

Irrationality exponents of certain reciprocal sums

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1 Introduction

For any sequence $\{x_n\}$ of positive integers such that $x_n^2 \mid x_{n+1}$ and $x_n^2 \neq x_{n+1}$ for all sufficiently large n and $\varepsilon_n \in \{-1, 1\}$, we define the sum

$$S = 1 + \sum_{n=1}^{\infty} \frac{\varepsilon_n}{x_n}.$$

In this paper we give the explicit continued fraction expansion of the sum and compute its irrationality exponent, where the irrationality exponent $\mu(\alpha)$ of a real number α is defined by the supremum of the set of numbers μ for which the inequality

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^\mu}$$

has infinitely many rational solutions p/q . Every irrational α has $\mu(\alpha) \geq 2$. If $\mu(\alpha) > 2$, then α is transcendental by Roth's theorem. Our result is as follows (see Theorem 2 in Section 3):

$$\mu(S) = \limsup_{n \rightarrow \infty} \frac{\log x_{n+1}}{\log x_n}.$$

For the proof of Theorem 2, we first expand the partial sums

$$S_n = 1 + \sum_{k=1}^n \frac{\varepsilon_k}{x_k}$$

in continued fractions in the generic case $x_1 \geq 3$, $x_n^2 \mid x_{n+1}$, and $x_n^2 \neq x_{n+1}$ ($n \geq 1$) (see Theorem 2 in Section 3), which were given by Hone [4] when $\varepsilon_n = 1$ for all $n \geq 1$. The continued fractions obtained in Theorem 1 have certain symmetric patterns; namely, if the continued fraction expansion of the n th partial sum is written using the standard notation as

$$S_n = [1; a_1, a_2, \dots, a_{l_n}]$$

with $a_{l_n} \neq 1$, then

$$\begin{aligned} S_{n+1} &= [1; a_1, a_2, \dots, a_{l_{n+1}}] \\ &= [1; a_1, a_2, \dots, a_{l_n}, x_{n+1}/x_n^2 - 1, 1, a_{l_n} - 1, a_{l_n-1}, \dots, a_1] \end{aligned}$$

if $\varepsilon_{n+1} = 1$, and otherwise,

$$S_{n+1} = [1; a_{l_{n+1}}, \dots, a_2, a_1]$$

(see the formula (11) in Theorem 1). By means of this recursive construction of the continued fraction expansions of S_n , we can compute the irrationality exponent of the sum $S = \lim_{n \rightarrow \infty} S_n$ using the following formula (cf., eg., [5, Theorem 1]): The irrationality exponent of the simple continued fraction $\alpha = [a_0; a_1, a_2, \dots]$ with the n th convergent $p_n/q_n = [a_0; a_1, a_2, \dots, a_n]$ is given by

$$\mu(\alpha) = 2 + \limsup_{n \rightarrow \infty} \frac{\log a_{n+1}}{\log q_n}. \quad (1)$$

The assumption $x_1 \geq 3$ in Theorem 1 is indispensable, since the minimal partial denominator in the continued fraction expansions of the sums S_n is $x_1 - 2$, which vanishes if $x_1 = 2$. In this degenerate case, we remove these zeros using the formula (6) below and obtain the simple continued fraction expansions, which will be exhibited in Theorem 3 in the final section 4.

2 Continued fraction expansion of the sums

We employ the standard notion for continued fractions:

$$[a_0; a_1, a_2, \dots] = \lim_{n \rightarrow \infty} [a_0; a_1, \dots, a_n],$$

where

$$[a_0; a_1, a_2, \dots] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}}}$$

The numerators p_n and denominators q_n of the n th convergent p_n/q_n satisfy the following relations:

$$\begin{cases} p_{-1} = 1, & p_0 = a_0, & p_n = a_n p_{n-1} + p_{n-2}, \\ q_{-1} = 0, & q_0 = 1, & q_n = a_n q_{n-1} + q_{n-2}, \end{cases} \quad (2)$$

$$\frac{q_n}{q_{n-1}} = [a_n; a_{n-1}, \dots, a_2, a_1], \quad (3)$$

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}. \quad (4)$$

We also use the formulas:

$$1 - [0; a_1, a_2, \dots, a_n] = [0; 1, a_1 - 1, a_2, \dots, a_n], \quad (5)$$

$$[\dots, a, 0, b, \dots] = [\dots, a + b, \dots]. \quad (6)$$

A continued fraction $[a_0; a_1, a_2, \dots]$ is said to be simple, if a_0 is an integer and $a_1, a_2, \dots$ are positive integers. We define the length of a finite continued fraction $[a_0; a_1, a_2, \dots, a_n]$ by n .

Theorem 1. *Let $\{x_n\}$ be a sequence of positive integers such that*

$$x_0 = 1, \quad x_1 \geq 3, \quad x_n^2 \mid x_{n+1}, \quad z_{n+1} = \frac{x_{n+1}}{x_n^2} \geq 2 \quad (n \geq 0), \quad (7)$$

and let $\varepsilon_n \in \{-1, 1\}$. Then the sums

$$S_n = 1 + \sum_{k=1}^n \frac{\varepsilon_k}{x_k} \quad (8)$$

have the following simple continued fraction expansions:

Case 1. Let $\varepsilon_1 = 1$. Then

$$S_2 = \begin{cases} [1; z_1 - 1, 1, z_2 - 1, z_1] & \text{if } \varepsilon_2 = 1, \\ [1; z_1, z_2 - 1, 1, z_1 - 1] & \text{if } \varepsilon_2 = -1. \end{cases} \quad (9)$$

For $n \geq 2$, if the expansion

$$S_n = [1; a_1, a_2, \dots, a_{l_n}] \quad (10)$$

with $a_{l_n} \neq 1$ and $l_n = 3 \cdot 2^{n-1} - 2$ ($n \geq 2$) is given, then

$$S_{n+1} = \begin{cases} [1, a_1, \dots, a_{l_n}, z_{n+1} - 1, 1, a_{l_n} - 1, a_{l_n-1}, \dots, a_1] & \text{if } \varepsilon_{n+1} = 1, \\ [1, a_1, \dots, a_{l_n-1}, a_{l_n} - 1, 1, z_{n+1} - 1, a_{l_n}, \dots, a_1] & \text{if } \varepsilon_{n+1} = -1 \end{cases} \quad (11)$$

with length $l_{n+1} = 2l_n + 2$.

Case 2. Let $\varepsilon_1 = -1$. Then

$$S_n = [0; 1, b_1 - 1, b_2, \dots, b_{l_n}] \quad (12)$$

with $b_{l_n} \neq 1$, where the expansion

$$[1; b_1, b_2, \dots, b_{l_n}] = 1 - \sum_{k=1}^n \frac{\varepsilon_k}{x_k}$$

is given by Case 1.

Corollary 1. *Make the same notations as in Theorem 1. Then*

$$1 + \sum_{n=1}^{\infty} \frac{\varepsilon_n}{x_n} = \begin{cases} \lim_{n \rightarrow \infty} [1; a_1, a_2, \dots, a_{l_n}] & \text{if } \varepsilon_1 = 1, \\ \lim_{n \rightarrow \infty} [0; 1, b_1 - 1, b_2, \dots, b_{l_n}] & \text{if } \varepsilon_1 = -1. \end{cases}$$

Theorem 1 follows immediately from the following formulas.

Lemma 1 (cf. [6]). *Let $A, a_1, a_2, \dots, a_k$ be positive real numbers and let $p_k/q_k = [0; a_1, a_2, \dots, a_k]$. Assume that $a_k > 1$ and $A > 1$. Then*

$$[0; a_1, a_2, \dots, a_k, A - 1, 1, a_k - 1, a_{k-1}, \dots, a_2, a_1] = \frac{p_k}{q_k} + \frac{(-1)^k}{Aq_k^2}, \quad (13)$$

$$[0; a_1, a_2, \dots, a_{k-1}, a_k - 1, 1, A - 1, a_k, \dots, a_2, a_1] = \frac{p_k}{q_k} - \frac{(-1)^k}{Aq_k^2}. \quad (14)$$

Proof of Theorem 1. The expansions (9) can be obtained by direct calculation. Noting that $x_k \mid x_{k+1}$, we have by (8) and (10) $x_n = q_{l_n}$ ($n \geq 1$).

Case 1. Let $\varepsilon_1 = 1$. Assume first that $\varepsilon_{n+1} = 1$. Applying the formula (13) with $k = l_n$, $A = z_{n+1}$, and $q_{l_n} = x_n$, we get

$$[1; a_1, \dots, a_{l_n}, z_{n+1} - 1, 1, a_{l_n} - 1, a_{l_n-1}, \dots, a_1] = \frac{p_{l_n}}{q_{l_n}} + \frac{(-1)^{l_n}}{z_{n+1}q_{l_n}^2} = S_n + \frac{1}{x_{n+1}} = S_{n+1}.$$

Similarly, we can prove (11) with $\varepsilon_{n+1} = -1$ using (14).

Case 2. Let $\varepsilon_1 = -1$. The expansion (12) follows from Case 1 and the formula (5), and the proof is completed. $\square$

3 Irrationality exponent of the sum

Theorem 2. *Let $\{x_n\}$ be a sequence of positive integers such that*

$$x_n^2 \mid x_{n+1}, \quad x_n^2 \neq x_{n+1} \quad (15)$$

for all sufficiently large n and let $\varepsilon_n \in \{-1, 1\}$. Then the irrationality exponent of the sum

$$S = 1 + \sum_{n=1}^{\infty} \frac{\varepsilon_n}{x_n} \quad (16)$$

is given by

$$\mu(S) = \limsup_{n \rightarrow \infty} \frac{\log x_{n+1}}{\log x_n}. \quad (17)$$

Corollary 2. *The sum S as in (16) is transcendental, if*

$$\mu(S) = \limsup_{n \rightarrow \infty} \frac{\log x_{n+1}}{\log x_n} > 2. \quad (18)$$

Corollary 3. Let $\{x_n\}$ and $\{\varepsilon_n\}$ be as in Theorem 2. Then

$$\mu\left(1 + \sum_{n=1}^{\infty} \frac{\varepsilon_n}{x_n^l}\right) = \limsup_{n \rightarrow \infty} \frac{\log x_{n+1}}{\log x_n} \quad (l = 1, 2, \dots).$$

For the proof of Theorem 2, we need the following lemma (cf., eg., [3, Lemma 1]):

Lemma 2. If α is an irrational number, then

$$\mu(\alpha) = \mu\left(\frac{a\alpha + b}{c\alpha + d}\right)$$

for any integers a, b, c , and d with $ad - bc \neq 0$.

Proof of Theorem 2. We may assume in view of Lemma 2 that $\{x_n\}$ fulfills (7). So we can apply Theorem 1 to the sum $S = \lim_{n \rightarrow \infty} S_n$.

Case 1. Let $\varepsilon_1 = 1$. Then by (10) and (11), we have

$$\max_{1 < k \leq l_{n+1}} \frac{\log a_k}{\log q_{k-1}} = \max \left\{ \max_{1 < k \leq l_n} \frac{\log a_k}{\log q_{k-1}}, \frac{\log(z_{n+1} - 1)}{\log q_{l_n}} \right\}$$

if $\varepsilon_{n+1} = 1$. Otherwise, namely, if $\varepsilon_{n+1} = -1$, the last formula holds with the equality replaced by the inequality $\leq$. Hence, we obtain

$$\limsup_{k \rightarrow \infty} \frac{\log a_k}{\log q_{k-1}} = \limsup_{n \rightarrow \infty} \frac{\log(z_{n+1} - 1)}{\log x_n} = \limsup_{n \rightarrow \infty} \frac{\log x_{n+1}}{\log x_n} - 2,$$

and the formula (1) yields (17).

Case 2. Let $\varepsilon_1 = -1$. Then

$$\mu(S) = \mu(2 - S) = \mu\left(1 + \frac{1}{x_1} + \sum_{k=2}^{\infty} \frac{-\varepsilon_k}{x_k}\right) = \limsup_{n \rightarrow \infty} \frac{\log x_{n+1}}{\log x_n}$$

by Lemma 2 and Case 1, and the proof is completed. $\square$

4 Continued fraction expansions in the degenerate case

In this section we give the continued fraction expansions of the sums S_n in the case $x_1 = 2$. We focus on the case $\varepsilon_1 = 1$, since the other case can be dealt with by using the formula (5). By the formulas (10) and (11) in Theorem 1, partial denominators a_k ($1 \leq k \leq l_{n+1}$) in the expansion of S_{n+1} consist of those in the expansion of S_n , namely, a_k ($1 \leq k \leq l_n$), plus 1, $z_{n+1} - 1 (\neq 0)$, and $a_{l_n} - 1$. We start with the expansions of S_3 with length $l_3 = 10$.

Example 1. *The continued fraction expansions of S_3 with $\varepsilon_1 = 1$.*

$$\begin{aligned} [1; z_1 - 1, 1, z_2 - 1, z_1, z_3 - 1, 1, z_1 - 1, z_2 - 1, 1, z_1 - 1] & \text{ if } (\varepsilon_2, \varepsilon_3) = (1, 1), \\ [1; z_1 - 1, 1, z_2 - 1, z_1 - 1, 1, z_3 - 1, z_1, z_2 - 1, 1, z_1 - 1] & \text{ if } (\varepsilon_2, \varepsilon_3) = (1, -1), \\ [1; z_1, z_2 - 1, 1, z_1 - 1, z_3 - 1, 1, z_1 - 2, 1, z_2 - 1, z_1] & \text{ if } (\varepsilon_2, \varepsilon_3) = (-1, 1), \\ [1; z_1, z_2 - 1, 1, z_1 - 2, 1, z_3 - 1, z_1 - 1, 1, z_2 - 1, z_1] & \text{ if } (\varepsilon_2, \varepsilon_3) = (-1, -1). \end{aligned}$$

Example 1 implies that, if $\varepsilon_2 = -1$, there is only one zero $z_1 - 2$ in the expansions of S_3 and $a_1 = a_{l_n} = z_1$ for all $n \geq 3$. Hence, since $a_{l_n} - 1 = z_1 - 1 \neq 0$, all zeros appearing in the expansion of S are generated from the zero $z_1 - 2$ in S_3 by the recursive procedure from (10) to (11). On the other hand, if $\varepsilon_2 = 1$, there is no zero in the expansions of S_3 . To study this case more precisely, we observe:

Example 2. *The continued fraction expansions of S_4 with $(\varepsilon_1, \varepsilon_2) = (1, 1)$.*

$$\begin{aligned} [1; a_1, \dots, a_9, z_1 - 1, z_4 - 1, 1, z_1 - 2, a_9, \dots, a_1] & \text{ if } \varepsilon_4 = 1, \\ [1; a_1, \dots, a_9, z_1 - 2, 1, z_4 - 1, z_1 - 1, a_9, \dots, a_1] & \text{ if } \varepsilon_4 = -1 \end{aligned}$$

with length $l_4 = 22$, where the expansions $S_3 = [1; a_1, \dots, a_{10}]$ are given in Example 1 with $\varepsilon_2 = 1$.

Example 1 and 2 with (11) imply that, if $\varepsilon_2 = 1$, there is only one zero $z_1 - 2$ in the expansions of S_4 and $a_1 = a_{l_n} = z_1 - 1$ for all $n \geq 3$. Hence each of the expansions of S_{n+1} ($n \geq 4$) contains zeros which come from that of S_n plus one new zero $a_{l_n} - 1 = z_1 - 2$.

In this way, we can locate all zeros, namely, $z_1 - 2$, in the expansions of S and remove them using the formula (6). Rewriting the continued fractions of the form $[\dots, 1, z_1 - 1]$ as $[\dots, 2]$, we obtain:

Theorem 3. *Let $\{x_n\}$ be a sequence of positive integers such that*

$$x_1 = 2, \quad x_n^2 | x_{n+1}, \quad z_{n+1} = \frac{x_{n+1}}{x_n^2} \geq 2 \quad (n \geq 1),$$

and let $\varepsilon_n \in \{-1, 1\}$. Then the sums

$$T_n = 1 + \sum_{k=1}^n \frac{\varepsilon_k}{x_k}$$

have the following simple continued fraction expansions:

Case 1.1. Let $(\varepsilon_1, \varepsilon_2) = (1, 1)$. Then

$$T_3 = \begin{cases} [1; 1, 1, z_2 - 1, 2, z_3 - 1, 1, 1, z_2 - 1, 2] & \text{ if } \varepsilon_3 = 1, \\ [1; 1, 1, z_2 - 1, 1, 1, z_3 - 1, 2, z_2 - 1, 2] & \text{ if } \varepsilon_3 = -1 \end{cases}$$

with length 9. For $n \geq 3$, if the expansion $T_n = [1; 1, 1, a_3, \dots, a_{t_n-1}, 2]$ with $t_n = 5 \cdot 2^{n-2} - 1$ is given, then

$$T_{n+1} = \begin{cases} [1; 1, 1, a_3, \dots, a_{t_n-1}, 1, 1, z_{n+1} - 1, 2, a_{t_n-2}, \dots, a_3, 2] & \text{if } \varepsilon_{n+1} = 1, \\ [1; 1, 1, a_3, \dots, a_{t_n-2}, 2, z_{n+1} - 1, 1, 1, a_{t_n-1}, \dots, a_3, 2] & \text{if } \varepsilon_{n+1} = -1 \end{cases}$$

with length $t_{n+1} = 2t_n + 1$.

Case 1.2. Let $(\varepsilon_1, \varepsilon_2) = (1, -1)$. Then

$$T_3 = \begin{cases} [1; 2, z_2 - 1, 1, 1, z_3 - 1, 2, z_2 - 1, 2] & \text{if } \varepsilon_3 = 1, \\ [1; 2, z_2 - 1, 2, z_3 - 1, 1, 1, z_2 - 1, 2] & \text{if } \varepsilon_3 = -1 \end{cases}$$

with length 8. For $n \geq 3$, if the expansion $T_n = [1; 2, a_2, \dots, a_{t_n-1}, 2]$ with length $t_n - 1$ is given, then

$$T_{n+1} = \begin{cases} [1; 2, a_2, \dots, a_{t_n-1}, 2, z_{n+1} - 1, 1, 1, a_{t_n-2}, \dots, a_2, 2] & \text{if } \varepsilon_{n+1} = 1, \\ [1; 2, a_2, \dots, a_{t_n-2}, 1, 1, z_{n+1} - 1, 2, a_{t_n-1}, \dots, a_2, 2] & \text{if } \varepsilon_{n+1} = -1 \end{cases}$$

with length $t_{n+1} - 1$.

Case 2. Let $\varepsilon_1 = -1$. Then

$$T_n = \begin{cases} [0; 2, b_2, \dots, b_{t_n}] & \text{if } \varepsilon_2 = -1, \\ [0; 1, 1, b_2, \dots, b_{t_n-1}] & \text{if } \varepsilon_2 = 1, \end{cases}$$

where the expansion

$$1 - \sum_{k=1}^n \frac{\varepsilon_k}{x_k} = [1; b_1, b_2, \dots, b_{u_n}]$$

with $u_n = t_n$ if $\varepsilon_2 = -1$, $= t_{n-1}$ if $\varepsilon_2 = 1$ is given by Case 1.1 or 1.2.

After the conference Amou kindly sent the last named author his joint paper [1] with Bugeaud, in which our Theorem 2 was already generalized (see [1, Lemma 3]). Recently, the authors proved the following theorems which improves the result in [1].

Theorem ([2, Theorem 1]). *Let $\{x_n\}$ be a sequence of rational numbers greater than one and let $\varepsilon_n \in \{1, -1\}$ with $\varepsilon_1 = 1$. Put $z_1 = x_1$, $z_{n+1} = x_{n+1}x_n^{-2}$ ($n \geq 1$) and define*

$$\delta_1 = \text{den } z_1, \quad \delta_{n+1} = \delta_n^2 \text{den } z_{n+1} \quad (n \geq 1). \quad (19)$$

Assume that the following two conditions hold:

- (i) $z_n \geq 1$ for all sufficiently large n ,
- (ii) $\log \delta_{n+1} = o(\log x_n)$.

Then the irrationality exponent of the number

$$S = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{x_n}$$

is equal to

$$\tau = \limsup_{n \rightarrow \infty} \frac{\log x_{n+1}}{\log x_n}.$$

Theorem ([2, Theorem 2]). *Make the same notations as in Theorem 2. Put $x_n = t_n/s_n$ with $t_n, s_n \in \mathbb{Z}_{>0}$. Assume that the following two conditions hold:*

- (i)' $s_n^2 | s_{n+1}, t_n^2 | t_{n+1}$ for all sufficiently large n ,
- (ii)' $\log s_{n+1} = o(\log t_n)$.

Then the irrationality exponent of the number S is equal to

$$\tau = \limsup_{n \rightarrow \infty} \frac{\log t_{n+1}}{\log t_n}.$$

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