

## Graphical illustration of “double Stokes phenomena”

By Kôichi UCHIYAMA

Sophia University, Tokyo

(上智大理工 内山康一)

### Abstract

We try to illustrate the formation of double Stokes phenomena or new Stokes lines by the hyperasymptotic method.

Stimulated by a series of talks by Dr Boyd and Dr Howls in this symposium, I would like to try to explain an example of double Stokes phenomena ([2]) in terms of hyperasymptotics. This is a revisit of the author's observation ([5], [6]).

Berk, Nevins and Roberts ([3]) gave an example of 3rd order differential equation with positive large parameter  $x$ :

$$(1) \quad \psi'''(q) + 3x^2\psi'(q) - ix^3q\psi(q) = 0,$$

which has so to call new Stokes lines. They analyzed formation of new Stokes lines in general setting. Aoki, Kawai and Takei ([1]) introduced a notion of a new turning point and proposed an Ansatz on the Stokes geometry. The author gave an illustration of the example (1) by saddle point method (the steepest descent method) and the Laplace integral representation of solutions (Balian - Bloch representation [7]). We give again an illustration of the same example but in a different view point.

As we did in [5], we use a basis of solutions by Laplace integral:

$$(2) \quad \Psi^{(i)}(q) = \int_{\Gamma_i(q)} \exp \left[ -x \left( -\frac{iu^4}{4} - \frac{3iu^2}{2} + uq \right) \right] du,$$

where  $\Gamma_i(q)$  is the steepest descent contour passing through a saddle point  $u_i(q)$  specified below.

The saddle points are given by the algebraic equation with parameter  $q$ :

$$(3) \quad u^3 + 3u + iq = 0.$$

This equation has no triple roots but a pair of double roots, if and only if  $q = \pm 2$ . The points  $q = -2$  and  $q = 2$  are simple turning points of the original differential equation (1). We cut the  $q$ -plane along the lines  $(-\infty, -2]$  and  $[2, +\infty)$  on the real axis and enumerate the three roots  $u_i(q)$ 's such that

$$(4) \quad u_1(0) = \sqrt{3}i, \quad u_2(0) = 0, \quad u_3(0) = -\sqrt{3}i.$$

We have

$$(5) \quad u_1(-2) = u_2(-2) = i, \quad u_3(-2) = -2i$$

and

$$(6) \quad u_2(2) = u_3(2) = -i, \quad u_1(2) = 2i.$$

We define the Stokes lines from the turning point  $\pm 2$  by the loci of  $q$  satisfying

$$(7) \quad \Im \int_{-2}^q u_1(q') dq' = \Im \int_{-2}^q u_2(q') dq'$$

and

$$(8) \quad \Im \int_2^q u_2(q') dq' = \Im \int_2^q u_3(q') dq'.$$

Two pairs of Stokes lines emanating from  $\pm 2$  have intersection points  $q_+$  and  $q_-$  on the imaginary axis. They are approximately  $q_{\pm} = \pm 0.3193i$  (Figure 1). When  $q = q_{\pm}$ , the steepest path from  $u_3(q)$  hits  $u_2(q)$  and another steepest path from  $u_2(q)$  hits  $u_1(q)$  (Figure 2). This is an example of a "double Stokes phenomenon" mentioned in [2] p666.

Our aim is illustrating by the resurgent formula in hyperasymptotic analysis that we have a Stokes phenomenon when  $q$  crosses over the so called new Stokes line  $q_- + i(-\infty, 0)$

and no Stokes phenomena when  $q$  crosses the line  $q_- + i(0, 2|q_-|)$  near  $q_-$ .

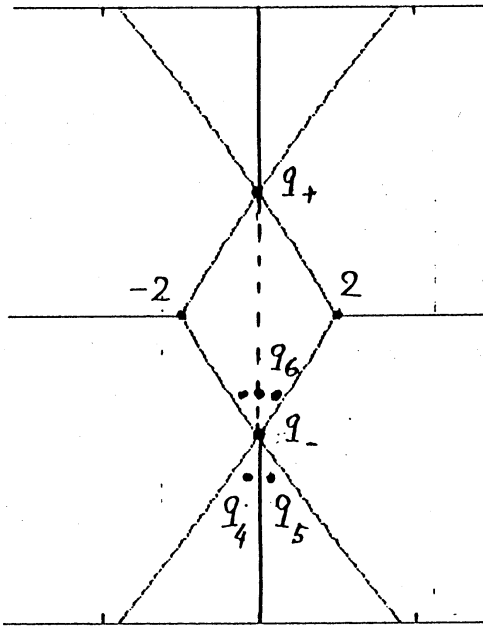


Figure 1: Stokes lines

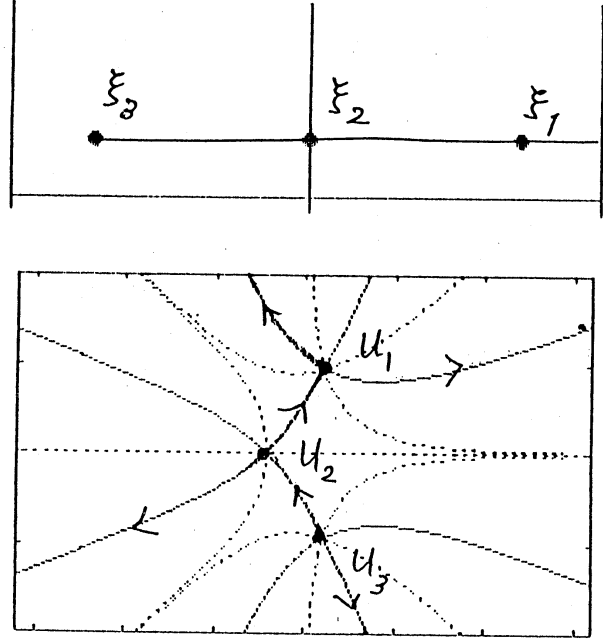


Figure 2: Steepest paths for  $q = q_-$

We put

$$H(u, q) = -\frac{iu^4}{4} - \frac{3iu^2}{2} + uq,$$

and

$$\xi_i(q) = H(u_i(q), q).$$

We define  $T^{(i)}(x, q)$  by

$$(9) \quad \Psi^{(i)}(q) = x^{-1/2} \exp[-x\xi_i(q)] T^{(i)}(x, q)$$

namely,

$$(10) \quad T^{(i)}(x, q) = x^{1/2} \int_{\Gamma_i(q)} du \exp\{-x[H(u, x) - \xi_i(q)]\}.$$

According to Berry and Howls ([2]), in virtue of an ingenious expression

$$(11) \quad T^{(i)}(x, q) = \frac{1}{2\pi i} \int_0^\infty dv \frac{e^{-v}}{\sqrt{v}} \int_{\Gamma_i(q)} du \frac{[H(u, q) - \xi_i(q)]^{1/2}}{H(u, q) - \xi_i - v/x},$$

we have a resurgence relation for any positive integer  $N$

$$(12) \quad T^{(i)}(x, q) = \sum_{r=0}^{N-1} \frac{T_r^{(i)}(q)}{x^r} + \frac{1}{2\pi i} \sum_j \frac{(-1)^{\gamma_{ij}}}{(x\xi_{ij}(q))^N} \int_0^\infty dv \frac{v^{N-1} \exp(-v)}{1 - \frac{v}{x\xi_{ij}(q)}} T^{(j)}\left(\frac{v}{\xi_{ij}(q)}, q\right)$$

where

$$(13) \quad T_r^{(i)}(q) = \frac{(r - \frac{1}{2})!}{2\pi i} \int^{(u_i(q)+)} du \frac{1}{[H(u, q) - \xi_i(q)]^{r+1/2}}.$$

In the formula (13),  $(u_i(q)+)$  denotes a sufficiently small circle with positive direction around  $u_i(q)$  in the complex  $u$ -plane. The summation  $\sum_j$  is crucial in the resurgence relation. The sum is taken for the *adjacent* saddles  $u_j(q)$  to the fixed saddle  $u_i(q)$  ([2], [4]).  $\gamma_{ij}$  is an orientation factor 0 or 1 ([2] p662).  $\xi_{ij}(q)$  is defined by

$$(14) \quad \xi_{ij}(q) = \xi_j(q) - \xi_i(q).$$

For simplicity,  $x$  has been restricted to a real large parameter. For the notion of adjacency, we have to consider in (10) all the steepest paths for the *complex* parameter  $x$  rotating its phase. A saddle  $u_j(q)$  is called adjacent to  $u_i(q)$ , if there is a phase of  $x$  such that the corresponding steepest path from  $u_i(q)$  hits  $u_j(q)$ .

In this example, the lines  $\{q; \Im q = 0\}$  near  $q_-$  satisfies  $\arg \xi_{31}(q) = 0$ . We assume in general the saddle point  $u_j(q)$  is adjacent to the saddle  $u_i(q)$ . By Cauchy's integral formula, when  $q$  crosses over the line  $\{q; \arg \xi_{ij}(q) = 0\}$ , we have a new term in the hyperasymptotic expansion (12), obtained by the residue calculus

$$\pm (-1)^{\gamma_{ij}} \exp(-x\xi_{ij}(q)) T^{(j)}(x, q).$$

The signature  $\pm$  depends on the direction of  $q$  when  $q$  crosses the Stokes line.

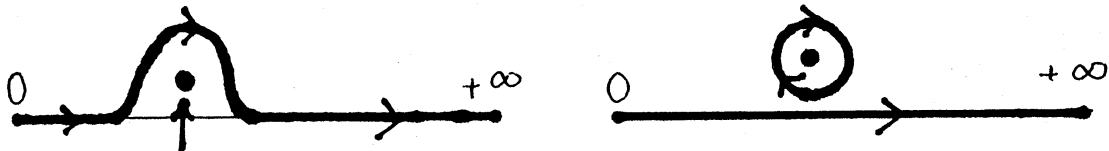


Figure 3: Deformation of integration contour.

Thus, in the hyperasymptotic theory, it is crucial to determine whether a saddle point is adjacent to another or not. In this report, the author did it numerically using computer graphics implemented by Visual Basic.

We follow the notations in [5], where figures of steepest paths through saddle points and location of  $\xi_i$ 's are given. In order to determine the adjacency, we need steepest paths for complex parameter  $x$  with given phase.

We put  $q_- = -3.193i$ ,  $q_4 = q_- + \exp(17\pi i/12)$ ,  $q_5 = q_- + \exp(19\pi i/12)$ ,  $q_6 = q_- + \exp(\pi i/2)$ ,  $q'_6 = q_- + \exp(7\pi i/12)$ ,  $q''_6 = q_- + \exp(5\pi i/12)$ .

By drawing steepest paths for  $\arg x = -\arg \xi_{ij}$ , we determine whether  $u_j(q_k)$  is adjacent to  $u_i(q_k)$ . In the following relations of saddle points, the arrow denotes adjacency.  $a \rightarrow b$  means the saddle  $b$  is adjacent to  $a$ . We omit  $q_k$  of  $u_j(q_k)$ .

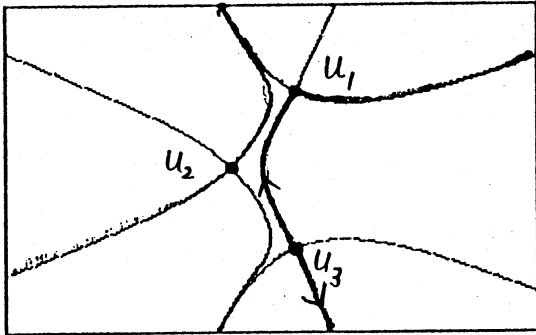
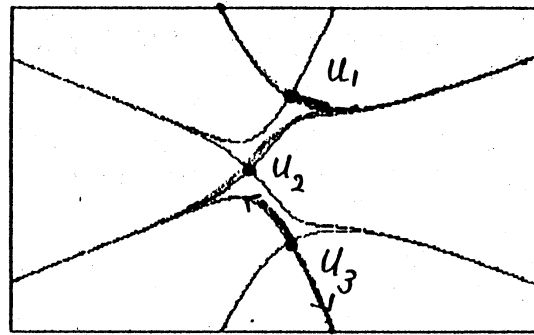
When  $q = q_4$  or  $q_5$ ,

$$u_1 \rightarrow u_2 \text{ and } u_1 \rightarrow u_3; \quad u_2 \rightarrow u_3 \text{ and } u_2 \rightarrow u_1; \quad u_3 \rightarrow u_1 \text{ and } u_3 \rightarrow u_2.$$

When  $q = q_6, q'_6, q''_6$ ,

$$u_1 \rightarrow u_2; \quad u_2 \rightarrow u_1 \text{ and } u_2 \rightarrow u_3; \quad u_3 \rightarrow u_2.$$

We observe from these data that there is a Stokes phenomenon for  $\Psi^{(3)}(q)$  when  $q$  crosses the new Stokes line  $q_- + (-\infty, 0)i$  from  $q_4$  to  $q_5$ . On the other hand, there are no Stokes phenomena when  $q$  crosses the line  $q_- + (0, 2|q_-|)i$  from  $q'_6$  to  $q''_6$ , since  $u_1$  is not adjacent to  $u_3$ . These facts are explained in [5] by the Riemann sheet structure of the functions  $\xi_i(q)$ 's.

Figure 4:  $q = q_4$ ,  $\arg x = -\arg \xi_{31}(q_4)$ Figure 5:  $q = q_6$ ,  $\arg x = -\arg \xi_{31}(q_6)$ 

## References

- [1] T.Aoki, T.Kawai and Y.Takei, New turning points in the exact WKB analysis for higher-order ordinary differential equations, "*Méthodes fésurgentes*" ed.by L.Boutet de Monvel, Hermann, 1994, 69 - 84
- [2] M.Berry and C.Howls, Hyperasymptotics for integrals with saddles, Proc. R. Soc. London A **434**(1991),657 - 675.
- [3] H. L. Berk, W. M. Nevins and K. V. Roberts, New Stokes line in WKB theory, J.Math. Phys. **23**(1982),988 - 1002.
- [4] W. Boyd, Error bounds for the method of steepest descents, Proc. R. Soc. London A **440**,(1993), 493-518.
- [5] K. Uchiyama, On examples of Voros analysis in complex WKB theory, "*Méthodes résurgentes*" ed. by L. Boutet de Monvel, Hermann, 1994, 115-134.
- [6] K. Uchiyama, Graphical illustration of new Stokes line for integrals with saddles. An informal seminar in the workshop "*Exponential Asymptotics*" at Cambridge, 1995, March.
- [7] A. Voros, The return of the quartic oscillator, The complex WKB method, Ann. Inst. Henri Poincaré, **39**(1983), 211-338.