

A continuous limit of the q -Garnier system

By

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Abstract

In a recent work [5], the q -Garnier system was formulated in a framework of an extended affine Weyl group with the aid of cluster mutations. This fact allows us to understand that the q -Garnier system is a q -difference system of multi-directions governed by the theory of the affine root systems. It is known that a direction of the q -Garnier system reduces to the Hamiltonian system proposed in [1, 7, 10]. In this article we give the group of symmetries of the Hamiltonian system from the extended affine Weyl group by taking a continuous limit $q \rightarrow 1$.

§ 1. Introduction

The main object of this article is the higher order Painlevé system proposed in [1, 7, 10]. It is expressed as a Hamiltonian system

$$(1.1) \quad \begin{aligned} t(t-1)\frac{dq_i}{dt} &= \{H, q_i\}, \quad t(t-1)\frac{dp_i}{dt} = \{H, p_i\} \quad (i = 1, \dots, n), \\ H &= \sum_{k=1}^n H_{\text{VI}} \left[-h + \sum_{j=0}^n a_{2j+1} - a_{2k-1}, \sum_{j=0}^{k-1} a_{2j}, \sum_{j=k}^n a_{2j}, a_{2k-1}h; q_k, p_k \right] \\ &\quad + \sum_{1 \leq k < l \leq n} (q_k - 1)(q_l - t) \{ (q_k p_k + a_{2k-1})p_l + (q_l p_l + a_{2l-1})p_k \}, \end{aligned}$$

where

$$\begin{aligned} H_{\text{VI}}[b_0, b_1, b_t, b_\infty; q, p] \\ = q(q-1)(q-t)p^2 - \{b_0(q-1)(q-t) + b_1q(q-t) + (b_t-1)q(q-1)\}p + b_\infty q, \end{aligned}$$

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with the Poisson bracket defined by

$$\{p_i, q_j\} = \delta_{i,j}, \quad \{p_i, p_j\} = \{q_i, q_j\} = 0 \quad (i, j = 1, \dots, n),$$

where the symbol $\delta_{i,j}$ stands for the Kronecker delta, and a relation between parameters

$$\sum_{j=0}^{2n+1} a_j = 1.$$

System (1.1) coincides with the Painlevé VI equation in the case $n = 1$. In this article we always assume that $n \geq 2$.

A group of symmetries of system (1.1) is also given in [1, 7, 10]. Let r_j ($j = 0, \dots, 2n+1$) and r'_0, π_1, π_2, ρ be birational transformations. They act on the independent variable t and the parameters h, a_l ($l = 0, \dots, 2n+1$) as the following table.

	r_j	r'_0	π_1	π_2	ρ
(1.2)	t	t^{-1}	t	t	t
	a_l	$a_l - a_{j,l}a_j$	a_l	a_{l+1}	$a_{l+2} \quad a_{-l}$
	a'_0	a'_0	$-a'_0$	$a'_0 + 1$	$a'_0 \quad -a'_0$

Here a matrix $(a_{j,l})_{j,l=0}^{2n+1}$ is the Cartan matrix of type $A_{2n+1}^{(1)}$, whose definition will be given in Section 2, and we use notations

$$a_{l+2n+2} = a_l, \quad a'_0 = 2h - \sum_{k=0}^n a_{2k+1}.$$

They also act on the dependent variables q_i, p_i ($i = 1, \dots, n$) as

$$\begin{aligned}
 r_{2j}(q_i) &= q_i, \quad r_{2j}(p_i) = p_i + a_{2j} \frac{\{q_j - q_{j+1}, p_i\}}{q_j - q_{j+1}} \quad (j = 0, \dots, n), \\
 r_{2j+1}(q_i) &= q_i + a_{2j+1} \frac{\{p_{j+1}, q_i\}}{p_{j+1}}, \quad r_{2j+1}(p_i) = p_i \quad (j = 0, \dots, n-1), \\
 r_{2n+1}(q_i) &= q_i + a_{2n+1} \frac{\{p_{n+1}, q_i\}}{p_{n+1} + \frac{a_{2n+1}}{t}}, \quad r_{2n+1}(p_i) = p_i + a_{2n+1} \frac{\{p_{n+1}, q_i\}}{p_{n+1}}, \\
 r'_0(q_i) &= \frac{1}{q_i}, \quad r'_0(q_i p_i) = -q_i p_i - a_{2i-1}, \\
 \pi_1(q_i) &= \frac{\sum_{k=1}^{n+1} t p_k}{\sum_{k=1}^i p_k + \sum_{k=i+1}^{n+1} t p_k}, \\
 \pi_1(q_i p_i) &= \frac{(q_i - q_{i+1})(\sum_{k=1}^i p_k + \sum_{k=i+1}^{n+1} t p_k)}{1 - t} - a_{2i} \\
 \pi_2(q_i) &= \frac{q_{i+1}}{q_1}, \quad \pi_2(q_i p_i) = q_{i+1} p_{i+1}, \\
 \rho(q_i) &= \frac{t q_1}{q_{n-i+2}}, \quad \rho(q_i p_i) = -q_{n-i+2} p_{n-i+2} - a_{2n-2i+3},
 \end{aligned}
 \tag{1.3}$$

where

$$q_0 = 1, \quad q_{n+1} = t, \quad p_{n+1} = -\frac{\sum_{k=1}^n q_k p_k + h}{t}.$$

Remark. A group of the transformations $F = \langle r_0, \dots, r_{2n+1}, r'_0, \pi_1, \pi_2, \rho \rangle$ coincides with that generated by all of the transformations given in [1, 7, 10]. More precisely, the transformations r_j ($j = 0, \dots, 2n+1$) and π given in §4.2 of [7] belong to F as the following table.

Trans. in [7]	Element of F
r_j	r_j
π	$\pi_1 r'_0$

Also, the transformations r_j, r'_j ($j = 0, \dots, n$) and $\pi, \rho, \eta_0, \eta_1, \zeta_{01}, \iota, \varphi$ given in §8 of [10] belong to F as the following table.

Trans. in [10]	Element of F
r_j	$r_{2j-1} r_{2j} r_{2j-1}$
r'_j	$r_{2n-2j} r_{2n-2j+1} r_{2n-2j}$
π	π_2
ρ	$r'_0 \pi_2^{-1} \rho$
η_0	$r_1 r_3 \dots r_{2n+1} \pi_1^{-1}$
η_1	$r_1 r_3 \dots r_{2n+1} \pi_1 \pi_2^{-1}$
ζ_{01}	r'_0
ι	$r_1 r_3 \dots r_{2n+1} \pi_2^{-1} \rho$
φ	$r_1 r_3 \dots r_{2n-1} \pi_2^{-1} \rho$

Here we use a notation $r_{-1} = r_{2n+1}$. Inversely, all of the elements of F can be expressed as compositions of the transformations given in [1, 7, 10]. We do not state its detail here.

A q -analogue of system (1.1) was proposed as a similarity reduction of the q -Drinfeld-Sokolov hierarchy of type $A_{2n+1}^{(1)}$ in [8, 9]. Its explicit formula will be given as system (2.1) in Section 2. Afterward, system (2.1) was formulated as an element called a translation of an extended affine Weyl group of type $A_{2n+1}^{(1)} \times A_1^{(1)} \times A_1^{(1)}$. More precisely, the extended affine Weyl group admits a birational representation and contains an abelian normal subgroup generated by translations. Hence we obtain a class of systems of $2n$ -th order q -difference equations from the abelian normal subgroup. This class is called the q -Garnier system and contains system (2.1).

The aim of this article is to derive the group of symmetries of system (1.1) from the extended affine Weyl group by taking a continuous limit $q \rightarrow 1$. However, system (1.1) has not been derived directly from system (2.1) yet. The limiting procedure was given only at the level of the Lax pairs in [8]. Therefore we first give a limiting procedure

which reduces system (2.1) to system (1.1). Then symmetries of system (2.1) reduce to those of (1.1) together. Consequently, we obtain transformations (1.2) and (1.3) in this article. This fact suggests that system (1.1) does not admit any other symmetry.

This article is organized as follows. In Section 2, we recall the definition of the affine Weyl group of type A and the formulation of the q -Garnier system. In Section 3, we give a limiting procedure which reduces system (2.1) to system (1.1). Then transformations (1.2) and (1.3) are also derived from the extended affine Weyl group which provides the q -Garnier system. We show this fact in Section 4.

§ 2. Affine Weyl group and q -Garnier system

We first recall the definition of the affine Weyl group of type A . Let $(a_{i,j})_{i,j=0}^N$ be the Cartan matrix of type $A_N^{(1)}$ defined by

$$a_{0,0} = a_{1,1} = 2, \quad a_{0,1} = a_{1,0} = -2$$

for $N = 1$ and

$$\begin{aligned} a_{i,i} &= 2 \quad (i = 0, \dots, N), \\ a_{i,i+1} &= a_{N,0} = a_{i+1,i} = a_{0,N} = -1 \quad (i = 0, \dots, N-1), \\ a_{i,j} &= 0 \quad (\text{otherwise}) \end{aligned}$$

for $N \geq 2$. Then the affine Weyl group of type $A_N^{(1)}$ is defined by generators r_i ($i = 0, \dots, N$) and the fundamental relations

$$r_i^2 = 1 \quad (N \geq 1), \quad (r_i r_j)^{2-a_{i,j}} = 1 \quad (N \geq 2, i \neq j).$$

We next give a birational representation of the affine Weyl group of type A . Let $\varphi_{j,i}$ ($j = 0, \dots, 2n+1, i = 0, 1$) be variables. We define birational transformations r_j ($j = 0, \dots, 2n+1$), s_i, s'_i ($i = 0, 1$) and π, π', ρ as automorphisms on $\mathbb{C}(\varphi_{j,i})$ by

$$\begin{aligned} r_j(\varphi_{j-1,i}) &= \varphi_{j-1,i} \frac{1 + \varphi_{j,i}}{1 + \frac{1}{\varphi_{j,i+1}}}, \quad r_j(\varphi_{j,i}) = \frac{1}{\varphi_{j,i+1}}, \quad r_j(\varphi_{j+1,i}) = \varphi_{j+1,i} \frac{1 + \varphi_{j,i+1}}{1 + \frac{1}{\varphi_{j,i}}}, \\ r_j(\varphi_{l,i}) &= \varphi_{l,i} \quad (l \neq j, j \pm 1), \\ s_i(\varphi_{j,i}) &= \frac{1}{\varphi_{j+1,i}} \frac{Q_{j,i}}{Q_{j+2,i}}, \quad s_i(\varphi_{j,i+1}) = \varphi_{j,i} \varphi_{j,i+1} \varphi_{j+1,i} \frac{Q_{j+2,i}}{Q_{j,i}}, \\ s'_i(\varphi'_{j,i}) &= \frac{1}{\varphi'_{j+1,i}} \frac{Q'_{j,i}}{Q'_{j+2,i}}, \quad s'_i(\varphi'_{j,i+1}) = \varphi'_{j,i} \varphi'_{j,i+1} \varphi'_{j+1,i} \frac{Q'_{j+2,i}}{Q'_{j,i}}, \\ \pi(\varphi_{j,i}) &= \varphi_{j+1,i+1}, \quad \pi'(\varphi_{j,i}) = \varphi_{j+1,i}, \quad \rho(\varphi_{j,i}) = \varphi'_{j,i}, \end{aligned}$$

where $\varphi_{j+2n+2,i} = \varphi_{j,i+2} = \varphi_{j,i}$, $\varphi'_{j,i} = \varphi_{-j,i-j}$ and

$$Q_{j,i} = \sum_{k=0}^{2n+1} \prod_{l=0}^{k-1} \varphi_{j+l,i}, \quad Q'_{j,i} = \sum_{k=0}^{2n+1} \prod_{l=0}^{k-1} \varphi'_{j+l,i}.$$

A group generated by these transformations can be regarded as an extended affine Weyl group of type $A_{2n+1}^{(1)} \times A_1^{(1)} \times A_1^{(1)}$.

Theorem 2.1 ([3, 5]). *Groups of the transformations $\langle r_0, \dots, r_{2n+1} \rangle$, $\langle s_0, s_1 \rangle$ and $\langle s'_0, s'_1 \rangle$ are isomorphic to the affine Weyl groups of type $A_{2n+1}^{(1)}$, $A_1^{(1)}$ and $A_1^{(1)}$ respectively. Moreover, any two groups are mutually commutative. Besides, the transformations π, π', ρ satisfy relations*

$$\begin{aligned} \pi^{2n+2} &= 1, \quad (\pi')^2 = \pi^2, \quad \rho^2 = 1, \quad \pi\pi' = \pi'\pi, \quad \pi\rho = \rho(\pi')^{-1}, \\ r_j\pi &= \pi r_{j-1}, \quad s_i\pi = \pi s_{i-1}, \quad s'_i\pi = \pi s'_i, \\ r_j\pi' &= \pi' r_{j-1}, \quad s_i\pi' = \pi' s_i, \quad s'_i\pi' = \pi' s'_{i+1}, \quad r_j\rho = \rho r_{-j}, \quad s_i\rho = \rho s'_i, \end{aligned}$$

where $r_{j+2n+2} = r_j$, $s_{i+2} = s_i$ and $s'_{i+2} = s'_i$.

Instead of the proof of this theorem, we introduce actions of the transformations on multiplicative simple roots. Let α_j ($j = 0, \dots, 2n+1$) and β_i, β'_i ($i = 0, 1$) be variables and q a constant defined by

$$\alpha_j = \varphi_{j,0}\varphi_{j,1}, \quad \beta_i = \prod_{l=0}^{2n+1} \varphi_{l,i}, \quad \beta'_i = \prod_{l=0}^{2n+1} \varphi_{l,i+l}, \quad q = \prod_{l=0}^{2n+1} \alpha_l = \beta_0\beta_1 = \beta'_0\beta'_1.$$

Then the transformations act on them as the following table.

	r_l	s_k	s'_k	π	π'	ρ
α_j	$\alpha_j \alpha_l^{-a_{l,j}}$	α_j	α_j	α_{j+1}	α_{j+1}	α_{-j}
β_i	β_i	$\beta_i \beta_k^{-b_{k,i}}$	β_i	β_{i+1}	β_i	β'_i
β'_i	β'_i	β'_i	$\beta'_i (\beta'_k)^{-b_{k,i}}$	β'_i	β'_{i-1}	β_i

Here $(a_{l,j})_{l,j=0}^{2n+1}$ and $(b_{k,i})_{k,i=0}^1$ are the Cartan matrices of type $A_{2n+1}^{(1)}$ and $A_1^{(1)}$ respectively. Also we use the notation $\alpha_{j+2n+2} = \alpha_j$, $\beta_{i+2} = \beta_i$ and $\beta'_{i+2} = \beta'_i$.

As is shown in [5], the group $G = \langle r_0, \dots, r_{2n+1}, s_0, s_1, s'_0, s'_1, \pi, \pi', \rho \rangle$ contains an abelian normal subgroup generated by translations, which provides the q -Garnier system. We now choose a translation $\tau_c = (\pi')^{-1} \pi s'_0 s_0$ as a direction of the q -Garnier system.

Theorem 2.2 ([5]). *The translation τ_c acts on the variables $\varphi_{2i,0}, \varphi_{2i+1,1}$ ($i =$*

$0, \dots, n)$ and α_j ($j = 0, \dots, 2n+1$) as

$$\begin{aligned}\tau_c(\varphi_{2i,0}) &= \varphi_{2i,0} \frac{\varphi_{2i-1,1} Q_{2i,1} Q'_{-2i,1}}{\alpha_{2i-1} \alpha_{2i} \varphi_{2i+1,1} Q_{2i+2,1} Q'_{-2i+2,1}}, \\ \tau_c(\varphi_{2i+1,1}) &= \varphi_{2i+1,1} \frac{\varphi_{2i+2,1} Q_{2i+3,1} Q'_{-2i-1,1} + (1-\beta_1)(1-\beta'_1)}{\varphi_{2i,1} Q_{2i+1,1} Q'_{-2i+1,1} + (1-\beta_1)(1-\beta'_1)}\end{aligned}$$

and $\tau_c(\alpha_j) = \alpha_j$. We also have

$$\tau_c(\beta_0) = \frac{\beta_0}{q}, \quad \tau_c(\beta_1) = q\beta_1, \quad \tau_c(\beta'_0) = \frac{\beta'_0}{q}, \quad \tau_c(\beta'_1) = q\beta'_1.$$

Hence we obtain a system of $2n$ -th order q -difference equations

$$\begin{aligned}(2.1) \quad \overline{\varphi_{2i,0}} &= \varphi_{2i,0} \frac{\varphi_{2i-1,1} Q_{2i,1} Q'_{-2i,1}}{\alpha_{2i-1} \alpha_{2i} \varphi_{2i+1,1} Q_{2i+2,1} Q'_{-2i+2,1}}, \\ \overline{\varphi_{2i+1,1}} &= \varphi_{2i+1,1} \frac{\varphi_{2i+2,1} Q_{2i+3,1} Q'_{-2i-1,1} + (1-\beta_1)(1-\beta'_1)}{\varphi_{2i,1} Q_{2i+1,1} Q'_{-2i+1,1} + (1-\beta_1)(1-\beta'_1)} \quad (i = 0, \dots, n), \\ \overline{\alpha_j} &= \alpha_j \quad (j = 0, \dots, 2n+1), \quad \overline{\beta'_0} = \frac{\beta'_0}{q}, \quad \overline{\left(\frac{\beta'_0}{\beta_0}\right)} = \frac{\beta'_0}{\beta_0}\end{aligned}$$

with an independent variable β'_0 , dependent variables $\varphi_{2i,0}, \varphi_{2i+1,1}$ ($i = 0, \dots, n-1$) and parameters $\frac{\beta'_0}{\beta_0}, \alpha_j$ ($j = 0, \dots, 2n+1$) satisfying $\prod_{j=0}^{2n+1} \alpha_j = q$. Here the notation $\overline{\varphi}$ stands for the q -shift defined by

$$\overline{\varphi(\beta'_0)} = \varphi(q^{-1}\beta'_0).$$

§ 3. Continuous limit

Let t be an independent variable, q_i, p_i ($i = 1, \dots, n$) dependent variables and h, a_j ($j = 0, \dots, 2n+1$) parameters satisfying $\sum_{j=0}^{2n+1} a_j = 1$. Under system (2.1), by introducing a small parameter ε such that $q = e^{-\varepsilon}$, we set

$$(3.1) \quad \begin{aligned}\varphi_{2i,0} &= -\frac{q_{i+1}}{q_i}, \quad \varphi_{2i+1,1} = -e^{\varepsilon q_{i+1} p_{i+1}} \quad (i = 0, \dots, n), \\ \alpha_j &= e^{-\varepsilon a_j} \quad (j = 0, \dots, 2n+1),\end{aligned}$$

where

$$q_0 = 1, \quad q_{n+1} = t, \quad p_{n+1} = -\frac{\sum_{k=1}^n q_k p_k + h}{t}.$$

Note that

$$\begin{aligned}\varphi_{2i,1} &= -\frac{q_i}{q_{i+1}} e^{-\varepsilon a_{2i}}, \quad \varphi_{2i+1,0} = -e^{\varepsilon(-q_{i+1} p_{i+1} - a_{2i+1})} \quad (i = 0, \dots, n), \\ \beta_0 &= t e^{\varepsilon(h - \sum_{k=0}^n a_{2k+1})}, \quad \beta_1 = \frac{1}{t} e^{\varepsilon(-h - \sum_{k=0}^n a_{2k})}, \quad \beta'_0 = t e^{-\varepsilon h}, \quad \beta'_1 = \frac{1}{t} e^{\varepsilon(h-1)}.\end{aligned}$$

Theorem 3.1. *System (2.1) reduces to system (1.1) in a continuous limit $\varepsilon \rightarrow 0$. Then the derivation is given by*

$$t \frac{df}{dt} = \lim_{\varepsilon \rightarrow 0} \frac{\bar{f} - f}{\varepsilon}.$$

We will prove this theorem in Section 3.1. In preparation for the proof, we introduce the following lemma.

Lemma 3.2. *Relation (3.1) implies*

$$(3.2) \quad Q_{2i,1} = \frac{t-1}{t} + A_i \varepsilon + O(\varepsilon^2), \quad Q_{2i+1,1} = B_i \varepsilon + O(\varepsilon^2) \quad (i = 0, \dots, n),$$

where

$$A_i = \frac{q_i(\sum_{k=1}^i p_k + \sum_{k=i+1}^{n+1} t p_k) + h + \sum_{k=0}^n a_{2k}}{t},$$

$$B_i = -\frac{q_{i+1}(\sum_{k=1}^i p_k + \sum_{k=i+1}^{n+1} t p_k)}{t}$$

and

$$(3.3) \quad Q'_{-2i,1} = \frac{t-1}{t} + A'_i \varepsilon + O(\varepsilon^2), \quad Q'_{-2i+1,1} = B'_i \varepsilon + O(\varepsilon^2) \quad (i = 0, \dots, n),$$

where

$$A'_i = -\frac{\sum_{k=1}^i t q_k (q_k p_k + a_{2k-1}) + \sum_{k=i+1}^{n+1} q_k (q_k p_k + a_{2k-1}) + (h-1)q_{i+1}}{t q_{i+1}},$$

$$B'_i = \frac{\sum_{k=1}^i t q_k (q_k p_k + a_{2k-1}) + \sum_{k=i+1}^{n+1} q_k (q_k p_k + a_{2k-1})}{t q_i}.$$

We will prove this lemma in Section 3.2.

Remark. It is shown in [4, 8] that system (2.1) admits a particular solution in terms of the basic hypergeometric series ${}_{n+1}\phi_n$ under a specialization

$$\varphi_{2i+1,1} = -1 \quad (i = 0, \dots, n).$$

Then $\varphi_{2i,0}$ is a ratio of two basic hypergeometric series. On the other hand, it is shown in [6] that system (1.1) admits a particular solution in terms of the generalized hypergeometric series ${}_{n+1}F_n$ under a specialization

$$p_i = 0 \quad (i = 0, \dots, n), \quad h = 0.$$

Then q_i is a ratio of two generalized hypergeometric series. Relation (3.1) is suggested by this fact.

§ 3.1. Proof of Theorem 3.1

We first show the derivation of q_i ($i = 1, \dots, n$). For any $i = 1, \dots, n-1$, we obtain

$$\frac{\varphi_{2i-1,1}}{\alpha_{2i-1}\alpha_{2i}\varphi_{2i+1,1}} = 1 + (q_i p_i - q_{i+1} p_{i+1} + a_{2i-1} + a_{2i})\varepsilon + O(\varepsilon^2)$$

from relation (3.1) and

$$\frac{Q_{2i,1}Q'_{-2i,1}}{Q_{2i+2,1}Q'_{-2i+2,1}} = 1 + \frac{t}{t-1}(A_i - A_{i+1} - A'_{i-1} + A'_i)\varepsilon + O(\varepsilon^2)$$

from Lemma 3.2. They imply

$$(3.4) \quad \overline{\varphi_{2i,0}} - \varphi_{2i,0} = \varphi_{2i,0} \left(\frac{\varphi_{2i-1,1}}{\alpha_{2i-1}\alpha_{2i}\varphi_{2i+1,1}} \frac{Q_{2i,1}Q'_{-2i,1}}{Q_{2i+2,1}Q'_{-2i+2,1}} - 1 \right) = -\frac{q_{i+1}}{q_i} X_i \varepsilon + O(\varepsilon^2),$$

where

$$X_i = q_i p_i - q_{i+1} p_{i+1} + a_{2i-1} + a_{2i} + \frac{t}{t-1}(A_i - A_{i+1} - A'_{i-1} + A'_i).$$

Similarly we obtain

$$(3.5) \quad \overline{\varphi_{0,0}} - \varphi_{0,0} = -q_1 X_0 \varepsilon + O(\varepsilon^2),$$

where

$$X_0 = q_{n+1} p_{n+1} - q_1 p_1 + a_{2n+1} + a_0 + \frac{t}{t-1}(A_0 - A_1 - A'_n + A'_0).$$

On the other hand, we obtain

$$(3.6) \quad \overline{\varphi_{2i,0}} - \varphi_{2i,0} = -t \frac{d \frac{q_{i+1}}{q_i}}{dt} \varepsilon + O(\varepsilon^2) \quad (i = 0, \dots, n-1)$$

from relation (3.1). Relations (3.4), (3.5) and (3.6) imply

$$t \frac{d \frac{q_{i+1}}{q_i}}{dt} = \frac{q_{i+1}}{q_i} X_i \quad (i = 0, \dots, n-1),$$

from which we obtain

$$(3.7) \quad t \frac{dq_i}{dt} = q_i \sum_{k=0}^{i-1} X_k \quad (i = 1, \dots, n).$$

Hence we only have to verify that system (3.7) is equivalent to the first equation of system (1.1). It can be shown by a direct calculation. We do not state its detail here.

We next show the derivation of $q_i p_i$ ($i = 1, \dots, n$), namely

$$\begin{aligned}
 & t(1-t) \frac{dq_i p_i}{dt} \\
 &= (q_i^2 - t) q_i p_i^2 + (h + a_{2i-1}) q_i^2 p_i + \left(-h - a_{2i-1} + \sum_{k=0}^n a_{2k+1} \right) t p_i + a_{2i-1} h q_i \\
 (3.8) \quad &+ \sum_{k=1}^{i-1} (q_k - 1) \{ t(q_k p_k + a_{2k-1}) p_i + (q_i p_i + a_{2i-1}) q_i p_k \} \\
 &+ \sum_{k=i+1}^n (q_k - t) \{ (q_k p_k + a_{2k-1}) p_i + (q_i p_i + a_{2i-1}) q_i p_k \}.
 \end{aligned}$$

We obtain

$$\begin{aligned}
 \varphi_{2i,1} Q_{2i+1,1} Q'_{-2i+1,1} &= -\frac{q_i}{q_{i+1}} B_i B'_i \varepsilon^2 + O(\varepsilon^3) \quad (i = 0, \dots, n), \\
 (1 - \beta_1)(1 - \beta'_1) &= \frac{(t-1)^2}{t^2} + O(\varepsilon)
 \end{aligned}$$

from relation (3.1) and Lemma 3.2. They imply

$$\begin{aligned}
 (3.9) \quad \overline{\varphi_{2i+1,1}} - \varphi_{2i+1,1} &= \varphi_{2i+1,1} \left(\frac{\varphi_{2i+2,1} Q_{2i+3,1} Q'_{-2i-1,1} + (1 - \beta_1)(1 - \beta'_1)}{\varphi_{2i,1} Q_{2i+1,1} Q'_{-2i+1,1} + (1 - \beta_1)(1 - \beta'_1)} - 1 \right) \\
 &= -Y_i \varepsilon^2 + O(\varepsilon^3) \quad (i = 0, \dots, n-1),
 \end{aligned}$$

where

$$Y_i = \frac{t^2}{(t-1)^2} \left(\frac{q_i}{q_{i+1}} B_i B'_i - \frac{q_{i+1}}{q_{i+2}} B_{i+1} B'_{i+1} \right).$$

On the other hand, we have

$$(3.10) \quad \overline{\varphi_{2i+1,1}} - \varphi_{2i+1,1} = -t \frac{dq_{i+1} p_{i+1}}{dt} \varepsilon^2 + O(\varepsilon^3) \quad (i = 0, \dots, n-1).$$

Relations (3.9) and (3.10) imply

$$(3.11) \quad t \frac{dq_i p_i}{dt} = Y_{i-1} \quad (i = 1, \dots, n).$$

Hence we only have to verify that system (3.11) is equivalent to system (3.8). It can be shown by a direct calculation. We do not state its detail here.

§ 3.2. Proof of Lemma 3.2

In this section we only show relation (3.2). Relation (3.3) can be shown in a similar manner.

We first consider $Q_{2i+1,1}$ ($i = 0, \dots, n$). Their definitions are rewritten as

$$\begin{aligned} Q_{2i+1,1} &= \sum_{k=i}^n \left(\prod_{l=i}^{k-1} \varphi_{2l+1,1} \varphi_{2l+2,1} \right) (1 + \varphi_{2k+1,1}) \\ &\quad + \sum_{k=0}^{i-1} \left(\prod_{l=i}^{n-1} \varphi_{2l+1,1} \varphi_{2l+2,1} \right) \varphi_{2n+1,1} \varphi_{0,1} \left(\prod_{l=0}^{k-1} \varphi_{2l+1,1} \varphi_{2l+2,1} \right) (1 + \varphi_{2k+1,1}). \end{aligned}$$

Then relation (3.1) implies

$$\begin{aligned} \sum_{k=i}^n \left(\prod_{l=i}^{k-1} \varphi_{2l+1,1} \varphi_{2l+2,1} \right) (1 + \varphi_{2k+1,1}) &= \sum_{k=i}^n \left(\frac{q_{i+1}}{q_{k+1}} + O(\varepsilon) \right) (-q_{k+1} p_{k+1} \varepsilon + O(\varepsilon^2)) \\ &= -q_{i+1} \sum_{k=i}^n p_{k+1} \varepsilon + O(\varepsilon^2) \end{aligned}$$

and

$$\begin{aligned} &\sum_{k=0}^{i-1} \left(\prod_{l=i}^{n-1} \varphi_{2l+1,1} \varphi_{2l+2,1} \right) \varphi_{2n+1,1} \varphi_{0,1} \left(\prod_{l=0}^{k-1} \varphi_{2l+1,1} \varphi_{2l+2,1} \right) (1 + \varphi_{2k+1,1}) \\ &= \sum_{k=0}^{i-1} \left(\frac{q_{i+1}}{t} + O(\varepsilon) \right) \left(\frac{1}{q_1} + O(\varepsilon) \right) \left(\frac{q_1}{q_{k+1}} + O(\varepsilon) \right) (-q_{k+1} p_{k+1} \varepsilon + O(\varepsilon^2)) \\ &= -\frac{q_{i+1}}{t} \sum_{k=0}^{i-1} p_{k+1} \varepsilon + O(\varepsilon^2), \end{aligned}$$

from which we obtain

$$Q_{2i+1,1} = B_i \varepsilon + O(\varepsilon^2) \quad (i = 0, \dots, n).$$

We next consider $Q_{2i,1}$ ($i = 0, \dots, n$). Their definitions imply

$$\begin{aligned} \varphi_{2i,1} Q_{2i+1,1} &= \sum_{k=0}^{2n+1} \prod_{l=0}^k \varphi_{2i+l,1} \\ &= \sum_{k=1}^{2n+1} \prod_{l=0}^{k-1} \varphi_{2i+l,1} + \prod_{l=0}^{2n+1} \varphi_{2i+l,1} \\ &= Q_{2i,1} - 1 + \beta_1 \quad (i = 0, \dots, n), \end{aligned}$$

from which we obtain

$$\begin{aligned} Q_{2i,1} &= \varphi_{2i,1} Q_{2i+1,1} + 1 - \beta_1 \\ &= \frac{t-1}{t} + \left(-\frac{q_i}{q_{i+1}} B_i + \frac{h + \sum_{k=0}^n a_{2k}}{t} \right) \varepsilon + O(\varepsilon^2) \\ &= \frac{t-1}{t} + A_i \varepsilon + O(\varepsilon^2) \quad (i = 0, \dots, n). \end{aligned}$$

§ 4. Affine Weyl group symmetry of system (1.1)

In this section we derive a group of symmetries of system (1.1) from the group G . For this purpose, we first determine the group of symmetries of system (2.1).

Proposition 4.1. *Elements $\omega \in G$ satisfying a relation $\omega\tau_c = \tau_c\omega$ or $\omega\tau_c = \tau_c^{-1}\omega$ generate a group*

$$F = \langle r_0, \dots, r_{2n+1}, r'_0, \pi_1, \pi_2, \rho \rangle,$$

where $r'_0 = (\pi')^{-1}\pi$, $\pi_1 = \pi s_0$ and $\pi_2 = \pi^2$.

We will prove this theorem in Section 4.1.

The actions of all elements of F are consistent with the limiting procedure given in Section 3. Hence the group F provides symmetries of system (1.1).

Theorem 4.2. *The group F reduces to that generated by transformations (1.2) and (1.3) in the continuous limit $\varepsilon \rightarrow 0$.*

We will prove this theorem in Section 4.2.

The fundamental relations of the group F follows from those of G given in Theorem 2.1 before the limiting procedure. Addition to those, a relation $\rho\pi_1\rho\pi_1 = 1$ is satisfied after the limiting procedure, since $\tau_c = \rho\pi_1\rho\pi_1$ turns to be the identity transformation. We now set

$$r'_1 = r'_0\pi_1^2\pi_2^{-1}.$$

Then the group F can be regarded as an extended affine Weyl group of type $A_{2n+1}^{(1)} \times A_1^{(1)}$.

Proposition 4.3. *After the limiting procedure, the group $\langle r_0, \dots, r_{2n+1} \rangle$ and $\langle r'_0, r'_1 \rangle$ is isomorphic to the affine Weyl group of type $A_{2n+1}^{(1)}$ and $A_1^{(1)}$ respectively. Moreover, those groups are mutually commutative. Besides, the transformations π_1, π_2, ρ satisfy relations*

$$\begin{aligned} \pi_2^{n+1} &= 1, \quad \rho^2 = 1, \quad \pi_1\pi_2 = \pi_2\pi_1, \quad \pi_1\rho = \rho\pi_1^{-1}, \quad \pi_2\rho = \rho\pi_2^{-1}, \\ r_j\pi_1 &= \pi_1r_{j-1}, \quad r_j\pi_2 = \pi_2r_{j-2}, \quad r_j\rho = \rho r_{-j} \quad (j = 0, \dots, 2n+1), \\ r'_0\pi_1 &= \pi_1r'_1, \quad r'_0\pi_2 = \pi_2r'_0, \quad r'_0\rho = \rho r'_0, \\ r'_1\pi_1 &= \pi_1r'_1r'_0r'_1, \quad r'_1\pi_2 = \pi_2r'_1, \quad r'_1\rho = \rho r'_0r'_1r'_0, \end{aligned}$$

where $r_{j+2n+2} = r_j$.

This proposition can be shown by direct calculations. We do not state its detail here.

As was seen in Section 1, parameters corresponding to simple roots of type $A_1^{(1)}$ are given by

$$a'_0 = 2h - \sum_{k=0}^n \alpha_{2k+1}, \quad a'_1 = 1 - a'_0 = -2h + \sum_{k=0}^n \alpha_{2k} + 2 \sum_{k=0}^n \alpha_{2k+1}.$$

The transformations r'_0, r'_1 act on them as

$$r'_0(a'_0) = -a'_0, \quad r'_0(a'_1) = a'_1 + 2a'_0, \quad r'_1(a'_0) = a'_0 + 2a'_1, \quad r'_1(a'_1) = -a'_1.$$

Remark. In [2], we give an action of the group

$$\tilde{F} = \langle r_0 r_1 r_0, r_1 r_2 r_1, \dots, r_{2n} r_{2n+1} r_{2n}, r_{2n+1} r_0 r_{2n+1}, \pi_1 r_1 r_3 \dots r_{2n+1}, \pi_2, \rho r'_0 \rangle$$

on the basic hypergeometric series ${}_{n+1}\phi_n$. Since the group $\tilde{F}$ is a subgroup of F , its action reduces to an action on the generalized hypergeometric series ${}_{n+1}F_n$ in the continuous limit $\varepsilon \rightarrow 0$. We do not state its detail here.

§ 4.1. Proof of Proposition 4.1

By using the fundamental relations of G given in Theorem 2.1, we can decompose any element $\omega \in G$ as

$$(4.1) \quad \omega = \omega_r ((\pi')^{-1} \pi)^{k_1} \rho^{k_2} (\pi^2)^{k_3} \pi^{l_1} (s'_0 s'_1)^{k_4} (s'_0)^{l_2} (s_0 s_1)^{k_5} s_0^{l_3},$$

where $\omega_r \in \langle r_0, \dots, r_{2n+1} \rangle$, $k_1, k_2 = 0, 1$, $k_3 = 0, \dots, n$, $k_4, k_5 \in \mathbb{Z}$ and $l_1, l_2, l_3 = 0, 1$. In the right-hand side of (4.1), some factors are elements of F as

$$(\pi')^{-1} \pi = r'_0, \quad \pi^2 = \pi_2, \quad s'_0 s'_1 = \rho \pi_1^{-2} \pi_2 \rho, \quad s_0 s_1 = \pi_1^{-2} \pi_2.$$

Hence we can rewrite relation (4.1) to

$$(4.2) \quad \omega = \nu (\pi s_0)^{l_1} (s'_0 s_0)^{l_2} s_0^{l_4},$$

where $\nu \in F$ and $l_4 = 0, 1$. Note that

$$\nu = \omega_r (r'_0)^{k_1} \rho^{k_2} \pi_2^{k_3} (\rho \pi_1^{-2} \pi_2 \rho)^{k_4} (\pi_1^{-2} \pi_2)^{(-1)^{l_1} k_5}$$

and

$$l_4 \equiv l_1 + l_2 + l_3 \pmod{2}.$$

In the right-hand side of (4.2), some factors are elements of F as

$$\pi s_0 = \pi_1, \quad s'_0 s_0 = \rho \pi_1^{-1} \rho r'_0 \pi_2^{-1} \pi_1.$$

Consequently we obtain

$$\omega = \nu' s_0^{l_4},$$

where $\nu' \in F$ and $l_4 = 0, 1$. Note that

$$\nu' = \nu \pi_1^{l_1} (\rho \pi_1^{-1} \rho r_0' \pi_2^{-1} \pi_1)^{l_2}.$$

We first consider the case $l_4 = 0$. Namely, we show that any element $\nu' \in F$ satisfies a relation $\nu' \tau_c = \tau_c \nu'$ or $\nu' \tau_c = \tau_c^{-1} \nu'$. The generator $r_0' \in F$ satisfies the non-commutative relation as

$$r_0' \tau_c = s_0' s_0 = \tau_c^{-1} r_0'.$$

In a similar manner, it can be shown that the other generators satisfy the commutative relations

$$r_j \tau_c = \tau_c r_j \quad (j = 0, \dots, 2n+1), \quad \pi_1 \tau_c = \tau_c \pi_1, \quad \pi_2 \tau_c = \tau_c \pi_2, \quad \rho \tau_c = \tau_c \rho.$$

We do not state its detail here.

We next consider the case $l_4 = 1$, or equivalently $\omega \notin F$. A compositions of the transformations $s_0 \tau_c$ acts on the variables β_0, β_0' as

$$s_0 \tau_c(\beta_0) = \frac{1}{q \beta_0}, \quad s_0 \tau_c(\beta_0') = \frac{\beta_0'}{q}$$

On the other hand, compositions $\tau_c s_0$ and $\tau_c^{-1} s_0$ act on β_0, β_0' as

$$\tau_c s_0(\beta_0) = \frac{q}{\beta_0}, \quad \tau_c s_0(\beta_0') = \frac{\beta_0'}{q}, \quad \tau_c^{-1} s_0(\beta_0) = \frac{1}{q \beta_0}, \quad \tau_c^{-1} s_0(\beta_0') = q \beta_0'.$$

It follows that $s_0 \tau_c \neq \tau_c s_0$ and $s_0 \tau_c \neq \tau_c^{-1} s_0$. Hence, if $\omega \notin F$, then ω does not satisfy both relations $\nu' \tau_c = \tau_c \nu'$ and $\nu' \tau_c = \tau_c^{-1} \nu'$.

§ 4.2. Proof of Theorem 4.2

Relation (3.1) implies

$$\begin{aligned} \varphi_{2i,0} &= -\frac{q_{i+1}}{q_i}, \\ \varphi_{2i+1,1} &= -1 + O(\varepsilon), \quad \frac{1 + \varphi_{2i+1,1}}{\varepsilon} = -q_{i+1} p_{i+1} + O(\varepsilon) \quad (i = 0, \dots, n), \\ \alpha_j &= 1 + O(\varepsilon), \quad \frac{1 - \alpha_j}{\varepsilon} = a_j + O(\varepsilon) \quad (j = 0, \dots, 2n+1). \end{aligned}$$

Recall that $q_0 = 1$, $q_{n+1} = t$ and $q_{n+1} p_{n+1} = -\sum_{k=1}^n q_k p_k - h$. Hence, if we rewrite the actions of the generators of F in $\mathbb{C}(\varphi_{2i,0}, \frac{1 + \varphi_{2i+1,1}}{\varepsilon}, \frac{1 - \alpha_j}{\varepsilon})$ before the limiting procedure,

the generators reduce to those acting on q_i, p_i, h, t, a_j in the continuous limit $\varepsilon \rightarrow 0$. Note that we can identify factors $\varphi_{2i+1,1}$ and α_j with integers -1 and 1 respectively in this procedure.

We first prove for the transformations r_{2j} ($j = 0, \dots, n$). Their actions in $\mathbb{C}(\varphi_{j,i})$ are rewritten as

$$\begin{aligned} r_{2j}(\varphi_{2j,0}) &= \frac{\varphi_{2j,0}}{\alpha_{2j}}, \quad r_{2j}(\varphi_{2i,0}) = \varphi_{2i,0} \quad (i \neq j), \\ r_{2j}\left(\frac{1 + \varphi_{2j-1,1}}{\varepsilon}\right) &= \frac{1 + \varphi_{2j-1,1}}{\varepsilon} - \varphi_{2j-1,1} \frac{\frac{1-\alpha_{2j}}{\varepsilon}}{1 + \varphi_{2j,0}}, \\ r_{2j}\left(\frac{1 + \varphi_{2j+1,1}}{\varepsilon}\right) &= \frac{1 + \varphi_{2j+1,1}}{\varepsilon} - \frac{\varphi_{2j+1,1}}{\alpha_{2j}} \frac{\frac{1-\alpha_{2j}}{\varepsilon} \varphi_{2j,0}}{1 + \frac{\varphi_{2j,0}}{\alpha_{2j}}}, \\ r_{2j}\left(\frac{1 + \varphi_{2i+1,1}}{\varepsilon}\right) &= \frac{1 + \varphi_{2i+1,1}}{\varepsilon} \quad (i \neq j-1, j) \end{aligned}$$

and

$$\begin{aligned} r_{2j}\left(\frac{1 - \alpha_{2j}}{\varepsilon}\right) &= -\frac{1}{\alpha_{2j}} \frac{1 - \alpha_{2j}}{\varepsilon}, \quad r_{2j}\left(\frac{1 - \alpha_{2j\pm 1}}{\varepsilon}\right) = \frac{1 - \alpha_{2j\pm 1}}{\varepsilon} + \alpha_{2j\pm 1} \frac{1 - \alpha_{2j}}{\varepsilon}, \\ r_{2j}\left(\frac{1 - \alpha_i}{\varepsilon}\right) &= \frac{1 - \alpha_i}{\varepsilon} \quad (i \neq j, j \pm 1). \end{aligned}$$

Hence we obtain

$$\begin{aligned} r_{2j}\left(-\frac{q_{i+1}}{q_i}\right) &= -\frac{q_{i+1}}{q_i} \quad (i = 0, \dots, n), \\ r_{2j}(-q_j p_j) &= -q_j p_j + \frac{a_{2j} q_j}{q_j - q_{j+1}}, \quad r_{2j}(-q_{j+1} p_{j+1}) = -q_{j+1} p_{j+1} - \frac{a_{2j} q_{j+1}}{q_j - q_{j+1}}, \\ r_{2j}(-q_{i+1} p_{i+1}) &= -q_{i+1} p_{i+1} \quad (i = 0, \dots, n, \ i \neq j-1, j), \end{aligned}$$

where $q_0 p_0 = q_{n+1} p_{n+1}$, and

$$\begin{aligned} r_{2j}(a_{2j}) &= -a_{2j}, \quad r_{2j}(a_{2j\pm 1}) = a_{2j\pm 1} + a_{2j}, \\ r_{2j}(a_i) &= a_i \quad (i = 0, \dots, 2n+1, \ i \neq j, j \pm 1), \end{aligned}$$

They are equivalent to the actions given in (1.2) and (1.3). Similarly we can prove for

the transformations r_{2j+1} ($j = 0, \dots, n$) by using

$$\begin{aligned} r_{2j+1}(\varphi_{2j,0}) &= \varphi_{2j,0} \frac{\frac{1+\varphi_{2j+1,1}}{\varepsilon} - \frac{1-\alpha_{2j+1}}{\varepsilon}}{\frac{1+\varphi_{2j+1,1}}{\varepsilon}}, \\ r_{2j+1}(\varphi_{2j+2,0}) &= \alpha_{2j+1} \varphi_{2j+2,0} \frac{\frac{1+\varphi_{2j+1,1}}{\varepsilon}}{\frac{1+\varphi_{2j+1,1}}{\varepsilon} - \frac{1-\alpha_{2j+1}}{\varepsilon}}, \\ r_{2j+1}(\varphi_{2i,0}) &= \varphi_{2i,0} \quad (i \neq j, j+1), \\ r_{2j+1}\left(\frac{1+\varphi_{2j+1,1}}{\varepsilon}\right) &= \frac{\frac{1+\varphi_{2j+1,1}}{\varepsilon} - \frac{1-\alpha_{2j+1}}{\varepsilon}}{\alpha_{2j+1}}, \\ r_{2j+1}\left(\frac{1+\varphi_{2i+1,1}}{\varepsilon}\right) &= \frac{1+\varphi_{2i+1,1}}{\varepsilon} \quad (i \neq j). \end{aligned}$$

We do not state its detail here.

We next prove for the transformation π_1 . Its action in $\mathbb{C}(\varphi_{j,i})$ is rewritten as

$$(4.3) \quad \begin{aligned} \pi_1(\varphi_{2i,0}) &= \frac{\varphi_{2i+2,0}}{\alpha_{2i+2}} \frac{Q_{2i+1,1}}{Q_{2i+3,1}}, \quad \pi_1\left(\frac{1+\varphi_{2i-1,1}}{\varepsilon}\right) = \frac{Q_{2i,1} + \alpha_{2i}\varphi_{2i+1,1}Q_{2i+2,1}}{\varepsilon Q_{2i,1}}, \\ \pi_1(\alpha_j) &= \alpha_j. \end{aligned}$$

In the first equation of (4.3), thanks to Lemma 3.2, we have

$$\frac{Q_{2i+1,1}}{Q_{2i+3,1}} = \frac{B_i + O(\varepsilon)}{B_{i+1} + O(\varepsilon)} \quad (i \neq n), \quad \frac{Q_{2n+1,1}}{Q_{1,1}} = \frac{B_n + O(\varepsilon)}{B_0 + O(\varepsilon)}.$$

It follows that

$$\pi_1\left(-\frac{q_{i+1}}{q_i}\right) = -\frac{q_{i+2}}{q_{i+1}} \frac{B_i}{B_{i+1}} \quad (i = 0, \dots, n-1), \quad \pi_1\left(-\frac{t}{q_n}\right) = -q_1 \frac{B_n}{B_0},$$

from which we obtain

$$(4.4) \quad \pi_1(q_i) = \frac{\frac{t}{q_1}B_0}{\frac{t}{q_{i+1}}B_i} \quad (i = 1, \dots, n), \quad \pi_1(t) = t.$$

In the second equation of (4.3), thanks to Lemma 3.2, we have

$$\frac{Q_{2i,1} + \alpha_{2i}\varphi_{2i+1,1}Q_{2i+2,1}}{\varepsilon} = A_i + \frac{t-1}{t}(a_{2i} - q_{i+1}p_{i+1}) - A_{i+1} + O(\varepsilon),$$

where $A_{n+1} = A_0$. It follows that

$$\begin{aligned} \pi_1(-q_i p_i) &= -q_{i+1} p_{i+1} + a_{2i} + \frac{t}{t-1}(A_i - A_{i+1}) \quad (i = 1, \dots, n), \\ \pi_1(-q_{n+1} p_{n+1}) &= -q_1 p_1 + a_0 + \frac{t}{t-1}(A_0 - A_1), \end{aligned}$$

from which we obtain

$$(4.5) \quad \begin{aligned} \pi_1(q_i p_i) &= q_{i+1} p_{i+1} - a_{2i} - \frac{t}{t-1} (A_i - A_{i+1}) \quad (i = 1, \dots, n), \\ \pi_1(h) &= h + \sum_{k=0}^n a_{2k} \end{aligned}$$

with

$$\frac{t}{t-1} (A_i - A_{i+1}) = -\frac{(q_i - q_{i+1})(\sum_{k=1}^i p_k + \sum_{k=i+1}^{n+1} t p_k)}{1-t} + q_{i+1} p_{i+1}.$$

It is trivial that the parameters a_j ($j = 0, \dots, 2n+1$) are invariant under the action of π_1 . Hence the actions (4.4) and (4.5) are equivalent to those given in (1.2) and (1.3).

We can prove for the transformations r'_0, π_2, ρ easily in a similar manner.

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