varphi_{\alpha} \rangle f_2(g_2).$$

Proof. By only calculation of the expansion.

Lemma 16. $W(\mathcal{F}_1 \otimes \mathcal{F}_1)$ is in the domain of $I \otimes T_2$ and

$$(13) \quad (I \otimes T_2)W(f_1 \otimes f_2) = W(Sf_1 \otimes Sf_2) \quad \text{for } \forall f_1, f_2 \in \mathcal{F}_1.$$

Proof. By Schmidt's orthogonalization, we can take all φ_{α} 's in (12) from $L^1(G) \cap L_c^{\infty}(G)$. Then by Lemmata 12 and 14, the function $\varphi_{\alpha}(g_1) \langle R_{g_2} f_1, \varphi_{\alpha} \rangle f_2(g_2)$ are in $D(I) \otimes D(T_2) \subset D(I \otimes T_2)$ (The domain of $I \otimes T_2$), and

$$\begin{aligned} (14) \quad (I \otimes T_2)(\varphi_{\alpha}(g_1) \langle R_{g_2} f_1, \varphi_{\alpha} \rangle f_2(g_2)) &= \\ &= \varphi_{\alpha}(g_1) T_{\infty}(\langle R_{g_2} f_1, \varphi_{\alpha} \rangle)(T_2 f_2)(g_2) \\ &= \varphi_{\alpha}(g_1) \langle R_{g_2} T_2 f_1, \varphi_{\alpha} \rangle (T_2 f_2)(g_2). \end{aligned}$$

Moreover, $\sum^N \varphi_{\alpha}(g_1) \langle R_{g_2} f_1, \varphi_{\alpha} \rangle f_2(g_2)$ and $(I \otimes T_2) \sum^N (\varphi_{\alpha}(g_1) \langle R_{g_2} f_1, \varphi_{\alpha} \rangle f_2(g_2)) = \sum^N \varphi_{\alpha}(g_1) \langle R_{g_2} T_2 f_1, \varphi_{\alpha} \rangle (T_2 f_2)(g_2)$ converge to $W(f_1 \otimes f_2)(g_1, g_2)$ and $W(Sf_1 \otimes Sf_2)$ in $L^2(G) \otimes L^2(G)$ respectively. This gives the results.

Lemma 17. $W^{-1}(\mathcal{F}_1 \otimes \mathcal{F}_1)$ is in the domain of $T_2 \otimes T_2$ and

$$(15) \quad (T_2 \otimes T_2)W^{-1}(f_1 \otimes f_2) = W^{-1}(I \otimes S)(f_1 \otimes f_2) \quad \text{for } \forall f_1, f_2 \in \mathcal{F}_1.$$

Proof. W^{-1} is given by $W^{-1}(f_1 \otimes f_2)(g_1, g_2) = f_1(g_1 g_2^{-1}) f_2(g_2)$.

Thus for $\forall f_1, f_2 \in \mathcal{F}_1$, $(S \otimes I)W^{-1}(f_1 \otimes f_2)(g_1, g_2) = (S \otimes I)(R_{g_2}^{-1} f_1(g_1) f_2(g_2)) = (S R_{g_2}^{-1} f_1)(g_1) f_2(g_2) = (R_{g_2}^{-1} S f_1)(g_1) f_2(g_2) =$
 $= (S f_1)(g_1 g_2^{-1}) f_2(g_2) = \sum \varphi_{\alpha}(g_1) \langle R_{g_2}^{-1}(S f_1), \varphi_{\alpha} \rangle f_2(g_2).$

This shows that $W^{-1}(\mathcal{F}_1 \otimes \mathcal{F}_1)$ is in $D(\overline{S \otimes I}) = D(T_2 \otimes I)$.

Next we shall show that $(S \otimes I)W^{-1}(\mathcal{F}_1 \otimes \mathcal{F}_1)$ is in $D(I \otimes T_2)$.

Indeed by Corollary 4 and the fact $Sf_1 \in \mathcal{G}_1 \subset L^1(G) \cap L_c^{\infty}(G)$, if we select the C.O.N.S $\{\varphi_{\alpha}\}$ as $\overline{\varphi_{\alpha}} \in \mathcal{F}_1$, $\langle R_{g_2}^{-1}(Sf_1), \varphi_{\alpha} \rangle \in D_{\infty}$ and hence $\varphi_{\alpha}(g_1) \langle R_{g_2}^{-1}(Sf_1), \varphi_{\alpha} \rangle f_2(g_2) \in D(I \otimes T_2)$.

Using (4), $(I \otimes T_2)(\varphi_\alpha(g_1) \langle R_{g_2}^{-1}(Sf_1), \varphi_\alpha \rangle f_2(g_2)) =$

$$\begin{aligned} & \varphi_\alpha(g_1) \langle R_{g_2} S \overline{\varphi_\alpha}, \overline{Sf_1} \rangle (Sf_2)(g_2) = \varphi_\alpha(g_1) \langle S(R_{g_2} \overline{\varphi_\alpha}), \overline{Sf_1} \rangle (Sf_2)(g_2) = \\ & = \varphi_\alpha(g_1) \langle R_{g_2} \overline{\varphi_\alpha}, \overline{f_1} \rangle (Sf_2)(g_2) = \varphi_\alpha(g_1) \langle R_{g_2}^{-1} f_1, \varphi_\alpha \rangle (Sf_2)(g_2). \end{aligned}$$

Obviously $\sum_{\alpha}^N \varphi_\alpha(g_1) \langle R_{g_2}^{-1}(Sf_1), \varphi_\alpha \rangle f_2(g_2)$ and

$\sum_{\alpha}^N \varphi_\alpha(g_1) \langle R_{g_2}^{-1} f_1, \varphi_\alpha \rangle (Sf_2)(g_2) (= (I \otimes T_2)(\sum_{\alpha}^N \varphi_\alpha(g_1) \langle R_{g_2}^{-1}(Sf_1), \varphi_\alpha \rangle \times f_2(g_2)))$ converge to $(S \otimes I)W^{-1}(f_1 \otimes f_2)$ and $W^{-1}(f_1 \otimes Sf_2)$ in $L^2(G) \otimes L^2(G)$ respectively. That is, $(S \otimes I)W^{-1}(f_1 \otimes f_2) \in D(I \otimes T_2)$ and $(I \otimes T_2)(S \otimes I)W^{-1}(f_1 \otimes f_2) = W^{-1}(f_1 \otimes Sf_2)$. Combining these, we get the wanted results.

Summalizing Lemmata 16 and 17, we conclude,

Proposition 4. The closed operator T_2 is admissible.

Now we are in the step to apply Proposition 3 together with Lemma 13 to our operator T_2 , and get,

Lemma 18. There exist an element g_0 with order 1 or 2 and an one-parameter subgroup $g(t)$ in the centre $Z(G)$ of G such that $T_2 = R_{g_0} e^H$, Here iH is the infinitesimal generator of $R_{g(t)}$.

Proof. The existence of g_0 and $g(t)$ are an direct results of the above arguments, so we must show that g_0 and $g(t)$ are in $Z(G)$ and the order of g_0 is atmost two.

But because (7) gives the polar decomposition of T_2 , and by Lemma 13, T_2 hence R_{g_0} and e^H must commute with R_g ($\forall g \in G$).

The relation (3) and the definitions of $\mathcal{F}_1, \mathcal{G}_1$ and S give

$\langle f_1, Sf_2 \rangle = \langle Sf_1, f_2 \rangle$ for $f_1, f_2 \in \mathcal{F}_1$. This concludes T_2 is symmetric. But since R_{g_0} is unitary and e^H is positive definite without kernel, R_{g_0} must be the form $P-(I-P)$ for some projection P .

Hence $(R_{g_0})^2 = I$, and $g_0^2 = e$.

The assertion of Lemma 18 talks about only operators on $L^2(G)$. However using [Assumption 2], we can extend this to the whole space as follows. That is, consider the operators on $\mathcal{H}$,

$H_0 \equiv (1/i)(d/dt)U_{g(t)} \big|_{t=0}$, $V \equiv e^{H_0}$ and $T^1 \equiv U_{g_0} V$ in which $g_0, g(t)$ are elements of G given in Lemma 18.

Lemma 19. $\langle \pi(B)\Omega, U_g \pi(A^*)\Omega \rangle = \langle U_{g_0} V^{1/2} \pi(A)\Omega, V^{1/2} \pi(B^*)\Omega \rangle$
for $\forall A, B \in \mathcal{A}_0$.

Proof. Let $\varphi(t) \equiv e^{-t^2}$ and for $c \in (0, \infty)$ and $A \in \mathcal{A}$,

$$A_{\varphi, c} \equiv (2c/\sqrt{\pi}) \int_{-\infty}^{\infty} \alpha_{g(t)}(A) \varphi(ct) dt.$$

Then it is easy to see $\pi(A_{\varphi, c})\Omega \in D(T^1)$ and $A_{\varphi, c} \xrightarrow{c \rightarrow \infty} A$ in $\mathcal{A}$, hence $\pi(A_{\varphi, c})\Omega \rightarrow \pi(A)\Omega$ and $\pi(A_{\varphi, c}^*)\Omega \rightarrow \pi(A^*)\Omega$ in $\mathcal{H}$. Denote $\mathcal{A}_1 \equiv \{A_{\varphi, c}; c \in (0, \infty), A \in \mathcal{A}_0\}$. Then direct calculations lead us to

$$R_{g_0} \left(\sum_n \frac{1}{n!} \left(\frac{1}{i} \frac{d}{dt} \right)^n (R_{g(t)} f) \bigg|_{t=0} \right)(g) = \langle U_g T^1 \pi(A)\Omega, \pi(B^*)\Omega \rangle - \langle \pi(A)\Omega, \Omega \rangle \langle \pi(B)\Omega, \Omega \rangle.$$

Since the convergences of $H_0 \pi(A)\Omega = (1/i) \lim_{t \rightarrow 0} t^{-1} (U_{g(t)} \pi(A)\Omega - \pi(A)\Omega)$ and $V \pi(A)\Omega = \sum_n (1/n!) H_0^n \pi(A)\Omega$ are in norm sense, the convergence of the left hand side is uniform in g . Generally $\mathcal{A}_1$ is not contained in $\mathcal{A}_0$, but all elements of $\mathcal{A}_1$ are norm limits of elements of

$\mathcal{A}_0$ and vice versa. Hence $f_{A,B}, g_{A,B}$ ($A, B \in \mathcal{A}_0 \cup \mathcal{A}_1$) are uniform limits of $f_{A_j, B_j}, g_{A_j, B_j}$ ($A_j, B_j \in \mathcal{A}_0$). And by (2)

$$(16) \quad \int (f_{AB}(g) f_1(g)) f_2(g) dg = \int (g_{AB}(g) S f_1(g)) S f_2(g) dg$$

for $\forall A, B \in \mathcal{A}_1 \cup \mathcal{A}_0, \forall f_1, f_2 \in \mathcal{F}_1$.

Now $f_1 \ni f_{A_j B_j} f_1 \rightarrow f_{AB} f_1$, $T_2(f_{A_j B_j} f_1) = (Sf_{A_j B_j})(Sf_1) = g_{A_j B_j}(Sf_1) \rightarrow g_{AB}(Sf_1)$ in $L^2(G)$, therefore $f_{AB} f_1 \in D_2$ and $T_2(f_{AB} f_1) = g_{AB} Sf_1$ for $\forall A, B \in \mathcal{A}_1 \cup \mathcal{A}_0$. And Lemma 18 assures $T_2(f_{AB} f_1) = R_{g_0}(\sum_n (1/n!)) (-i)^n (d/dt)^n (R_{g(t)} f_{AB}) (Sf_1)$ for $\forall A \in \mathcal{A}_1$, thus $g_{AB} = R_{g_0}(\sum_n (1/n!)) (-i)^n (d/dt)^n (R_{g(t)} f_{AB})$ (converges in $L^\infty(G)$). Therefore

$$\langle U_g T^1 \pi(A) \Omega, \pi(B^*) \Omega \rangle = \langle \pi(B) \Omega, U_g \pi(A^*) \Omega \rangle \text{ for } \forall A \in \mathcal{A}_1,$$

$\forall B \in \mathcal{A}$. Especially for $A, B \in \mathcal{A}_0$, $A_{\varphi, c} B_{\varphi, c} \in \mathcal{A}_1$, hence

$$\langle U_{g_0} V^{1/2} \pi(A_{\varphi, c_1}) \Omega, V^{1/2} \pi(B_{\varphi, c_2}^*) \Omega \rangle = \langle \pi(B_{\varphi, c_2}) \Omega, U_g \pi(A_{\varphi, c_1}^*) \Omega \rangle.$$

Put $g = g_0^{-1}$. When c tends to ∞ , $A_{\varphi, c} \rightarrow A$ in $\mathcal{A}$, $A_{\varphi, c}^* \rightarrow A^*$,

$\pi(A_{\varphi, c}) \Omega \rightarrow \pi(A) \Omega$, and $\pi(A_{\varphi, c}^*) \Omega \rightarrow \pi(A^*) \Omega$. Taking the limit,

$$\begin{aligned} \text{we obtain } \lim_{\substack{c_1 \rightarrow \infty \\ c_2 \rightarrow \infty}} \langle V^{1/2} \pi(A_{\varphi, c_1}) \Omega, V^{1/2} \pi(B_{\varphi, c_2}^*) \Omega \rangle &= \\ &= \langle \pi(B) \Omega, U_{g_0}^{-1} \pi(A^*) \Omega \rangle. \text{ Hence} \end{aligned}$$

$$\begin{aligned} \lim_{\substack{c_1 \rightarrow \infty \\ c_2 \rightarrow \infty}} \|V^{1/2} \pi(A_{\varphi, c_1}) \Omega - V^{1/2} \pi(A_{\varphi, c_2}) \Omega\|^2 &= \lim_{i, j=1, 2} \sum (-1)^{i+j} \times \\ &\times \langle V^{1/2} \pi(A_{\varphi, c_i}) \Omega, V^{1/2} \pi(A_{\varphi, c_j}) \Omega \rangle = \\ &= \sum_{i, j=1, 2} (-1)^{i+j} \langle \pi(A) \Omega, U_{g_0}^{-1} \pi(A) \Omega \rangle = 0. \end{aligned}$$

That is, $\{V^{1/2} \pi(A_{\varphi, c}) \Omega\}_{c \rightarrow \infty}$ is a Cauchy sequence in $\mathcal{H}$, so

$\pi(A) \Omega \in D(V^{1/2})$, $V^{1/2} \pi(A) \Omega = \lim_{c \rightarrow \infty} V^{1/2} \pi(A_{\varphi, c}) \Omega$, and

$$\begin{aligned} \langle U_{g_0} V^{1/2} \pi(A) \Omega, V^{1/2} \pi(B^*) \Omega \rangle &= \\ &= \lim_{c \rightarrow \infty} \langle U_{g_0} V^{1/2} \pi(A_{\varphi, c}) \Omega, V^{1/2} \pi(B_{\varphi, c}^*) \Omega \rangle \\ &= \langle \pi(B) \Omega, U_g \pi(A^*) \Omega \rangle = g_{AB}(g). \end{aligned}$$

$\mathcal{A}_0$ is norm dense in $\mathcal{A}$, therefore $\mathcal{A}_1$ is norm dense in $\mathcal{A}$,

and $\{\pi(A) \Omega; A \in \mathcal{A}_1\}$ is dense in $\mathcal{H}$ too. Thus,

Corollary 6. There exists a norm dense subalgebra $\mathcal{A}_1$ in

$\mathcal{A}$, and a closed operator T^1 on $\mathcal{H}$ such that $\{\pi(A) \Omega; A \in \mathcal{A}_1\} \subset D(T^1)$,

and $\langle U_g T^1 \pi(A) \Omega, \pi(B^*) \Omega \rangle = \langle \pi(B) \Omega, U_g \pi(A^*) \Omega \rangle$
for $\forall g \in G, \forall A \in \mathcal{A}_1, \forall B \in \mathcal{A}$.

Lemma 20. In Lemma 18, the element g_0 is equal to e .

Proof. Consider two positive definite functions

$$\begin{aligned}\psi_1(g) &= \langle U_g \pi(B) \Omega, \pi(B) \Omega \rangle, \\ \psi_2(g) &= \langle U_g v^{1/2} \pi(B^*) \Omega, v^{1/2} \pi(B^*) \Omega \rangle.\end{aligned}$$

In Lemma 19, putting $A = B^*$, we obtain

$$(17) \quad \psi_1(g) = \psi_2(gg_0) \quad \text{for } \forall g \in G.$$

But, $\psi_1(g_0^{-1}) = \psi_2(e) = \|\psi_2\|_\infty = \|\psi_1\|_\infty = \psi_1(e)$, therefore
 $\langle U_{g_0} \pi(B) \Omega, \pi(B) \Omega \rangle = \|\pi(B) \Omega\|^2$ and $U_{g_0} \pi(B) \Omega = \pi(B) \Omega$
for $\forall B \in \mathcal{A}_0$. That is $U_{g_0} = I$, hence g_0 is in $K = \{e\}$.

Thus the results of Lemmata 18 - 20 give a proof of our Main theorem.

[Acknowledgement]. Professors H. Araki, D. Kastler, M. Takesaki gave to the author their idea to use general duality theorem for proving the KMS-property of a C^* -system. The author wishes to express his thanks to thire hospitality and kind discussion, especially to Prof. D. Kastler who gave the opportunity of discussion in Ellmau.

Reference

[1] N. Tatsuuma : A duality theorem for locally compact groups, J. Math. Kyoto Univ. , 6 (1967), 187 - 293.