

Double covers and vector bundles of rank two

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Introduction

In this article, we discuss the correspondence between vector bundles of rank 2 (say *2-bundles* for short) and line bundles on double covers.

In the study of the embedded topology of curves on the complex projective plane $\mathbb{P}^2$, it is effective to consider the irreducibility of ϕ^*C for an irreducible curve $C \subset \mathbb{P}^2$ and a Galois cover $\phi : X \rightarrow \mathbb{P}^2$ (cf. [3], [15], [16]). For example, let $B, C \subset \mathbb{P}^2$ be two plane curves such that $\deg B$ is even and C is irreducible with $\deg B \neq \deg C$, and let $\phi : X \rightarrow \mathbb{P}^2$ be the double cover branched at B ; then the embedded topology of $B + C$ changes depending on whether ϕ^*C is irreducible or not. In the case where ϕ is a cyclic cover and C is smooth, a criterion for irreducibility of ϕ^*C is known in [8]. This criterion is intensively used to distinguish embedded topology of plane curves (cf. [1], [2], [4], [16]). In the case where ϕ is the double cover branched at a smooth conic and C is a nodal curve, a criterion for irreducibility of ϕ^*C is known in [5]. However, for general $C \subset \mathbb{P}^2$, it is still a problem to determine the irreducibility of ϕ^*C even if ϕ is a double cover. We consider a new approach to the problem, which is constructing various curves on X (which correspond to irreducible components of ϕ^*C) and studying property of their images (which correspond to C). The purpose of this article is a preparation for this new approach by studying a correspondence between line bundles on X and vector bundles of rank 2 (say *2-bundles* for short) on Y in the case where ϕ is a double cover. As bi-products, we obtain approaches to studying the Picard group of double covers and to constructing 2-bundles. This article gives statements of results without proofs. For the proofs, see [17].

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1 Setting

Let $\phi : X \rightarrow Y$ be a non-singular double cover, i.e., a finite morphism of degree two between non-singular complex varieties X, Y . Let $B_\phi \subset Y$ and $R_\phi \subset X$ be the branch locus and the ramification locus of $\phi : X \rightarrow Y$, respectively. Let $F \in H^0(Y, \mathcal{O}_Y(B_\phi))$ and $t \in H^0(X, \mathcal{O}_X(R_\phi))$ be sections defining B_ϕ and R_ϕ , respectively, such that $t^2 = F$. There is a divisor L on Y such that $2L \sim B_\phi$

and $\phi_* \mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathcal{O}_Y(-L)$ as $\mathcal{O}_Y$ -modules. Note that we have

$$\phi_* \mathcal{O}_X \cong \left(\bigoplus_{n=0}^{\infty} t^n \mathcal{O}_Y(-nL) \right) / (t^2 - F)$$

as $\mathcal{O}_Y$ -algebras. In 2017, Catanese–Perroni proved the following lemma.

Lemma 1.1 ([7]). *If $\mathcal{M}$ is a 2-bundle on Y , and $M : \mathcal{M}(-L) \rightarrow \mathcal{M}$ is a morphism satisfying $M^2 = F \cdot \text{id}_{\mathcal{M}}$ (i.e., the composition of $M(-L) : \mathcal{M}(-2L) \rightarrow \mathcal{M}(-L)$ and $M : \mathcal{M}(-L) \rightarrow \mathcal{M}$ is the multiplication by F), then the pair $(\mathcal{M}, M)$ determines naturally a line bundle $\mathcal{L}_{(\mathcal{M}, M)}$ on X such that $\phi_* \mathcal{L}_{(\mathcal{M}, M)} \cong \mathcal{M}$, and M corresponds to the multiplication by t in $\mathcal{L}_{(\mathcal{M}, M)}$.*

Definition 1.2. Let $(\mathcal{M}, M)$ be a pair of a 2-bundle $\mathcal{M}$ on Y and a morphism $M : \mathcal{M}(-L) \rightarrow \mathcal{M}$.

- (i) We call $(\mathcal{M}, M)$ an *admissible pair for ϕ* if $M^2 = F \cdot \text{id}_{\mathcal{M}} : \mathcal{M}(-2L) \rightarrow \mathcal{M}$.
- (ii) Two admissible pairs $(\mathcal{M}, M)$ and $(\mathcal{N}, N)$ are *equivalent* if there exists an isomorphism $\Psi : \mathcal{M} \rightarrow \mathcal{N}$ such that $\Psi \circ M = N \circ \Psi(-L)$, and write $(\mathcal{M}, M) \sim (\mathcal{N}, N)$.
- (iii) $\text{AD}_{\phi}(Y) := \{(\mathcal{M}, M) : \text{an admissible pair for } \phi\} / \sim$.

Remark 1.3. We have the following facts.

- (i) In [7], it is shown that $(\mathcal{M}, M) \sim (\mathcal{N}, N)$ if and only if $\mathcal{L}_{(\mathcal{M}, M)} \cong \mathcal{L}_{(\mathcal{N}, N)}$.
- (ii) $\text{AD}_{\phi}(Y)$ has a group structure induced by $\text{Pic}(X)$.

2 The group structure of $\text{AD}_{\phi}(Y)$

To describe a correspondence between admissible pairs for a non-singular double cover $\phi : X \rightarrow Y$ and line bundles on X , we introduce some notation. Let $(\mathcal{M}, M)$ be an admissible pair for ϕ , and let $\mathfrak{U} := \{U_i\}_{i \in I}$ be an affine open covering of Y such that

$$\begin{aligned} \phi_* \mathcal{O}_X|_{U_i} &\cong \mathcal{O}_{U_i} \oplus \mathcal{O}_{U_i} t_i && \text{as } \mathcal{O}_{U_i}\text{-algebras,} \\ \varphi_i : \mathcal{M}_i &:= \mathcal{M}|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}^{\oplus 2} && \text{as } \mathcal{O}_{U_i}\text{-modules} \end{aligned}$$

for any $i \in I$, where $t_i := t|_{U_i}$. Note that $t_j = t_i \xi_{ij}$ for $i, j \in I$, where $\xi_{ij} \in \mathcal{O}_{U_i \cap U_j}^{\times}$ correspond to transition functions of $\mathcal{O}_Y(-L)$:

$$\begin{array}{ccc} \mathcal{O}_Y(-L)|_{U_j} & \xlongequal{\quad} & \mathcal{O}_Y(-L)|_{U_i} \\ \downarrow \wr & & \downarrow \wr \\ \mathcal{O}_{U_j} & \xrightarrow{\quad \times \xi_{ij} \quad} & \mathcal{O}_{U_i} \end{array}$$

Then we have transition functions $G_{ij} \in \mathrm{GL}(2, \mathcal{O}_{U_i \cap U_j})$ of $\mathcal{M}$ for $i, j \in I$:

$$G_{ij} = \begin{pmatrix} g_{ij,11} & g_{ij,12} \\ g_{ij,21} & g_{ij,22} \end{pmatrix} := \varphi_i \circ \varphi_j^{-1} : \mathcal{O}_{U_j}^{\oplus 2}|_{U_i \cap U_j} \rightarrow \mathcal{O}_{U_i}^{\oplus 2}|_{U_i \cap U_j} \quad (1)$$

satisfying $G_{ik} = G_{ij}G_{jk}$ and $G_{ii} = E$ for each $i, j, k \in I$, where E is the identity matrix. The restriction of $M : \mathcal{M}(-L) \rightarrow \mathcal{M}$ to U_i corresponds to a matrix M_i :

$$M_i = \begin{pmatrix} a_{i0} & a_{i2} \\ a_{i1} & -a_{i0} \end{pmatrix} := \varphi_i \circ (\varphi_i(-L))^{-1} : \mathcal{O}_{U_i}^{\oplus 2} \rightarrow \mathcal{O}_{U_i}^{\oplus 2} \quad (2)$$

satisfying $a_{i0}^2 + a_{i1}a_{i2} = F_i := F|_{U_i}$ and $M_j = \xi_{ij}G_{ij}^{-1}M_iG_{ij}$ as elements of $\mathrm{GL}(2, \mathbb{C}(X))$ for each $i, j \in I$:

$$\begin{array}{ccccc} \mathcal{M}_j(-L) & \xrightarrow{\quad M \quad} & \mathcal{M}_j & & \\ \parallel & \searrow \varphi_j(-L) & \mathcal{O}_{U_j}^{\oplus 2} & \xrightarrow{M_j} & \mathcal{O}_{U_j}^{\oplus 2} \swarrow \varphi_j \\ & \xi_{ij}G_{ij} \downarrow & & & \downarrow G_{ij} \\ \mathcal{M}_i(-L) & \xrightarrow{\quad M \quad} & \mathcal{M}_i & & \\ \parallel & \swarrow \varphi_i(-L) & \mathcal{O}_{U_i}^{\oplus 2} & \xrightarrow{M_i} & \mathcal{O}_{U_i}^{\oplus 2} \swarrow \varphi_i \\ & & & & \end{array}$$

Definition 2.1. With the above notation, we call $(\{G_{ij}\}, \{M_i\})_{\mathcal{U}}$ a *representation* of the admissible pair $(\mathcal{M}, M)$. A representation $(\{G_{ij}\}, \{M_i\})_{\mathcal{U}}$ of $(\mathcal{M}, M)$ is said to be *good* if a_{i1} is a unit on U_i for each $i \in I$.

We can prove the following lemma by taking a ‘fine’ affine open covering of Y and a certain local basis of $\mathcal{M}$.

Lemma 2.2 ([17, Lemma 1.3]). *Any admissible pair $(\mathcal{M}, M)$ for ϕ has a good representation.*

We describe the group structure of $\mathrm{AD}_{\phi}(Y)$. Let $(\mathcal{M}^{(k)}, M^{(k)})$ be an admissible pair for a non-singular double cover $\phi : X \rightarrow Y$ for each $k = 1, \dots, m$. Let $(\{G_{ij}^{(k)}\}, \{M_i^{(k)}\})_{\mathcal{U}}$ be a good representation of $(\mathcal{M}^{(k)}, M^{(k)})$ for each $k = 1, \dots, m$, where

$$G_{ij}^{(k)} = \begin{pmatrix} g_{ij,11}^{(k)} & g_{ij,12}^{(k)} \\ g_{ij,21}^{(k)} & g_{ij,22}^{(k)} \end{pmatrix}, \quad M_i^{(k)} = \begin{pmatrix} a_{i0}^{(k)} & a_{i2}^{(k)} \\ a_{i1}^{(k)} & -a_{i0}^{(k)} \end{pmatrix} \quad (k = 1, \dots, m).$$

Let $(\{G_{ij}^{(0)}\}, \{M_i^{(0)}\})_{\mathcal{U}}$ be the good representation corresponding to $\mathcal{O}_X$, where

$$G_{ij}^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & \xi_{ij} \end{pmatrix}, \quad M_i^{(0)} = \begin{pmatrix} 0 & F_i \\ 1 & 0 \end{pmatrix}.$$

For 2×2 matrices $A_1, \dots, A_m$, put $\prod_{k=1}^m A_k := A_1 A_2 \dots A_m$. Then the following theorem holds.

Theorem 2.3 ([17, Theorem 2.1]). *With the above notation, put*

$$K_{ij}^{(k)+} := \frac{1}{a_{i1}^{(k)}} \left(\left(a_{i1}^{(k)} g_{ij,11}^{(k)} - a_{i0}^{(k)} g_{ij,21}^{(k)} \right) E + g_{ij,21}^{(k)} M_i^{(0)} \right), \quad (3)$$

$$K_{ij}^{(k)-} := \frac{\xi_{ij}}{a_{i1}^{(k)} \det(G_{ij})} \left(\left(a_{i1}^{(k)} g_{ij,11}^{(k)} - a_{i0}^{(k)} g_{ij,21}^{(k)} \right) E - g_{ij,21}^{(k)} M_i^{(0)} \right) \quad (4)$$

for each $k = 1, \dots, m$. Let $n_1, \dots, n_m$ be m integers, and let $[n]$ be the list $[n_1, \dots, n_m]$. Then $\mathcal{L}^{[n]} := \mathcal{L}_{(\mathcal{M}^{(1)}, M^{(1)})}^{\otimes n_1} \otimes \dots \otimes \mathcal{L}_{(\mathcal{M}^{(m)}, M^{(m)})}^{\otimes n_m}$ is associated to the normal representation $(\{G_{ij}^{[n]}\}, \{M_i^{[n]}\})_{\mathfrak{U}}$ with

$$G_{ij}^{[n]} := \prod_{k=1}^m \left(K_{ij}^{(k)}(n_k) \right)^{|n_k|} G_{ij}^{(0)}, \quad M_i^{[n]} := M_i^{(0)},$$

where $K_{ij}^{(k)}(n_k) := K_{ij}^{(k)+}$ if $n_k \geq 0$, and $K_{ij}^{(k)}(n_k) := K_{ij}^{(k)-}$ otherwise.

In the proof of Theorem 2.3, we have the following proposition.

Proposition 2.4 ([17, Proposition 2.5]). *Let $(\mathcal{M}, M)$ be an admissible pair for a non-singular double cover $\phi : X \rightarrow Y$, and let $(\{G_{ij}\}, \{M_i\})_{\mathfrak{U}}$ be a good representation of $(\mathcal{M}, M)$. Let $\iota : X \rightarrow X$ be the covering transformation of ϕ . Then the followings hold:*

$$\iota^* \mathcal{L}_{(\mathcal{M}, M)} \cong \mathcal{L}_{(\mathcal{M}, -M)}, \quad (5)$$

$$\mathcal{L}_{(\mathcal{M}, M)} \otimes \iota^* \mathcal{L}_{(\mathcal{M}, M)} \cong \phi^* \left((\det \mathcal{M}) \otimes \mathcal{O}_Y(L) \right), \quad (6)$$

$$\mathcal{L}_{(\mathcal{M}, M)}^{-1} \cong \phi^* \left((\det \mathcal{M})^{-1} \otimes \mathcal{O}_Y(-L) \right) \otimes \mathcal{L}_{(\mathcal{M}, -M)}. \quad (7)$$

Moreover, $\mathcal{L}_{(\mathcal{M}, M)}^{-1}$ is associated to

$$\left(\left\{ \frac{\xi_{ij}}{\det(G_{ij})} G_{ij} \right\}, \{-M_i\} \right)_{\mathfrak{U}}.$$

3 A subgroup of $\text{Pic}(X)$

We have seen the correspondence between admissible pairs for a non-singular double cover $\phi : X \rightarrow Y$ and line bundles on X . Hence it is effective for understanding $\text{Pic}(X)$ to study $\text{AD}_{\phi}(Y)$. However, it seems difficult to find a morphism $M : \mathcal{M}(-L) \rightarrow \mathcal{M}$ satisfying $M^2 = F \cdot \text{id}_{\mathcal{M}}$ for a general 2-bundle $\mathcal{M}$ on Y . In the case where $\mathcal{M} \cong \mathcal{O}_Y(D_1) \oplus \mathcal{O}_Y(D_2)$ for some divisors D_1, D_2 on Y , such a morphism M can be represented as

$$M = \begin{pmatrix} a_0 & a_2 \\ a_1 & -a_0 \end{pmatrix} : \mathcal{O}_Y(D_1 - L) \oplus \mathcal{O}_Y(D_2 - L) \rightarrow \mathcal{O}_Y(D_1) \oplus \mathcal{O}_Y(D_2), \quad (8)$$

where a_0, a_1 and a_2 are global sections of $\mathcal{O}_Y(L)$, $\mathcal{O}_Y(L - D_1 + D_2)$ and $\mathcal{O}_Y(L + D_1 - D_2)$, respectively, satisfying $a_0^2 + a_1 a_2 = F$.

Definition 3.1. Let $\phi : X \rightarrow Y$ be a non-singular double cover.

- (i) We say that a line bundle $\mathcal{L}$ on X *splits* with respect to ϕ if $\phi_*\mathcal{L}$ is the direct sum of two line bundles on Y .
- (ii) Let $\text{sPic}_\phi(X)$ denote the subgroup of $\text{Pic}(X)$ generated by line bundles which split with respect to ϕ (“s” of sPic means “sub” or “split”);

$$\text{sPic}_\phi(X) := \left\langle [\mathcal{L}] \in \text{Pic}(X) \mid \mathcal{L} \text{ splits with respect to } \phi \right\rangle.$$

Remark 3.2. If $\phi_*\mathcal{L} \cong \mathcal{O}_Y(D_1) \oplus \mathcal{O}_Y(D_2)$, then

$$\phi_*(\mathcal{L} \otimes \phi^*\mathcal{O}_Y(-D_2)) \cong \mathcal{O}_Y(D_1 - D_2) \oplus \mathcal{O}_Y$$

by projection formula. Since $\phi^*\mathcal{O}_Y(D_2) \in \text{sPic}_\phi(X)$, the subgroup $\text{sPic}_\phi(X)$ is generated by $\phi^*(\text{Pic}(Y))$ and line bundles $\mathcal{L}$ satisfying $\phi_*\mathcal{L} \cong \mathcal{O}_Y(D') \oplus \mathcal{O}_Y$ on X for some divisor D' on Y .

Lemma 3.3 ([17, Lemma 3.3]). *If Y is an open subset of a smooth projective variety $\bar{Y}$ with $\text{codim}_{\bar{Y}}(\bar{Y} \setminus Y) \geq 2$, then $H^0(Y, \mathcal{O}_Y) = \mathbb{C}$, and $\text{sPic}_\phi(X)$ is generated by $\phi^*(\text{Pic}(Y))$ and line bundles $\mathcal{L}$ with $\phi_*\mathcal{L} \cong \mathcal{O}_Y(D') \oplus \mathcal{O}_Y$ such that either $\mathcal{O}_Y(D') \cong \mathcal{O}_Y$ or $H^0(Y, \mathcal{O}_Y(D')) = 0$.*

Proof. For an irreducible divisor C on Y , let $\bar{C}$ denote the closure of C on $\bar{Y}$. Let D be a divisor on Y , and put $\bar{D} := \sum_{i=1}^k \bar{C}_i$, where $D = \sum_{i=1}^k C_i$ is the irreducible decomposition of D . Then $\mathcal{O}_Y(D) = \mathcal{O}_{\bar{Y}}(\bar{D})|_Y$. By [9, Proposition 1.6], we have $H^0(Y, \mathcal{O}_Y(D)) = H^0(\bar{Y}, \mathcal{O}_{\bar{Y}}(\bar{D}))$. Thus

$$H^0(Y, \mathcal{O}_Y) = H^0(\bar{Y}, \mathcal{O}_{\bar{Y}}) = \mathbb{C}.$$

For a line bundle $\mathcal{L}$ on X with $\phi_*\mathcal{L} \cong \mathcal{O}_Y(D) \oplus \mathcal{O}_Y$, if $\mathcal{O}_Y(D) \not\cong \mathcal{O}_Y$ and $H^0(Y, \mathcal{O}_Y(D)) \neq 0$, then $H^0(Y, \mathcal{O}_Y(-D)) = 0$ and

$$\phi_*(\mathcal{L} \otimes \phi^*(\mathcal{O}_Y(-D))) \cong \mathcal{O}_Y(-D) \oplus \mathcal{O}_Y.$$

Therefore the assertion holds true by Remark 3.2. $\square$

Theorem 3.4 is a criterion for splitting of the push-forward for a line bundle on X (see Remark 3.2 and Lemma 3.3).

Theorem 3.4 ([17, Theorem 2]). *Let D^+ be an effective divisor on X , and let D be the effective divisor on Y defined by $f = 0$ for $f \in H^0(Y, \mathcal{O}_Y(D))$ such that $\phi^*D = D^+ + \iota^*D^+$. Assume that $H^0(Y, \mathcal{O}_Y) = \mathbb{C}$. If $\phi_*\mathcal{O}_X(D^+) \cong \mathcal{O}_Y(D') \oplus \mathcal{O}_Y$ for a divisor D' on Y satisfying either $\mathcal{O}_Y(D') \cong \mathcal{O}_Y$ or $H^0(Y, \mathcal{O}_Y(D')) = 0$, then D and D' satisfy the following two conditions;*

- (i) D' is linearly equivalent to $D - L$, i.e., $\mathcal{O}_Y(D') \cong \mathcal{O}_Y(D - L)$; and
- (ii) there are global sections a_0 and a_1 of $\mathcal{O}_Y(L)$ and $\mathcal{O}_Y(2L - D)$, respectively, such that $F = a_0^2 + fa_1$.

Moreover, in the case where D^+ is irreducible, the converse holds true.

Remark 3.5. In the case of $Y = \mathbb{P}^1$, Jacobi [10] have studied the correspondence between line bundles on the hyperelliptic curve X and equations of the form $F = a_0^2 + a_1 a_2$ via the Jacobian variety. The study of [10] relates to Hitchin theory (cf. [18]).

Theorem 3.4 implies that generators of $\mathrm{sPic}_\phi(X)$ correspond to equations of the form $F = a_0^2 + a_1 a_2$. Hence we can expect that $\mathrm{sPic}_\phi(X)$ reflects the arrangement of B_ϕ in Y enough to describe the structure of $\mathrm{Pic}(X)$. In several examples below, the equation $\mathrm{sPic}_\phi(X) = \mathrm{Pic}(X)$ holds.

Example 3.6. Let X be a hyperelliptic curve, and let $\phi : X \rightarrow \mathbb{P}^1$ be a non-singular double cover. Since any rank 2-bundle on $\mathbb{P}^1$ splits, we have $\mathrm{Pic}(X) = \mathrm{sPic}_\phi(X)$.

Example 3.7. Let $\phi : X \rightarrow \mathbb{P}^2$ be a double cover branched along a smooth conic B_ϕ . Note that $\deg(L) = 1$. Then $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathrm{Pic}(X) \cong \mathbb{Z} \oplus \mathbb{Z}$. Let D^+ be a ruling of $X \cong \mathbb{P}^1 \times \mathbb{P}^1$. The image $D = \phi(D^+)$ is a tangent line of B_ϕ . Let $f \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ be a section defining D . Then B_ϕ is given by $a_0^2 + f a_1 = 0$ for some $a_0, a_1 \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. By Proposition 2.4 and Theorem 3.4, we obtain $\phi_* \mathcal{O}_X(D^+) \cong \phi_* \mathcal{O}_X(\iota^* D^+) \cong \mathcal{O}_Y^{\oplus 2}$. Since $\mathrm{Pic}(X)$ is generated by $\mathcal{O}_X(D^+)$ and $\mathcal{O}_X(\iota^* D^+)$, we have $\mathrm{Pic}(X) = \mathrm{sPic}_\phi(X)$.

Example 3.8. Let $\phi : X \rightarrow \mathbb{P}^2$ be a double cover branched along a smooth quartic B_ϕ . Note that $\deg(L) = 2$. Then X is isomorphic to the blowing-up of $\mathbb{P}^2$ at 7 points in general position. Moreover $\mathrm{Pic}(X)$ is generated by 8 (-1) -curves $E_0, \dots, E_7$, where E_0 is the strict transform of a line passing through two blowing-up centers, and $E_1, \dots, E_7$ are the exceptional divisors. The images $\phi(E_0), \dots, \phi(E_7)$ are 8 of 28 bitangent lines of B_ϕ . Let $f_j \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ be a section defining $\phi(E_j)$ for each $j = 0, \dots, 7$. Then there exist global sections $a_{j,k}$ ($k = 0, 1$) of $\mathcal{O}_{\mathbb{P}^2}(k+2)$ such that B_ϕ is defined by $a_{j,0}^2 + f_j a_{j,1} = 0$. Hence $\phi_* \mathcal{O}_X(E_j) \cong \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}$ for each $j = 0, \dots, 7$ by Theorem 3.4. Therefore we obtain $\mathrm{Pic}(X) = \mathrm{sPic}_\phi(X)$.

The following conjecture arises.

Conjecture 3.9. If $H^0(Y, \mathcal{O}_Y) = \mathbb{C}$, then $\mathrm{Pic}(X) = \mathrm{sPic}_\phi(X)$.

Remark 3.10. In the cases of Examples 3.7 and 3.8, it is known that $\phi_* \mathcal{L}$ is indecomposable for general line bundles $\mathcal{L}$ on X in [13].

4 An idea to generate 2-bundles

In this section, we give an idea to generate 2-bundles through double covers. Schwarzenberger proved the following theorem.

Theorem 4.1 (Schwarzenberger [13, Theorem 3]). *Let Y be a non-singular surface. For any 2-bundle $\mathcal{E}$ on Y , there exist a non-singular double cover $\phi : X \rightarrow Y$ and a line bundle $\mathcal{L}$ on X such that $\phi_* \mathcal{L} \cong \mathcal{E}$.*

We give a generalization of this theorem. We call $\phi : X \rightarrow Y$ a *normal double cover* if ϕ is a finite surjective morphism of degree two from a normal variety X to a smooth variety Y over $\mathbb{C}$. Let $\text{Cl}(X)$ be the divisor class group of X . Note that there is a canonical one-to-one correspondence between $\text{Cl}(X)$ and the set of divisorial sheaves on X (cf. [14]). See [9] for general results on reflexive sheaves. By modifying the proof of [13, Theorem 3], we can prove the following theorem which is a generalization of [13, Theorem 3].

Theorem 4.2 ([17, Theorem 3]). *Let $\mathcal{E}$ be a 2-bundle on a smooth projective variety Y of dimension n over $\mathbb{C}$. There exist a normal double cover $\phi : X \rightarrow Y$ and a divisorial sheaf $\mathcal{L}$ on X such that $\mathcal{E} \cong \phi_*\mathcal{L}$.*

Proof. Let $\mathbb{P}_{\mathcal{E}}$ be the $\mathbb{P}^1$ -bundle $\mathbf{Proj}(S(\mathcal{E}))$, and let $p : \mathbb{P}_{\mathcal{E}} \rightarrow Y$ be the projection, where $S(\mathcal{E})$ is the symmetric algebra of $\mathcal{E}$. Let $\mathcal{H}$ be a very ample line bundle on Y . The line bundle $\mathcal{H}$ gives the embedding $\Phi_{\mathcal{H}} : Y \hookrightarrow \mathbb{P}^s$ with $s+1 = \dim H^0(Y, \mathcal{H})$. For k large enough, we have the following exact sequence:

$$\mathcal{H}^{\oplus r+1} \rightarrow \mathcal{E} \otimes \mathcal{H}^k \rightarrow 0$$

for some $r > 0$. This induces an embedding $i : \mathbb{P}_{\mathcal{E}} \hookrightarrow \mathbb{P}^N$ for $N = rs + r + s$ via the Segre embedding $\mathbb{P}^r \times \mathbb{P}^s \hookrightarrow \mathbb{P}^N$. Put $\tilde{\mathcal{L}} := i^* \mathcal{O}_{\mathbb{P}^N}(1)$. Note that $i(p^{-1}(P))$ is a line in $\mathbb{P}^N$ for each $P \in Y$, and $p_*\tilde{\mathcal{L}} \cong \mathcal{E} \otimes \mathcal{H}^k$. Hence $i : \mathbb{P}_{\mathcal{E}} \hookrightarrow \mathbb{P}^N$ induces an embedding $i' : Y \hookrightarrow \text{Gr}_1(N)$ by $P \mapsto i(p^{-1}(P))$, where $\text{Gr}_1(N)$ is the Grassmannian consisting of lines in $\mathbb{P}^N$.

For a quadratic hypersurface $Q \subset \mathbb{P}^N$, let $V(Q)$ be the subscheme of $\text{Gr}_1(N)$ consisting of lines on Q . Note that $\text{PGL}(N, \mathbb{C}) := \text{Aut}(\mathbb{P}^N)$ acts transitively on both of $\mathbb{P}^N$ and $\text{Gr}_1(N)$ such that $V(g(Q)) = g(V(Q))$ for any $g \in \text{PGL}(N, \mathbb{C})$ and $Q \subset \mathbb{P}^N$. Since $\dim \text{Gr}_1(N) = 2N - 2$, $\dim V(Q) = 2N - 5$ (cf. [6]) and $\dim Y = n$, we obtain $\dim Y \cap V(Q) = n - 3$ for a general hypersurface $Q \subset \mathbb{P}^N$ of degree 2 by [11, Theorem 2]. Put $X' := \mathbb{P}_{\mathcal{E}} \cap Q$ for a general quadratic hypersurface $Q \subset \mathbb{P}^N$ such that $\dim i'(Y) \cap V(Q) = n - 3$ and X' is smooth. Let $\mathcal{L}'$ be the restriction of $\tilde{\mathcal{L}}$ to X' . For an affine open set U of Y , the Künneth formula for sheaves implies that

$$R^q p_* \tilde{\mathcal{L}}^{-1}(U) = H^q(U \times \mathbb{P}^1, \mathcal{O}_U \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$$

for all $q \geq 0$. Since $\tilde{\mathcal{L}} \otimes \mathcal{I}_{X'} \cong \tilde{\mathcal{L}}^{-1}$ for the ideal sheaf $\mathcal{I}_{X'} \cong \tilde{\mathcal{L}}^{-2}$ of X' , we have $p'_*\mathcal{L}' \cong \mathcal{E} \otimes \mathcal{H}^k$ by [13, Proposition 5].

Then the restriction $p' := p|_{X'} : X' \rightarrow Y$ is a generically finite morphism of degree 2. Let $U' := \{P \in Y \mid (p')^{-1}(P) \text{ is finite}\}$. By Stein factorization of p' , we obtain a birational morphism $f' : X' \rightarrow X''$ and a finite morphism $g' : X'' \rightarrow Y$ such that $g' \circ f' = p'$. Take the normalization $\kappa : X \rightarrow X''$, and put $\phi := g' \circ \kappa : X \rightarrow Y$, which is a normal double cover. Let $\mathcal{L}$ be the double dual $(\kappa^* f'_* \mathcal{L}')^{\vee\vee}$ of $\kappa^* f'_* \mathcal{L}'$. Then $\mathcal{L}$ is a divisorial sheaf on X , and $\phi_*\mathcal{L}|_{U'} \cong p'_*\mathcal{L}'|_{U'}$ since f' and κ are isomorphic over U' . Since $\phi_*\mathcal{L}$ is reflexive by [9, Corollary 1.7] and $\text{codim}_Y(Y \setminus U') = 3$, $\phi_*\mathcal{L} \cong p'_*\mathcal{L}' \cong \mathcal{E} \otimes \mathcal{H}^k$. Therefore, $\phi_*(\mathcal{L} \otimes \phi^*\mathcal{H}^{-k}) \cong \mathcal{E}$. $\square$

As the idea of [7], we can apply our method to divisorial sheaves on normal double covers as follows. Let $\phi : X \rightarrow Y$ be a normal double cover. Let X° be the smooth locus $X \setminus \text{Sing}(X)$ of X , and put $Y^\circ := \phi(X^\circ)$. Then the restriction $\phi^\circ : X^\circ \rightarrow Y^\circ$ of ϕ is a non-singular double cover. For a divisorial sheaf $\mathcal{L}$ on X , the restriction $\mathcal{L}^\circ$ of $\mathcal{L}$ to X° is a line bundle on X° , and $i_*\mathcal{L}^\circ = \mathcal{L}$ and $j_*\phi_*^\circ\mathcal{L}^\circ = \phi_*\mathcal{L}$ hold, where $i : X^\circ \rightarrow X$ and $j : Y^\circ \rightarrow Y$ are the inclusion maps. Hence computation of push-forwards of line bundles on X° can be applied to that of divisorial sheaves on X via j_* .

If Conjecture 3.9 is true, then Theorem 4.2 implies that any 2-bundle on $\mathbb{P}^n$ can be generated by the following method:

- (i) Take a reduced divisor $B : F = 0$ of even degree on $\mathbb{P}^n$ with several representation of the form $F = a_0^2 + a_1a_2$;
- (ii) let $\phi : X \rightarrow \mathbb{P}^n$ be the normal double cover branched at B , and let $\phi^\circ : X^\circ \rightarrow Y^\circ$ be the non-singular double cover as above;
- (iii) take several line bundles $\mathcal{L}_1, \dots, \mathcal{L}_m$ on X° such that $\phi_*^\circ\mathcal{L}_i$ is split, and compute 2-bundles $\phi_*^\circ(\mathcal{L}_1^{n_1} \otimes \dots \otimes \mathcal{L}_m^{n_m})$ on Y° by Theorem 2.3;
- (iv) then $j_*\phi_*^\circ(\mathcal{L}_1^{n_1} \otimes \dots \otimes \mathcal{L}_m^{n_m})$ are reflexive sheaves of rank two.

Remark 4.3. Let $\phi : X \rightarrow \mathbb{P}^2$ be a non-singular double cover branched along smooth curve of degree $2r$. In [13], $\phi_*\mathcal{L}$ was studied for several line bundles $\mathcal{L}$ on X in the case of $r = 1, 2$. Ottaviani [12] and Vallès [19] studied the direct images of line bundles on X .

If $\phi : X \rightarrow \mathbb{P}^n$ is non-singular, then the reflexive sheaves in (iv) are 2-bundles. A problem of this method is when a reflexive sheaf of (iv) is a 2-bundle.

Problem 4.4. Give a condition for a reflexive sheaf $j_*\phi_*^\circ(\mathcal{L}_1^{n_1} \otimes \dots \otimes \mathcal{L}_m^{n_m})$ in (iv) to be a 2-bundle.

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