

PL DENSITY INVARIANT FOR TYPE II DEGENERATING K3 SURFACES, MODULI COMPACTIFICATION AND HYPERKÄHLER METRICS

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ABSTRACT. A protagonist here is a new-type invariant for type II degenerations of K3 surfaces, which is explicit *PL (piecewise linear) convex function* from the interval with at most 18 non-linear points. Forgetting its actual function behaviour, it also classifies the type II degenerations into several combinatorial types, depending on the type of root lattices as appeared in classical examples.

From differential geometric viewpoint, the function is obtained as the density function of the limit measure on the collapsing hyperKähler metrics to conjectural segments, as in [HSZ19]. On the way, we also reconstruct a moduli compactification of elliptic K3 surfaces by [Brun15, AB19, ABE20] in a more elementary manner, analyze the cusps more explicitly.

We also interpret the glued hyperKähler fibration of [HSVZ18] as a special case from our viewpoint, discuss other cases, and possible relations with Landau-Ginzburg models in the mirror symmetry context.

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1. INTRODUCTION

In this paper, to each type II degeneration of polarized K3 surfaces $\pi: (\mathcal{X}, \mathcal{L}) \rightarrow \Delta = \{t \in \mathbb{C} \mid |t| < 1\}$, we associate an explicit piecewise linear convex function $V = V_\pi: [0, 1] \rightarrow \mathbb{R}_{\geq 0}(\cup \infty)$ over the interval, as a new type invariant and discuss its geometric meanings from various geometric perspectives. The non-differential points of V are at most 18 points and anyhow the behaviour of V is completely classified. If $\mathcal{L}$ are assumed to be of relative Hodge (integral) class as in algebraic geometry, the function L is rational while if we extend to relative Kähler class on (not necessarily algebraic) $\mathcal{X}$, then we obtain not necessarily rational bend points.

From differential geometric perspective, this is done by considering the behaviour of hyperKähler metrics on the fibers $\mathcal{X}_t = \pi^{-1}(t)$ with the Kähler class in $\mathbb{R}_{>0}c_1(\mathcal{L}_t = \mathcal{L}|_{\mathcal{X}_t})$ with diameter bounded rescale, as our function V is the density function of a limit measure on the conjectural limit interval as predicted in recent [HSZ19]. As inferred from such background, we can actually define V for not only holomorphic one parameter degeneration but for more general sequences “of type II”.

The ends behaviour of V is encoded in the root lattices of type D or E while the open part is reflected in type A lattices. This root lattice-theoretic information has classically appeared and studied at least in lower degree case e.g., in [Fri84], and also in recent [AET19, §3B, 9.10], [LO19, §1], and [ABE20]. Our exploration aims to reveal their hidden meanings.

History of this work. This paper originally stems out as a part of the series for ongoing joint work with Y.Oshima on collapsing of hyperKähler metrics, with recent focus on K3 surfaces to segments, with great inspirations input from [HSZ19] and [ABE20] as well. Our whole framework depends on the one initiated in our previous joint paper [OO21] (its short summary is [OO18]), whose particular focus of the latter part was on type III degenerations and associated collapsing to spheres.

Also the recent log KSBA style explicit compactification work of moduli of *elliptic* K3 surfaces by [Brun15, ABE20, AB19] has much to do with our work. In particular, V implicitly appears in [ABE20] in the form of their integral affine spheres construction, and used in the projective moduli variety construction, much to our surprise then. Next, we give the comparison, partially to give an overview of this paper.

Comparison and Organization. While [ABE20, §7A] implicitly obtained the definition of V in the form of its “graphs” as integral affine spheres, Oshima ([Osh]) also had definition of V independently, as a function for the collapsing K3 surfaces to segment. Then he proves that it is the limit measure of the McLean metric on $\mathbb{P}^1$, from periods calculation along explicit 2-cycles.

Part 2 of this paper provides another algebro-geometric proof of the theorem of Oshima, which is the heart of the paper. Before that, in preparatory Part 1, we give an elementary reproof and analysis of the stable reduction corresponding to [ABE20]. The reproof has virtue for the arguments in Part 2. More precisely speaking, the information of asymptotic behaviour of singular fibers analyzed in Part 1, not only the location of limits of discriminants, is crucially used in Part 2.

In Part 2, we also connect our work with [HSVZ18]. More precisely, we interpret *loc.cit* as a special case when the lattice theoretic label (combinatorial information) of our PL function V is of “type EAE”. Note that there are other interesting cases whose label include “type D” at the ends and also the label can be longer. Indeed, it can involve at most 20 lattices as the longest possibility, as it readily follows from [ABE20] or our discussions.

Part 1 can and does work over an arbitrary algebraically closed field K of characteristic neither 2 nor 3, unless otherwise stated. The assumption on characteristic is frequently used, especially for the Weierstrass standard form description of elliptic curves and the reducedness of the finite group schemes μ_2 and μ_3 over K . On the other hand, Part 2 works over $\mathbb{C}$ as, for instance, discussions involve hyperKähler metrics.

Acknowledgement. As noted above, this paper stems out as a part of collaboration with Y.Oshima, and we plan for more sequels. So first of all, the author thanks Y.Oshima for the ongoing fruitful and enjoyable discussions, as well as the permission to emit this part of results in this form.

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Part 1. Moduli of elliptic K3 surfaces revisited

2. REVIEW OF [OO21, §7] AND ANALYSIS OF CUSPS

In the work [OO21, §7] on collapsing of K3 surfaces, the moduli $M_W(\mathbb{C})$ of complex Weierstrass elliptic K3 surfaces played an important role as it

parametrizes real 2-dimensional collapses (“tropical K3 surfaces”) of Kähler K3 surfaces. Still keeping it as one of the motivations, we first make further analysis on M_W in this paper. It also naturally extends to other field K . First, we set up or recall the notation.

We set $\mathbb{A}^{22}$, which parametrizes the coefficients of degree 8 polynomial g_8 and the coefficients of degree 12 polynomial g_{12} .

Recall from [OO21, §7.1] that $\overline{M}_W$ is nothing but the GIT quotient of $\mathbb{A}_{g_8, g_{12}}^{22} \setminus \{0\}$ by the action of $\mathrm{GL}(2)$, or in other words, that of

$$(1) \quad \mathbb{P}(\underbrace{2, 2, 2, 2, 2, 2, 2, 2, 2}_9, \underbrace{3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3}_{{13}})$$

$$(2) \quad (= (\mathbb{A}_{g_8, g_{12}}^{22} \setminus \{0\}) / \mathbb{G}_m(K))$$

by the further action of $SL(2)$. We denote the homogeneous coordinates of the base $\mathbb{P}^1$ as s_1 and s_2 , and set $s := \frac{s_1}{s_2}$.

Recall that [OO21, §7.2.1] shows $\overline{M}_W$ is isomorphic to the Satake-Baily-Borel compactification for an appropriate $O(2, 18)$ orthogonal symmetric variety (and also has the structure as a double (*anti-(!)*holomorphic) covering over the boundary component $\mathcal{M}_{K3}(a)$ of $\overline{M}_{K3}$ in [OO21, §6]), which appears in the context of F-theory e.g., as *classical F-theory moduli space* in [CM05]. See [OO21, §6.1], in particular its last discussion for the proof of Theorem 6.6 of §7.3.7 in *loc.cit* for the details.

Now we head towards more explicit understanding of cusps of the compactification. From the uniformization structure $M_W \simeq \Gamma \backslash \mathcal{D}$, with orthogonal symmetric domain $\mathcal{D}$, there is the natural branch divisor B in M_W with the standard coefficients. From [Mum77, Proposition 3.4], it follows that $(\overline{M}_W, \overline{B})$ is the log canonical model and the three cusps M_W^{nn} and M_W^{seg} , $M_W^{\mathrm{nn}} \cap M_W^{\mathrm{seg}}$ are the set of (all) log canonical centers.

An important point to notice is that $\mathrm{Supp}(\overline{B})$ actually contain both of M_W^{nn} and M_W^{seg} . Indeed, as we see below later, M_W (without the branch divisor) are log terminal around both log canonical centers.

More direct way to see it is as follows. Recall from [OO21, §7.1.5] that the locus S_b corresponding to (b) in *loc.cit*, i.e., the surface in M_W isomorphic to $\mathbb{A}^1 \times \mathbb{A}^1$, parametrize Kummer surfaces for the product of elliptic curves $E_1 \times E_2$ and the closure include both M_W^{nn} and M_W^{seg} . As [OO21, Proposition 7.8] shows, for all such Kummer surfaces, the corresponding Weierstrass models contain four D_4 -singularities which are ordinary cusps fiberwise, as a birational transform of $(E_1 \times E_2)/(\mathbb{Z}/2\mathbb{Z})$. The Heegner divisor of M_W , which corresponds to their partial smoothings with a single A_1 -singularity, contain the locus S_b obviously.

2.0.1. *Around M_W^{nn} .* As the locus $M_W^{nn} \setminus M_W^{\text{seg}}$ locates inside the (strictly) stable locus inside the GIT quotient $\bar{M}_W$ (cf., [OO21, §7.1.1]) it follows that the stabilizer of the $GL(2)$ -action on $\mathbb{A}^{22}$ which represents a point inside M_W^{nn} is finite. Furthermore, it is generically the Klein four group, i.e., $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ and becomes larger only at finite points in M_W^{nn} (e.g., when the corresponding degree 4 polynomial G_4 is $s_1 s_2 (s_1 - s_2)(s_1 + s_2)$ (or $s^3 - s$ in the way written in [OO21]) so that the corresponding stabilizer group is $(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$).

Before our statements, we define the following singularity.

Definition 2.1. A canonical Gorenstein 3-fold singularity whose germ is written as

$$(3) \quad \vec{0} \in [X^2 = YZW] \subset \mathbb{A}^4$$

are denoted as $\mathcal{A}_1^{(3)}$ in this paper. Indeed, each component of the singular locus meeting at $\vec{0}$,

- $X = Z = W = 0, Y \neq 0$
- $X = Y = W = 0, Z \neq 0$
- $X = Y = Z = 0, W \neq 0$

are transversally 2-dimensional A_1 -singularity (cA_1), hence the name. It is also easy to see that this coincides with the quotient singularity by $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2} = K_4$ of $\mathbb{A}_K^3$ acting by the eigenvalues

$$\begin{array}{ll} (1, 1, 1, 1) & \text{(by the unit } e \text{ of } K_4), \\ (1, -1, 1, -1) & \text{(by an element } a \text{ of } K_4), \\ (-1, -1, 1, 1) & \text{(by an element } b \text{ of } K_4), \\ (-1, 1, 1, -1) & \text{(by the element } ab \text{ of } K_4). \end{array}$$

Theorem 2.2. *At general points in M_W^{nn} , M_W is formally (hence also analytically if $K = \mathbb{C}$) isomorphic to*

$$(4) \quad (\mathcal{A}_1^{(3)} \times \mathcal{A}_1^{(3)} \times \mathcal{A}_1^{(3)} \times \mathcal{A}_1^{(3)}) \times \mathbb{A}^6,$$

hence canonical Gorenstein singular in particular.

It is interesting as, with the branch divisor, it becomes one of strictly log canonical locus.

Proof. We use the Luna slice theorem [Luna73] (see also the exposition [Dre04, 5.3]). Take a general point p in M_W^{nn} and its lift $\tilde{p}$ to $\mathbb{A}_{g_8, g_{12}}^{22}$ as (P_4^2, P_4^3) , where $P_4 \in \mathcal{O}_{\mathbb{P}^1}(4)$ is of the form $(s_1^2 - \epsilon^2 s_2^2)(s_2^2 - \epsilon^2 s_1^2)$ so that its stabilizer is K_4 generated by

$$\begin{array}{ll} (\text{switch})_\iota: s_1 \mapsto s_2, & s_2 \mapsto s_1, \\ (-1)_{s_1}: s_1 \mapsto -s_1, & s_2 \mapsto s_2. \end{array}$$

Now we construct slice at the above point in $\mathbb{A}_{g_8, g_{12}}^{22}$ with respect to the natural $\mathrm{SL}(2)$ -action as follows. Consider following regular parameter system (or holomorphic coordinates at neighborhood) around $(P_4^2, P_4^3) \in \mathbb{A}_{g_8, g_{12}}^{22}$: they are formed by coeff P_4 , the coefficients of the polynomial P_4 , which is introduced before, and those of

$$(5) \quad R^{\mathrm{rfn}} \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(8)),$$

$$(6) \quad Q^{\mathrm{rfn}} \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4)),$$

$$(7) \quad R'^{\mathrm{rfn}} \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(12)),$$

each of which are linear combinations of:

- (for R^{rfn})

$$\begin{aligned} s_1^3 s_2^5 \pm s_1^5 s_2^3, \\ s_1^2 s_2^6 \pm s_1^6 s_2^2 \end{aligned}$$

- (for Q^{rfn})

$$\begin{aligned} s_1^4 \pm s_2^4, \\ s_1^3 s_2 \pm s_1 s_2^3, \\ s_1^2 s_2^2 \end{aligned}$$

- (for R'^{rfn})

$$\begin{aligned} s_1^{10} s_2^2 \pm s_1^2 s_2^{10}, \\ s_1^9 s_2^3 \pm s_1^3 s_2^9, \\ s_1^8 s_2^4 \pm s_1^4 s_2^8, \\ s_1^7 s_2^5 \pm s_1^5 s_2^7 \end{aligned}$$

and we consider the points

$$(8) \quad (g_8 = P_4^2 + R^{\mathrm{rfn}}, g_{12} = P_4^3 + (3P_4)^2 Q^{\mathrm{rfn}} + R'^{\mathrm{rfn}}),$$

for those $R^{\mathrm{rfn}}, Q^{\mathrm{rfn}}, R'^{\mathrm{rfn}}$ which are generated by special ones above. Then this forms a $\mathrm{stab}(\tilde{p})$ -invariant étale slice. And the action of $\mathrm{stab}(\tilde{p}) \simeq K_4$ whose generators we recall as

$$(9) \quad (\mathrm{switch})_t: s_1 \mapsto s_2, \quad s_2 \mapsto s_1,$$

$$(10) \quad (-1)_{s_1}: s_1 \mapsto -s_1, \quad s_2 \mapsto s_2,$$

act with eigenvalues -1 or 1 on each basis vector above. Looking at the eigenvalues, the assertion readily follows. $\square$

2.0.2. *Around M_W^{seg} .* Now, take a point $p \in M_W^{\text{seg}}$ and its lift $\tilde{p}$ as $(c_1 s_1^4 s_2^4, c_2 s_1^6 s_2^6)$ for some $c_1, c_2 \in K$, and consider the stabilizer group at the point with respect to the natural $GL(2)$ -action, which we denote as $\text{stab}(\tilde{p})$. It is simply isomorphic to $\mathbb{G}_m(K) \rtimes \mu_2(K)$ which acts as either

$$(11) \quad \{s_1 \mapsto cs_1, s_2 \mapsto c^{-1}s_2 \mid c \neq 0\} \text{ or}$$

$$(12) \quad \{s_1 \mapsto cs_2, s_2 \mapsto c^{-1}s_1 \mid c \neq 0\}.$$

From the easy calculation of the tangent space to the orbit $GL(2)\tilde{p}$, we can take $\text{stab}(\tilde{p})$ -invariant étale slice at $\tilde{p}$ as

$$\mathcal{S}(\tilde{p}) := \tilde{p} + \{(\oplus_{0 \leq i \leq 8, i \neq 3, 4, 5} k \cdot s_1^i s_2^{8-i}, \oplus_{0 \leq j \leq 12} k \cdot s_1^j s_2^{12-j})\} \subset \mathbb{A}_{g_8, g_{12}}^{22}.$$

Here we apply the Luna slice theorem [Luna73, Dre04] again to see the local structure around M_W^{seg} . From above description of the slice $\mathcal{S}(\tilde{p})$, it is locally

$$(13) \quad (\mathcal{S}(\tilde{p}) // (\mathbb{G}_m(K) \times \mu_2(K))) \equiv ((\mathbb{A}_K^{18} // \mathbb{G}_m(K)) / \mu_2(K)) \times K.$$

The weights for the $\mathbb{G}_m(K)$ -action on $\mathbb{A}_K^{18}$ are twice the following

$$(14) \quad -4, -3, -2, 2, 3, 4,$$

as which corresponds to the coefficients of g_8 , further followed by

$$(15) \quad -6, -5, -4, -3, -2, -1, 1, 2, 3, 4, 5, 6$$

as which correspond to the coefficients of g_{12} . Recall that in general, affine toric variety is characterized as GIT quotient of affine space by a linear action of some algebraic torus [Cox95, §2]. By applying it to our situation conversely, it follows that $\mathbb{A}_K^{18} // \mathbb{G}_m(K)$ is isomorphic to¹ the 17-dimensional affine toric variety U_σ corresponding to $\mathcal{S}_\sigma = \sigma^\vee \cap M$ defined as follows:

2.0.3. *Cone description.* If we consider $w: \mathbb{R}_{\geq 0}^{18} \rightarrow \mathbb{R}$ the inner product with the above vector $(-4, -3, -2, 2, 3, 4, -6, -5, -4, -3, -2, -1, 1, 2, 3, 4, 5, 6)$, then for $\mathcal{S}_\sigma := \mathbb{Z}^{18} \cap w^{-1}(0)$ and $\sigma := \mathcal{S}_\sigma^\vee$ in the dual vector space $(\mathbb{R}^{18})^\vee$, above GIT quotient corresponds to this $\sigma \subset N \otimes \mathbb{R}$.

It is easy to see this is nothing but the affine cone of self product of weighted projective space

$$(16) \quad \mathbb{P}^8(1, 2, 2, 3, 3, 4, 4, 5, 6) \times \mathbb{P}^8(1, 2, 2, 3, 3, 4, 4, 5, 6)$$

with respect to the $(\mathbb{Q})$ -line bundle $\mathcal{O}(1, 1)$. Therefore, germ at any point in M_W^{seg} in $\bar{M}_W$ is isomorphic to the product of smooth curve with the

¹this isomorphism is also easy to see directly, in this special case since the weights of the $\text{stab}(p)(\simeq \mathbb{G}_m(K))$ -action involve 1 and the acting algebraic torus is one dimensional.

affine cone of $\text{Sym}^2(\mathbb{P}^8(1, 2, 2, 3, 3, 4, 4, 5, 6))$ with respect to the descent of $\mathcal{O}(1, 1)$.

Hence, if we blow up M_W^{seg} with the descent of the vertex, we get

$$(17) \quad \text{Sym}^2(\mathbb{P}^8(1, 2, 2, 3, 3, 4, 4, 5, 6))$$

as fibers over any point at M_W^{seg} . We suspect this corresponds to the variation of two rational elliptic surfaces.

Remark 2.3. Looijenga [Looi76] (cf., also Friedman-Morgan-Witten [FMW97, p.681-682]) proves the following by use of the Weyl formula for affine root systems (Macdonald). We wonder if one can explain somewhat mysterious coincidence of the appeared exponents and those in (16) and (17), in a more systematic manner.

Theorem 2.4 ([Looi76], [BS78], cf., also Pinkham [Pin77], [FMW97]). *For each elliptic curve E , and root lattice Q and its dual root lattice $Q^\vee$, $(E \otimes Q^\vee)/W(Q)$ is isomorphic to the weighted projective space of dimension $\text{rk}(Q)$. The weights are e.g.*

$$(18) \quad \mathbb{P}(\underbrace{1, 1, 1, 1}_4, \underbrace{2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2}_l)$$

for D_l

$$(19) \quad \mathbb{P}(1, 2, 2, 3, 3, 4, 4, 5, 6)$$

for E_8 . Note that if Q is of A, D, E, F, G type, then $Q = Q^\vee$ by their self-duality.

3. ALGEBRO-GEOMETRIC COMPACTIFICATION AFTER [ABE20]

- ELEMENTARY RECONSTRUCTION -

3.1. Introduction to this section. In this section, we reconstruct and analyze one of the algebro-geometric compactifications of M_W recently studied in [ABE20, especially §4C, §7], denoted F^{rc} in *loc.cit.* There was also a preceding work [Brun15] before that, and there is also a closely related independent work [AB19, especially §5 and §9]. In this paper, we call the compactification $\overline{M}_W^{\text{ABE}}$. [ABE20] shows its normalization $\overline{M}_W^{\text{ABE}, \nu}$ is a toroidal compactification, whose corresponding admissible rational polyhedral fan is what they call *rational curves fan* Σ_{rc} ([ABE20, §4C]), as introduced as “ $\mathcal{J}$ ” in [Brun15, Chapter 12], because the considered boundary on K3 surfaces are weighted sum of rational curves in the polarization, as in [YZ96, BL00].

We briefly describe the points of our re-construction of $\overline{M}_W^{\text{ABE}}$, especially the difference with [ABE20]. Our methods certainly overlap with the

discussions in [ABE20] and even some exposition of this section §3 also parallel theirs, but the main point of our logic here is to replace some of essential parts of [ABE20] (especially the implicit/indirect stable reductions) by a simple elementary analysis of Weierstrass normal forms so that the properness follows directly and construction extends even over $\mathbb{Z}[1/6]$. Also there is an independent nice work by [AB19] which constructs $\overline{M}_W^{\text{ABE}}$ and described the boundary components in *loc.cit* section 9 (of version3), mainly from the viewpoints of the minimal model program again and twisted stable maps of [AV02].

In turn, our method based on explicit analysis of Weierstrass equations also helps to answer certain differential geometric questions shared with Y.Oshima after the paper [HSZ19] and fruitful discussions with S.Honda. More precisely, as it is culminated in §4.2, we show and determine the very rich nontrivial moduli of all the *limit measures* of (further) Gromov-Hausdorff collapses from tropical K3 surfaces to an interval. For algebraic geometers, one can say that this gives a new invariant for type II degenerations of K3 surfaces, as a PL function of one real variable.

As another virtue for algebraic perspective of the reconstruction, we also do *not* rely on the general theory of Kollár-Shepherd-Barron-Alexeev moduli of semi-log-canonical models, which in turn depends on the Minimal Model Program (3-dimensional relative semistable MMP in this case). Furthermore, from our construction, the presence of fibration structures on each degenerate surface come for free, which [ABE20, §7C] proved by some discussions on periods and deformation theory.

Furthermore, our (re-)proof also do *not* logically use the tropical K3 surfaces or the key PL functions although we finally aim to clarify the meaning of those tropics appeared in [ABE20] and [Osh]. We expect that this reconstruction also provides convenience for future study of limits of K3 metrics at *different rescale*.

In this section, we first briefly review the irreducible components of stable degenerations introduced in [ABE20] (see also [AB19, 8.13]) and give alternative description to each.

3.2. Preparation.

3.2.1. Notations and Convention.

- (recall) the base $\mathbb{P}^1$ of elliptic K3 surfaces in our concern, has homogeneous coordinates s_1, s_2 and $s := s_1/s_2$.
- $g_8 = \sum_i a_i s^i \in H^0(\mathbb{P}_s^1, \mathcal{O}(8) = \mathcal{O}(8[\infty]))$,
- $g_{12} = \sum_i b_i s^i \in H^0(\mathbb{P}_s^1, \mathcal{O}(12) = \mathcal{O}(12[\infty]))$,
- $\Delta_{24} = \sum_i d_i s^i \in H^0(\mathbb{P}_s^1, \mathcal{O}(24) = \mathcal{O}(24[\infty]))$.
- $g_4 \in H^0(\mathbb{P}_s^1, \mathcal{O}(4) = \mathcal{O}(4[\infty]))$,

- $g_6 \in H^0(\mathbb{P}_s^1, \mathcal{O}(6) = \mathcal{O}(6[\infty]))$,
- Following [FriMrg94, e.g. §1.4], $\mathbb{P}_{\mathbb{P}_s^1}(\mathcal{E})$ for a locally free coherent sheaf $\mathcal{E}$ denotes the projective bundle as the *covariant* projectivization, rather than Grothendieck contravariant notation.
- (Notes added in the revision) 8 months after the first version of this paper on arXiv, the preceded preprint [ABE20], which had been a great inspiration for us, is revised to its version 4 in which the authors change some notation. In particular, the notation of the types of the components of the degenerations from $\mathbb{A}, \mathbb{D}, \mathbb{E}, \tilde{\mathbb{D}}, \tilde{\mathbb{E}}$ to $A, C, E, \tilde{C}, \tilde{E}$. We use and keep the notation after their previous version as we explicitly and independently describe in §3.2.3, §3.2.4, §3.2.5, and §3.2.6.

3.2.2. *Degenerate surfaces over the compactified moduli by [ABE20].* We briefly recall that the degenerate surfaces over the boundary of $\overline{M}_W^{\text{ABE}}$. We explore and classify the prime divisors later in §3.3.

First we focus on the type III degenerations parametrized on the normalization of $\overline{M}_W^{\text{ABE}}$ i.e., the toroidal compactification $\overline{M}_W^{\text{toroidal}, \Sigma_{\text{rc}}}$ with respect to the rational curves cone Σ_{rc} ([Brun15, §12], [ABE20, §4C]), which first parametrizes special Kulikov degenerations up to the flops of the “Kulikov type” either:

$$\begin{cases} XI \cdots IX \\ XI \cdots IY \\ YI \cdots IY. \end{cases}$$

Each symbol refers to a irreducible components, but they are not all the components. We omitted the subindices (called “charge” as invariant of the integral affine singularities, in [AET19, ABE20]), whose sum is 24. When we pass to the ultimate KSBA degeneration, then many of the components are contracted so that we get a surface of the “stable type”:

$$\begin{cases} \mathbb{D}\mathbb{A} \cdots \mathbb{A}\mathbb{D} \\ \mathbb{D}\mathbb{A} \cdots \mathbb{A}\mathbb{E} \\ \mathbb{E}\mathbb{A} \cdots \mathbb{A}\mathbb{E}, \end{cases}$$

respectively, as X turns to $\mathbb{E}$ with subindex 3 less, $Y_2 Y_{d+2}$ turns to $\mathbb{D}$ with (total) subindex 4 less, and I turns to $\mathbb{A}$ with subindex 1 less during this contraction process. These $\mathbb{A}, \mathbb{D}, \mathbb{E}$ corresponds to the root lattices of the same symbols.

From here, we recall some of the surface components including Type II case, and give some different elementary descriptions for our purpose of the reconstruction of $\overline{M}_W^{\text{ABE}}$.

3.2.3. $\mathbb{A}$ -type surface. About the $\mathbb{A}$ -type surface ([ABE20, §7G]), we have nothing new to add to [ABE20, §7G] so we simply recall it for readers' convenience. For the nodal rational curve C i.e., the rational curve with only one singularity which is the node, consider $C \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ with marked k fibers over the points which are neither over 0 nor ∞ . The normalization is $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$.

3.2.4. $\mathbb{D}$ -type surface. For any square-free quadric polynomial P_2 of s , regarded as an element of $H^0(\mathbb{P}_s^1, \mathcal{O}(2) = \mathcal{O}(2[\infty]))$, the fibers of

$$(20) \quad X_{3P_2^2, P_2^3}^W := [y^2z = 4x^3 - 3P_2^2xz^2 + P_2^3z^3] \rightarrow \mathbb{P}_s^1$$

$$(21) \quad = [y^2z = (2x - P_2z)^2(x + P_2z)]$$

$$(22) \quad \subset \mathbb{P}_{\mathbb{P}_s}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}),$$

as fibration over $\mathbb{P}_s^1$, are generically (irreducible) nodal rational curves, with at most 2 cuspidal rational curves over the roots of P_2 . Recall that our notation for projective bundle is, as §3.2.1 declares, the covariant projectivization of a locally free sheaf.

The normalization of this surface is the $\mathbb{P}^1$ -fiber bundle with fiber coordinates $[y : x - P_2z]$, which is $\mathbb{P}_{\mathbb{P}_s^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}(1))$, the Hirzebruch surface $\mathbb{F}_1$. So, as P_2 is square-free, the surface coincides with the underlying fibered surface of $\mathbb{D}_k$ -surface (k only makes difference of the boundary divisors) which [ABE20, §7G] writes. The fiberwise ordinary cusps are simply pinch points as [ABE20].

As we note in §3.2.1, our notation follows version 3 of [ABE20] and version 4 of *loc.cit* refers to the above surface as type C . Recall that the lattice generated by the root system of C_n is the same as that corresponding to D_n , although the lengths of roots are not all same.

3.2.5. $\mathbb{E}$ -type and $\tilde{\mathbb{E}}$ -type surface. For general g_4, g_6 ,

$$(23) \quad X_{g_4, g_6}^W := [y^2z = 4x^3 - g_4xz^2 + g_6z^3] \subset \mathbb{P}_{\mathbb{P}_s}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1})$$

becomes a rational elliptic surface with only ADE singularities. (cf., [Mir81, Kas77]). We specify the I_{9-k} Kodaira type fiber ([Kod63]) as the boundary, then we call this type of log surface $\mathbb{E}_k$ ($k = 1$ has two types). If k reaches 9, we rather denote $\tilde{E}_9$ which is nothing but the rational elliptic surface minus a smooth elliptic curve fiber.

Here, we allude to the fact that this $\mathbb{E}_k (k \leq 8)$ surface (resp., $\tilde{E}_9$) is exactly the Landau-Ginzburg model for Del Pezzo surfaces (resp., rational elliptic surface) in the context of mirror symmetry as [AKO06] showed the homological mirror symmetry type statement. Furthermore, the associated lattices coincides with those of Del Pezzo surfaces ([Manin, Chapter IV, §25]). See [CJL21] for related work.

3.2.6. $\tilde{\mathbb{D}}$ -type surface. We discuss $\tilde{\mathbb{D}}_{16}$ -type surface similarly to above §3.2.4. For a square-free quartic polynomial $G_4 \in H^0(\mathbb{P}_s^1, \mathcal{O}(4[\infty]))$, we consider as in [OO21, §7] the explicit surface

$$(24) \quad X_{3G_4^2, G_4^3}^W : [y^2 z = 4x^3 - 3G_4^2 x z^2 + G_4^3 z^3] \rightarrow \mathbb{P}_s^1$$

$$(25) \quad = [y^2 z = (2x - G_4 z)^2 (x + G_4 z)]$$

$$(26) \quad \subset \mathbb{P}_{\mathbb{P}_s}(\mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(6) \oplus \mathcal{O}_{\mathbb{P}^1}).$$

This is a generically nodal curve fibration, with exactly 4 cuspidal rational curves degenerations over the roots of G_4 (see [OO21, §7.1.1 and §7.1.3] for details). The normalization of this surface is the $\mathbb{P}^1$ -fiber bundle with fiber coordinates $[y : x - G_4 z]$, which is $\mathbb{P}_{\mathbb{P}_s^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$, the Hirzebruch surface of degree 2, i.e., $\mathbb{F}_2$.

We remark here that the log KSBA surface parametrized along the same strata as [ABE20, §7F] consists of 18 components and the middle ruled components are all not open K-polystable in the sense of [Od20a], unless the 16 $\mathbb{P}^1$ s on the top components all the way flopped down to the bottom components so that all the middle components become trivial $\mathbb{P}^1$ -bundle over the elliptic curve.

As we note in §3.2.1, our notation follows version 3 of [ABE20] and version 4 of *loc.cit* refers to the above surface as type $\tilde{C}$.

3.2.7. Mutations of Y -surfaces. Recall from [ABE20] that two type of parts of Kulikov degenerations $(Y_2)Y_2(I_a \cdots)$ and $(Y_2)Y_2'(I_a \cdots)$, modulo corner blowups, parametrized at the toroidal compactification $\overline{M}_W^{\text{toroidal}, \Sigma_{\text{rc}}}$ are *not* distinguished once we contract them to the KSBA models (they become $\mathbb{D}_0 \mathbb{A}_{a-1} \cdots$ type) parametrized at $\overline{M}_W^{\text{ABE}}$. This is the main reason of non-normality of $\overline{M}_W^{\text{ABE}}$, as explained in [ABE20, §7I].

Here, we reinterpret this by *elementarily* (by explicit equations) construct a one parameter family of fibered surfaces at one parameter family level, hence total space 3-dimensional $\tilde{\pi}$ at (31) soon by using only pure algebraic geometry of algebraic surfaces and simple birational geometry. At one end of the one parameter family, we have $Y_2(I_a \cdots)$ surface while the other end we see degeneration to $Y_2'(I_a \cdots)$. The generic fiber is $Y_3(I_a \cdots)$. This is

the transition we should observe at the outer (and left) part of the [ABE20, §7] type Kulikov degeneration.

For that purpose, recall the Hirzebruch surface $\mathbb{F}_1$ and $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^1$ -bundles over the common base $\mathbb{P}^1$ are elementary transforms of each other. Therefore, there is a common non-corner blow up which we write as $\varphi_S: S \rightarrow \mathbb{P}^1$ (this corresponds to Y_3 in [ABE20]) and we denote their centers in $\mathbb{F}_i$ are $p_i (i = 0, 1)$. We denote the projections as $\pi_i: \mathbb{F}_i \rightarrow \mathbb{P}^1$ which satisfies $\pi_0 \circ \varphi_0 = \pi_1 \circ \varphi_1$.

In general, if we take a general conic in $\mathbb{P}^2$ and its strict transform D_1 in $S, \mathbb{F}_i (i = 0, 1)$, then the projection to $\mathbb{P}^1$ has two ramifying points as [ABE20, §7B] write. It is easy to see that after the automorphism, we can and do assume that $p_i \in \mathbb{F}_i$ is one of two points $D_1 \cap \pi_S^{-1}(\infty)$ for both i .

Here we use the construction of [Ohno21, §3.1], which originally aimed to partially establish the CM degree minimization conjecture (cf., [Ohno21, Od20c]) in the context of K-stability, by the author. One main point is we consider extra direction by introducing $\mathbb{A}_t^1$. We consider the blow up of $\mathbb{P}^1 \times \mathbb{A}_t^1$ at $(\infty, (t=)0)$ (resp., $(\infty, 1)$), which we denote by

$$(27) \quad \beta_i: B_i \rightarrow \mathbb{P}^1 \times \mathbb{A}^1.$$

Then take the fibre product with

$$(28) \quad \Pi_i = (\pi_i \times id): \mathbb{F}_i \times \mathbb{A}^1 \rightarrow \mathbb{P}^1 \times \mathbb{A}^1,$$

for $i = 0$ (resp., $i = 1$) and further blow up the total space along a smooth closed curve $(\{\infty\} \times (\mathbb{A}^1 \setminus \{i\})) (\simeq \mathbb{A}_K^1)$. Then we obtain²

$$(29) \quad \tilde{\Pi}_i: \mathcal{F}_i \rightarrow \text{Bl}_{(\infty, i)}(\mathbb{P}^1 \times \mathbb{A}^1).$$

We can glue these two for $i = 0, 1$, since the blow ups of $\mathbb{F}_0$ at p_0 and $\mathbb{F}_1$ at p_1 coincides, and obtain

$$(30) \quad \tilde{\Pi} : \mathcal{F} \rightarrow \text{Bl}_{(\infty, 0) \cup (\infty, 1)}(\mathbb{P}^1 \times \mathbb{A}^1),$$

$$(31) \quad \tilde{\pi} := \text{pr} \circ \tilde{\Pi} : \mathcal{F} \rightarrow \text{Bl}_{(\infty, 0) \cup (\infty, 1)}(\mathbb{P}^1 \times \mathbb{A}^1) \rightarrow \mathbb{A}^1.$$

We denote the fiber over t by $\bar{F}_t$. Then, $\mathcal{F}_t$ is

$$(32) \quad (\mathbb{F}_0 \rightarrow \mathbb{P}^1) \cup (S \rightarrow \mathbb{P}^1) = Y'_2 I_a \text{ for } t = 0$$

$$(33) \quad S \rightarrow \mathbb{P}^1 = Y_3 \text{ for } t \neq 0, 1$$

$$(34) \quad (\mathbb{F}_1 \rightarrow \mathbb{P}^1) \cup (S \rightarrow \mathbb{P}^1) = Y_2 I_a \text{ for } t = 1.$$

This interesting family $\{\mathcal{F}_t\}_t$ with two different degenerations at $t = 0$ and $t = 1$ exactly describes the switch between $Y_2 Y_2$ and $Y_2 Y'_2$ in the context of

²the author also used this construction in a joint work with R.Thomas on K-stability in 2013.

[ABE20]. Recall from [ABE20] (also see [Osh]) that the corresponding PL functions to each of (32), (33), (34) starts with slope 8, 7, 8 respectively.

3.2.8. Slight extension of ADE lattices. In [ABE20], over $K = \mathbb{C}$, they used the periods and corresponding Torelli theorems for components of the degeneration of elliptic K3 surfaces after [GHK15, Fri15].

The convention of denoting each components by $\mathbb{A}, \mathbb{D}, \mathbb{E}$ comes from it but for such description, they indirectly used the following slight extension of the usual ADE lattices; allows D_i for $i = 1, 2, 3$ and also E_i for $i = 1, 2, 3, 4, 5$. We logically do not need it until §5 but for the convenience of readers, we clarify here.

The lattice D_i for $i < 4$ is constructed in the same way as those with $i \geq 4$. Simply,

$$D_i := \{(x_1, \dots, x_i) \in \mathbb{Z}^i \mid \sum_j x_j \in 2\mathbb{Z}\}.$$

In our context, with respect to the fundamental domain, these D type lattices are naturally realized in Λ_{seg} as

- $\langle \alpha_1 - \alpha_3 \rangle$ for $i = 1$,
- $\langle \alpha_1, \alpha_3 \rangle$ for $i = 2$,
- $\langle \alpha_1, \alpha_3, \alpha_4 \rangle$ for $i = 3$.

On the other hand, the following inductive construction of E_i (from $i = 1$) is essentially due to Manin [Manin].

We construct a little extended lattice E'_i for $i = 1, 2, \dots$ with $E_i \subset E'_i$ which has corank 1 and orthogonal to K_i . (Geometrically it is fairly simple i.e., $E'_i = H^2(S_{9-i}, \mathbb{Z})$ where S_{9-i} stands for Del Pezzo surface of degree $9-i$ and $c_1(S_{9-i})^\perp = E_i$.) Here is more elementary construction (through “blow up”):

$$\begin{aligned} E'_i &:= \mathbb{Z}l(l^2 = 1), & -K_1 &= 3l. \\ E'_{i+1} &:= E'_i \oplus \mathbb{Z}e_i(e_i^2 = -1) & K_{i+1} &= K_i + e_i. \end{aligned}$$

Allowing above type D lattices and E lattices with lower indices, we call these such A, D, E lattices and their direct sum as *slightly generalized root lattice*. See [LO19, §1] for related discussions.

3.3. Re-construction of [ABE20]. In our logic for the re-construction of the compactification of [ABE20], first we readily construct the desired moduli stack $\overline{\mathcal{M}}_W^{\text{ABE}}$ and then, we show the desired properties especially the properness as well as the presence of projective coarse moduli spaces $\overline{M}_W^{\text{ABE}}$ (F^{rc} in [ABE20]) later.

Our discussion uses the degenerations of the elliptic K3 surfaces parametrized by $\overline{M}_W^{\text{ABE}}$ *simply as a set(!)* and denote them by $(X, R) \in \overline{M}_W^{\text{ABE}}$. First we fix large enough positive integers m and d so that for any $(X, R = s + m \sum f_i) \in \overline{M}_W^{\text{ABE}}$, R is ample and dR is very ample without high cohomology. Obviously, $\chi(X, \mathcal{O}_X(dR))$ does not depend on (X, R) s. Then we take the corresponding Hilbert scheme H' . Naturally, $G := \text{SL}(H^0(X, dR))$ acts on H .

We take a subset H of H' parametrizing the surfaces X parametrized by $\overline{M}_W^{\text{ABE}}$ embedded by dR . Since the subset is characterized as those $\mathcal{O}_{\mathbb{P}(1)}|_X = \mathcal{O}_X(dR)$ (closed condition) as well as the reduced semi-log-canonical-Gorenstein properties of X (open condition), H is a locally closed subset of H' .

Then we put reduced scheme structure on H and set

$$(35) \quad \overline{\mathcal{M}}_W^{\text{ABE}} := [H/G],$$

the quotient (a priori only Artin) stack. It is straightforward to see that, by expanding the meaning, the corresponding moduli contravariant functor (35) represents is the following which we denote by the same symbol $\overline{\mathcal{M}}_W^{\text{ABE}}$: For a scheme T with all residue characteristics other than 2, 3,

$$\overline{\mathcal{M}}_W^{\text{ABE}}(T) := \{(\mathcal{X}, \mathcal{R}) \rightarrow \mathcal{P} \rightarrow T^{\text{red}}\} / \sim,$$

where T^{red} is the reduced subscheme of T i.e., the same topological space as T with the structure sheaf replaced by the reduced quotient, $\mathcal{P} \rightarrow T^{\text{red}}$ is a proper morphism whose geometric fibers are all either $\mathbb{P}^1$ or the chain of $\mathbb{P}^1$ s, $\mathcal{X} \rightarrow \mathcal{P}$ is a flat proper family of surfaces, $\mathcal{R}$ is its relative Cartier divisor such that for all $t \in T$, the fiber $(\mathcal{X}_t, \mathcal{R}_t) \rightarrow \mathcal{P}_t$ is a fiber of universal family over H , and $\sim$ refers to the natural isomorphism as an equivalence relation. See also [PS21, Definitions 4.1, 4.3 (also 1.1, 1.2)] (cf., also [ASD73, §1]) for the description of open subfunctor, in which $\mathcal{R}$ is a redundant data.

Now we prove this is actually a proper Deligne-Mumford stack (i.e., stable reduction type statements) case by case, so that we reprove the following in an elementary way. (Of course, we do not mean to be short arguments, by the word “elementary”.)

Theorem 3.1 (cf., [ABE20]). *The moduli algebraic stacks (constructed above) $\mathcal{M}_W \subset \overline{\mathcal{M}}_W^{\text{ABE}}$ of elliptic K3 surfaces and their degenerations over $\text{Spec}(\mathbb{Z}[1/6])$, (the former is an open substack of the latter) both admit the coarse moduli varieties $M_W \subset \overline{M}_W^{\text{ABE}}$ (the former is an open subvariety of the latter) such that $\overline{M}_W^{\text{ABE}}$ is projective.*

The reason we work away from characteristics 2 and 3 are that our following arguments rely on the (uniform) description of equation of the Weierstrass models (cf., [Mir81], also [Sil85, Appendix A], [PS21]) and their degenerations.

Elementary direct reproof. The existence of coarse moduli spaces as algebraic spaces follows from [KeMo97], since the inertia groups of the moduli stack are nothing but the automorphism of log canonical model $(X, \epsilon R)$ which is finite cf., [Iit82, Chapter 11], [Amb05, Proposition 4.6]). The projectivity follows from the ampleness of the determinant of direct image sheaves of pluri-log-canonical bundles [KP17],[Fjn18].

Therefore, to reprove Theorem 3.1, it remains to show the following key claim from the valuative criterion of properness *relative to* $\mathrm{Spec}(\mathbb{Z})[1/6]$ (e.g., [LM00, §7]). In particular, the uniqueness part shows that the reconstructed compactification in this section and [ABE20] are identical.

Theorem 3.2 (stable reduction cf., [ABE20]). *For any field K of characteristic different from 2 and 3, and any $(X, R) \rightarrow \mathbb{P}_s^1$ parametrized in $\overline{\mathcal{M}}_W^{\mathrm{ABE}}(K((t)))$, $(X, R) \rightarrow \mathbb{P}_s^1$ has a unique (explicit) model $(\mathcal{X}, \mathcal{R}) \rightarrow \mathcal{B}$ over $K[[t]]$ in $\overline{\mathcal{M}}_W^{\mathrm{ABE}}(K[[t]])$.*

We fix further notations before giving the details of the proof.

3.3.1. Some further notations.

- K denotes the field we take in Theorem 3.2, whose characteristic is coprime to 6. Recall that we use s for the corresponding coordinate, virtually valued in K .
- Since we only wish to prove properness of the above quotient algebraic stack, we can and do assume the field K is actually algebraically closed, just for simpler exposition.
- We denote the obvious trivial model $\mathbb{P}_s^1 \times \mathrm{Spec}(K[[t]])$ of $\mathbb{P}_s^1 \times \mathrm{Spec}(K((t)))$ as $\mathcal{B}_{\mathrm{triv}}$. We make birational transforms of this $\mathcal{B}_{\mathrm{triv}}$ to other model $\mathcal{B}$.
- Discriminant locus of $[(X, R) \rightarrow \mathbb{P}_s^1] \in \mathcal{M}_W(K((t)))$ as $D \subset \mathcal{B}$. The fibers over its reduction $\overline{D} \cap (t = 0) \subset \mathcal{B}$ are called *really singular* in [ABE20] which we follow to use. We call their underlying closed points in the base as *real discriminant (points)*.

Proof of Theorem 3.2. The uniqueness part follows from a standard birational geometric result i.e., the general uniqueness of relative log canonical model, not relying on the existence of difficult birational models nor the minimal model program. As this argument is well-known to experts, we only sketch the proof for the normal case. Non-normal case is similarly proven after normalization. Suppose we have a priori two different models

$[\pi_i: (\mathcal{X}_i, \mathcal{R}_i) \rightarrow \mathcal{P}_i \rightarrow S] \in \overline{\mathcal{M}}_W^{\text{ABE}}(S)$ for $i = 1, 2$ over a DVR S with the same generic normal fiber, we take a bigger normal model $\tilde{\pi}: \tilde{\mathcal{X}} \rightarrow S$ which dominates both $\mathcal{X}_i \rightarrow S$ by morphisms $f_i: \tilde{\mathcal{X}} \rightarrow \mathcal{X}_i$ as S -schemes. We denote the exceptional divisors of f_i with all coefficients 1 as E_i . Then, since $(\mathcal{X}_i, \mathcal{R}_i)$ ($i = 1, 2$) are both relative log canonical models over S , it follows that $\text{Proj}_S(\tilde{\pi}_* \oplus_{m \geq 0} \mathcal{O}(mK_{\tilde{\mathcal{X}}/S} + (f_i)_*^{-1}\mathcal{R}_i + E_i))$ is isomorphic to $\mathcal{X}_i$ as S -scheme for both $i = 1, 2$. Hence $\mathcal{X}_1 \simeq \mathcal{X}_2$ so that $\mathcal{R}_i$ and $\mathcal{P}_i$ also coincides for $i = 1$ and $i = 2$.

Hence, we focus on the explicit construction of the desired stable reduction to each punctured families lying on $\mathcal{M}_W$. By lifting to $\mathbb{A}^{22}$, reduce to the following four cases: Case 1 to Case 4.

Case 1 (Type III degenerations from M_W). This case amounts to show the following claim:

Claim 3.3 (Maximally degenerating stable reduction). *Given any $g_8(s)$ in $\Gamma(H^0(\mathbb{P}_s^1, \mathcal{O}(8))) \otimes K[[t]]$ (resp., $g_{12}(s)$ in $\Gamma(H^0(\mathbb{P}_s^1, \mathcal{O}(12))) \otimes K[[t]]$) such that*

$$(36) \quad X_{g_8, g_{12}}^W|_{t \neq 0} := [y^2z = 4x^3 - g_8(t)xz^2 + g_{12}(t)z^3]$$

$$(37) \quad \subset \mathbb{P}_{\mathbb{P}_s^1}(\mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(6) \oplus \mathcal{O}_{\mathbb{P}^1}),$$

as in [OO21, §7.1] is an elliptic K3 surfaces parametrized in $M_W(K((t)))$ i.e., only with ADE singularities and $g_8|_{t=0} = 3s^4, g_{12}|_{t=0} = s^6$ (i.e., converging to $M_W^{nm, seg}$ in the Satake-Baily-Borel compactification (cf., [OO21, §7]), the corresponding $X \rightarrow \mathbb{P}_s^1$ (resp., $\mathbb{P}_s^1 \times \text{Spec}(K((t)))$) over $K((t))$ has another model $\mathcal{X}$ (resp., connected proper scheme $\mathcal{B}$ of relative dimension 1) over $K[[t]]$ so that $\mathcal{X}|_{t=0} \rightarrow \mathcal{B}|_{t=0}$ is (the only possible) one of those parametrized in $\overline{M}_W^{\text{ABE}}$.

Step 1 (End surfaces). To prove the above Claim 3.3, first we take finitely ramified base change from $K[[t]]$ to $K[[t^{1/d}]]$ for some $d \in \mathbb{Z}_{>0}$, so that we can and do assume the roots of $g_8, g_{12}, \Delta_{24} := g_8^3 - 27g_{12}^2$ are Laurent (not only Puiseux), i.e., there are $\xi_i \in K((t))$ ($i = 1, \dots, 8$), $\eta_i \in K((t))$ ($i = 1, \dots, 12$), $\chi_i \in K((t))$ ($i = 1, \dots, 24$) in the descending order of the valuations $v_t(-)$ along coordinates s with respect to t (or additive inverse of the valuation of s^{-1}).

We first set

$$(38) \quad e(0) := \min\{\text{val}_t(\xi_1), \dots, \text{val}_t(\xi_4), \text{val}_t(\eta_1), \dots, \text{val}_t(\eta_6)\},$$

(39)

$$e(\infty) := \min\left\{\mathrm{val}_t\left(\frac{1}{\xi_5}\right), \dots, \mathrm{val}_t\left(\frac{1}{\xi_8}\right), \mathrm{val}_t\left(\frac{1}{\eta_7}\right), \dots, \mathrm{val}_t\left(\frac{1}{\eta_{12}}\right)\right\}.$$

and after an appropriate elementary transform of the trivially extended $\mathbb{P}^1$ -bundle over $\mathbb{P}_s^1 \times_K K[[t]]$ (we fix this ambiguity below soon), further blow it up to $\mathcal{B}_1 \rightarrow \mathcal{B}_{\mathrm{triv}}$ by the coherent ideal sheaf

$$(40) \quad \langle s, t^{e(0)} \rangle \cdot \langle s', t^{e(\infty)} \rangle \cdot \mathcal{O}_{\mathcal{B}_{\mathrm{triv}}}.$$

Here, $s' := \frac{s_2}{s_1}$ and it is also a local uniformizer at $[s_1 : s_2] = [1 : 0]$ (“ ∞ -point”) in the base $\mathbb{P}_s^1$. Then, the special fibre of $\mathcal{B}_1$ over $t = 0$ is

$$(41) \quad \mathbb{P}_{\frac{s}{t^{e(0)}}}^1 \cup \mathbb{P}_s^1 \cup \mathbb{P}_{\frac{s'}{t^{e(\infty)}}}^1$$

where the two ends are exceptional curves.

Accordingly, we can naturally degenerate the ambient space $\mathbb{P}_{\mathbb{P}_s^1}(\mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(6) \oplus \mathcal{O}_{\mathbb{P}^1})$ over $K((t))$ to over $K[[t]]$ so that the special fiber over $t = 0$ is a connected union of the following three irreducible components:

- (i) $\mathbb{P}_{\frac{s}{t^{e(0)}}}^1(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1})$ over $\mathbb{P}_{\frac{s}{t^{e(0)}}}^1$
- (ii) trivial $\mathbb{P}^2$ -bundle over $\mathbb{P}_s^1$ (i.e., $\mathbb{P}^2 \times \mathbb{P}_s^1$)
- (iii) $\mathbb{P}_{\frac{s'}{t^{e(\infty)}}}^1(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1})$ over $\mathbb{P}_{\frac{s'}{t^{e(\infty)}}}^1$.

Inside the first component (i), the closure of X (“limit component”) appears as

$$(42) \quad X_{g_4^\nu, g_6^\nu}^W := [y^2 z = 4x^3 - g_4^\nu|_{t=0} x z^2 + g_6^\nu|_{t=0} z^3],$$

where $g_4^\nu = c_4 \prod_{i=1}^4 (s - \xi_i)$, $g_6^\nu = c_6 \prod_{i=1}^6 (s - \eta_i)$, with replaced roots ξ s and η s. Recall that construction of the model $\mathcal{B}_1$ above had an ambiguity modulo elementary transform with respect to $t = 0$ but we fix it by assuming $(c_4, c_6) \in K^2 \setminus \vec{0}$. From the construction, g_4^ν and g_6^ν are strictly degree 4 and 6 respectively with coefficients 3 and 1 respectively, $\Delta_{12}^\nu := (g_4^\nu)^3 - 27(g_6^\nu)^2$ has degree at most 11. This means the component $X_{g_4^\nu, g_6^\nu}^W$ has singular fiber over ∞ , which corresponds to the fact that the degeneration is of type III.

Also, from the definition of $e(0)$, not all of ξ_i s and η_i s vanish. Similarly, in the last component (iii), the closure of X (“limit component”) appears as

$$(43) \quad X_{h_4^\nu, h_6^\nu}^W := [y^2 z = 4x^3 - h_4^\nu|_{t=0} x z^2 + h_6^\nu|_{t=0} z^3],$$

where $h_4^\nu = \prod_{i=5}^8 (s - \xi_i)$, $h_6^\nu = \prod_{i=7}^{12} (s - \eta_i)$, again with newly replaced roots ξ s and η s. From the construction, due to [Kas77, Lemma1], if Weierstrass surfaces are generically smooth, they automatically only have ADE singularities (at non-zero finite base coordinates).

When $K = \mathbb{C}$, in comparison with our asymptotic analysis of McLean's real Monge-Ampère metrics in [OO21, §7.3.3], these “end surfaces” are where the term (denominator of the second term in [OO21, Lemma 7.16])

$$(44) \quad \log(|g_8|^3 + 27|g_{12}|^2)$$

becomes dominant. On the other hand, the following next step is relevant to expand the divergence of the $\log(|\Delta_{24}|)$ term.

Step 2 (Separating “middle” χ_i s). Next step we consider toric model $\mathcal{B}$ with respect to some combinatorial data coming from the Newton polygon, as the method used classically by [Mum72b, AN99, Don02] as follows. We consider the Newton polygon $\text{Newt}(\Delta_{24})$ of Δ_{24} i.e., the convex hull of

$$(45) \quad \{(i, v_t(d_i)) \mid 0 \leq i \leq 24\} + \mathbb{R}_{\geq 0}(0, 1).$$

We regard it as a graph of PL convex function $\varphi_\Delta: [0, 24] \rightarrow \mathbb{R} \cup \{\infty\}$. Then we modify this as follows (this process aims at including the previous step when we consider the toric models):

Set

$$(46) \quad i_{e(0)} := \max\{i \mid \varphi_\Delta(i) - \varphi_\Delta(i+1) \geq e(0)\},$$

$$(47) \quad i_{e(\infty)} := \min\{i \mid \varphi_\Delta(i+1) - \varphi_\Delta(i) \geq e(\infty)\},$$

where $e(0)$ and $e(\infty)$ as (38) and (39). We modify φ_Δ to $\bar{\varphi}_\Delta: [0, 24] \rightarrow \mathbb{R} \cup \{\infty\}$ defined as follows:

$$(48) \quad \bar{\varphi}_\Delta(i) := \begin{cases} \varphi_\Delta(i_{e(0)}) - e(0)(i_{e(0)} - i) & (\text{if } 0 \leq i \leq i_{e(0)}) \\ \varphi_\Delta(i) & (\text{if } i_{e(0)} \leq i \leq i_{e(\infty)}) \\ \varphi_\Delta(i_{e(\infty)}) + e(\infty)(i - i_{e(\infty)}) & (\text{if } i_{e(\infty)} \leq i \leq 24). \end{cases}$$

Then, consider the toric model (test configuration of $\mathbb{P}^1$) $\mathcal{B}$ over $\mathbb{A}^1$ (hence also over $K[[t]]$), corresponding to $\bar{\varphi}_\Delta$, i.e., for

$$(49) \quad P_{\Delta,c} := \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 24, -c \leq y \leq -\bar{\varphi}_\Delta(x)\}$$

for some $c \geq 0$ the moment polytope of (the natural compactification of) $\mathcal{B}$ becomes $P_{\Delta,c}$.

In particular, the normal fan of the graph of $\bar{\varphi}_\Delta$ gives $\mathcal{B}$ by usual toric construction. We fix and take the natural c such that the obtained $\mathcal{B}$ has the same end components as $\mathcal{B}_1$ in the previous Step 1, i.e., the end components of $\mathcal{B}|_{t=0}$ are the bases $\mathbb{P}^1_{\frac{s}{t^{e(0)}}}$ and $\mathbb{P}^1_{\frac{s'}{t^{e(\infty)}}}$ of the ends at (63). Indeed, it is possible by our modification (73) of the PL function.

Furthermore, as desired, every other components of $\mathcal{B}|_{t=0}$ has at least one point of $D_0 (= \bar{D} \cap (t = 0))$. Here, recall that D denotes the discriminant locus defined after Theorem 3.2 whose closure is denoted as $\bar{D}$. This

ensures the ampleness of the boundary R in the corresponding irreducible components of the Weierstrass (reducible) fibred surface.

Step 3 (About end surfaces again). If the end surface $X_{g_4^\nu, g_6^\nu}^W \rightarrow \mathbb{P}_s^1$ is generically smooth, it is nothing but a rational elliptic surface i.e., type $\mathbb{E}_k$ in [ABE20]. In that case, because of the construction, $\deg \Delta_{12}^\nu = 12$ (Δ_{12}^ν does not vanish at ∞) so that the fiber over $\frac{s}{t^{e(0)}} = \infty$ can not be singular.

On the other hand, if the end surface $X_{g_4^\nu, g_6^\nu}^W$ has singular general fibers, it means that there is $P_2 \in H^0(\mathcal{O}(2[\infty]))$ such that

$$(50) \quad g_4^\nu = 3P_2^2, g_6^\nu = P_2^3.$$

$\deg(P_2)$ can not be less than 2 from the construction. If this P_2 is square-free, then from our discussion in §3.2.4, we get the surface $\mathbb{D}$ type and end the step here. If P_2 is *not* square-free, we continue to next step.

Step 4 (Modifying almost $\mathbb{D}$ type end). Depending on formulation, this process may be included in Step1 but nevertheless we separated it to make the steps clearer. From here, we treat the “left end” surfaces in the original sense of Step1 i.e., those maps to $s = 0$ i.e., defined by g_4^ν and g_6^ν . (For the right end surface which maps to $s = \infty$, the completely similar arguments work by symmetry so we avoid repetition of the details of the arguments.)

We continue from the previous step, so suppose P_2 is *not* square free. Nevertheless, since our generic fiber at $t \neq 0$, $X_{g_8, g_{12}}^W$ was originally at worst ADE, among those (a priori at total 10) roots of g_4^ν or g_6^ν i.e., $\xi_i (1 \leq i \leq 4), \eta_j (1 \leq j \leq 6)$, at least two of them do not coincide as elements of $K[[t]]$ (before substitution $t = 0$). Suppose that they are $\{p, q\} \subset \{\xi_i (1 \leq i \leq 4), \eta_j (1 \leq j \leq 6)\}$ with respect to the new coordinates after Step 1. Write the local uniformizer at $p(0) = q(0)$ for the component $\mathbb{P}_{\frac{s}{t^{e(0)}}}^1$, as $s_{p,q}$.

We make Puiseux expansions of p, q and set $e_{p,q} := v_t(p - q)$, where v_t denotes the t -adic (additive) valuation. Then do blow up of $\mathcal{B}$ (which was the outcome of processes until the previous step) along $\langle s_{p,q}, t^{e_{p,q}} \rangle \mathcal{O}_{\mathcal{B}}$ whose cosupport is in $\mathbb{P}_{\frac{s}{t^{e(0)}}}^1 \times \{t = 0\}$, and blow down the surface without $\bar{D} \cap (t = 0)$ if necessary, we obtain the situation with squarefree P_2 . Note that by this last step, the resulting model $\mathcal{B}$ may *not* be toric, while toroidal, with respect to the original coordinates (since $p(0) = q(0)$ may not be zero).

Case 2 (Type II degenerations). These cases are essentially done in [CM05, §3] via deformation theory and more Hodge-theoretic viewpoint, while the degenerations are slightly modified in [ABE20] (see also [Fri84, Kon85] including non-elliptic case).

Here we again recover them by our elementary method using the Weierstrass form as below.

Subcase. (to $\widetilde{\mathbb{D}_{16}}$) This case essentially follows from the GIT picture in [OO21, §7] by applying the GIT stable reduction. Recall that the Satake-Baily-Borel compactification $\overline{M}_W^{\text{SBB}}$ coincides with the GIT compactification with respect to the Weierstrass expression [OO21, §7.2.1]. As [OO21, §7.1] shows, the locus M_W^{nn} is in the strictly stable locus, which parametrizes the semi-log-canonical surface of the form (24), which is nothing but $\widetilde{\mathbb{D}_{16}}$ -type in [ABE20].

If we have $(g_8, g_{12}) \in H^0(\mathcal{O}(8)) \times H^0(\mathcal{O}(12))$ over the base $K[[t]]$, with reduction sits in the stable locus mapping down to M_W^{nn} , then the GIT stable reduction proves that after finite base change if necessary, if we apply an element of $SL(2)$ in the coefficient $K((t))$, we get reduction with special fiber of the surface of type (24). This completes the required process.

Remark 3.4. By comparing with toroidal compactification, recall that Type II locus does not depend on the choice of admissible rational polyhedral decompositions (cf., e.g., [Fri84]). Furthermore, the preimage of M_W^{nn} in it which we write as $M_W^{\text{nn}, \text{tor}}$ is a $\text{Aut}(D_{16})$ -quotient of the 16-th self fiber-product of the (coarse moduli of) universal elliptic curve over $M_W^{\text{nn}} \simeq \mathbb{A}_j^1$ (j stands for the j -invariant of E). There is a very clear geometric meaning to this phenomenon - by [CM05, §3] and [ABE20, 7.20, 7.22, 7.44], the 16 real discriminants are arbitrary (for each fixed E), which give the difference of this M_W^{nn} and $M_W^{\text{nn}, \text{tor}}$.

Note that the parametrized degeneration is slightly different between that in [CM05, §3] and [ABE20] (i.e., the former has two components one of which is those parametrized in [ABE20] - $\mathbb{D}_{16}$ -surface), but this is unsubstantial difference. Indeed, the relation is by a simple birational transform (at the total space level) as explained in [OO21, §7.1.3].

Subcase. (to $\widetilde{\mathbb{E}_8\mathbb{E}_8}$) We treat the case of degenerating from M_W to $M_W^{\text{seg}} \subset M_W^{\text{SBB}}$, which we recall to be the $\tilde{E}_8^{\oplus 2}$ -type 1-cusp (see also its GIT interpretation in [OO21, §7]).

Take $(X, R) \twoheadrightarrow B$ in $\mathcal{M}_W(K((t)))$ which degenerates to M_W^{seg} at the closed point. From [OO21, §7], it follows that we can lift this data to $(g_8, g_{12}) \in H^0(\mathcal{O}(8)) \times H^0(\mathcal{O}(12))$ with coefficients in $K[[t]]$ so that its reduction is (cs^4, s^6) for $c \neq 3$.

Then we can exploit the same procedure as Case1 Step 1, to replace the reduction as the reducible fibered surface

$$(51) \quad (X_1 \cup X_2) \rightarrow \mathbb{P}^1 \cup \mathbb{P}^1$$

where X_1 (resp., X_2) is a hypersurface of the $\mathbb{P}^2$ -bundle $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(3))$ over the first $\mathbb{P}^1$ (resp., the second $\mathbb{P}^1$), defined by

$$(52) \quad [y^2z = 4x^3 - g_4^\nu xz^2 + g_6^\nu z^3],$$

$$(53) \quad [y^2z = 4x^3 - h_4^\nu xz^2 + h_6^\nu z^3],$$

respectively. Then, from our assumption that $c \neq 3$, it follows that the double locus $X_1 \cap X_2$ is smooth elliptic curve fiber, hence this is of $\tilde{E}_8\tilde{E}_8$ -type surface as desired. We have 12 real discriminant points in each base.

Case 3 (Further degenerations from Type III degenerations). Below, we study the occuring degeneration componentwise. We proceed as follows. In the notations below, we promise that

- (i) $\sum l_i = l$,
- (ii) all the subindices are nonnegative,
- (iii) We call the images of really singular fibers (cf., notations below Theorem 3.2) on any of *possibly singular* $[(X, R) \rightarrow B(\simeq \mathbb{P}^1 \cup \dots \cup \mathbb{P}^1)] \in \overline{M}_W^{\text{ABE}}(K)$ or $\overline{M}_W^{\text{ABE}}(K((t)))$ as $\chi_1, \dots, \chi_{24}$ (which extends the original meaning in the realm of M_W) and continue to call them real discriminant points.
- (iv) Further, before each disucssion below, we lift this data $[(X, R) \rightarrow B \simeq \mathbb{P}^1 \cup \dots \cup \mathbb{P}^1]$ by fixing gauge i.e., the isomorphism of every rational component with $\mathbb{P}^1$ so that their nodal points have coordinate 0 or ∞ .

Subcase. ($\mathbb{A}_{l-1}$ to $\mathbb{A}_{l_1-1}\mathbb{A}_{l_2-1}\dots\mathbb{A}_{l_m-1}$) We now concentrate on the base of component of A -type in the degenerated

$$(54) \quad [(X, R) \rightarrow \mathbb{P}^1 \cup \dots \cup \mathbb{P}^1] \in \overline{M}_W^{\text{ABE}}(K((t)))$$

which we denote as $X_A \rightarrow \mathbb{P}^1$ here, with coordinate s_A . The real discriminant points $\chi_{a+1}, \dots, \chi_{a+l}$ can be seen as formal Puiseux series i.e., elements of $K((t))$. Note that any of χ_{a+i} is *not* 0 nor ∞ (as element of $\mathbb{P}^1(K((t)))$). Hence, after finite base change, we can suppose they all lie in $K((t))$ and we write $\Delta_A(s_A) := \prod_{1 \leq i \leq l} (s_A - \chi_{a+i})$.

Similarly to Step 2 of Case 1, we take Newton polygon $\text{Newt}(\Delta_A)$, its supporting function φ_A and the toric degeneration model $\mathcal{B}_A$ over $\mathbb{A}^1$ (hence also over $K[[t]]$) whose corresponding fan is the normal fan of the graph of φ_A . Or in other words, the natural compactification has moment polytope

$$(55) \quad \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq l, -c \leq y \leq -\varphi_A(x)\}$$

for a constant $c \gg 0$. This is one component of our desired $\mathcal{B}$ i.e., the closure of $\mathbb{P}_{s_A}^1$. Then, accordingly, we degenerate the ambient space $\mathbb{P}^2 \times \mathbb{P}_{s_A}^1 \simeq$

$\mathbb{P}_{s_A}^1(\mathcal{O}^{\oplus 3})$ to still trivial $\mathbb{P}_{s_A}^2$ -bundle over $\mathcal{B}$ so that we obtain the (semi-log-canonical) union of $\mathbb{A}$ -type log surfaces as the closure of X_A inside the ambient model $\mathcal{B} \times \mathbb{P}^2$.

Subcase. ($\mathbb{D}_{k+l}$ to $\mathbb{D}_k \mathbb{A}_{l_1-1} \cdots \mathbb{A}_{l_m-1}$) Next we consider the base of component of D -type in the degenerated

$$(56) \quad [(X, R) \rightarrow \mathbb{P}^1 \cup \cdots \cup \mathbb{P}^1] \in \overline{M_W}^{\text{ABE}}(K((t)))$$

which we denote as $X_D \rightarrow \mathbb{P}^1$ here, with coordinate s_D . We can and do suppose that the only double curve in X_D which is the intersection of next surface component, has coordinates $s_D = \infty$.

Recall from §3.2.4 that we have explicit Weierstrass type equation for the $\mathbb{D}$ -type surface, (20) in terms of a quadratic polynomial $P_2(s_D)$ whose coefficients live in $K((t))$. By quadratic base change if necessary, we can further suppose its two roots are also both in $K((t))$. Then by multiplying appropriate powers t^{2c}, t^{3c} of t to g_4^ν and g_6^ν which does not change the isomorphism class of original $X_D \rightarrow \mathbb{P}^1$ (over $t \neq 0$), we can and do assume that coefficients of both lie in $K[[t]]$ and do not vanish at $t = 0$ generically (with respect to s_D).

If some of real discriminants χ_i in the base of X_D (including two roots of P_2) converges to ∞ , whose fiber is in the double locus of the surface, then we do weighted blow up of the model finite times so that all χ_i in the base of X_D never diverge to ∞ when $t = 0$. Furthermore, in a similar manner, if two distinct roots of P_2 converges to the same point for $t \rightarrow 0$, then we do further weighted blow up at the point so that the two roots converge to different points. After these composition of weighted blow ups of the base surface, we contract all irreducible components of $t = 0$ which do not contain any real discriminant.

Then, we degenerate the bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(3)$ on $\mathcal{B}_{t \neq 0}$ to the whole model obtained above, so that it restricts to $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(3)$ on the component where the roots of P_2 converge, to $\mathcal{O}_{\mathbb{P}^1}^3$ otherwise. We consider its projectivization as the ambient model and take closure of X to get desired model of type $\mathbb{D}\mathbb{A} \cdots \mathbb{A}$.

Subcase. ($\mathbb{E}_{k+l}$ to $\mathbb{E}_k \mathbb{A}_{l_1-1} \cdots \mathbb{A}_{l_m-1}$) Next we consider the base of component of $\mathbb{E}$ -type in the degenerated

$$(57) \quad [(X, R) \rightarrow \mathbb{P}^1 \cup \cdots \cup \mathbb{P}^1] \in \overline{M_W}^{\text{ABE}}(K((t)))$$

which we denote as $\pi_E: X_E \rightarrow \mathbb{P}_{s_E}^1$ here, with coordinate s_E . We can and do suppose that the only double curve in X_E which is the intersection of next surface component, has coordinates $s_E = \infty$. We consider stable reduction of generic fiber thus over $K(s_E)$, which is from elliptic curve to either

elliptic curve or (irreducible) nodal rational curve over whole $K(s_E)[[t]]$. Correspondingly, we realize this model by multiplying t^{2c} (resp., t^{3c})

$$g_4 \in H^0(\mathbb{P}_{s_E}^1, \mathcal{O}(4)) (\text{resp.}, g_6 \in H^0(\mathbb{P}_{s_E}^1, \mathcal{O}(6)))$$

with appropriate c (we fix this normalization from now on), so that g_4, g_6 both become non-zero at $t = 0$.

In this subcase, we focus when the generic fiber at $t = 0$ is smooth i.e., elliptic curve, which we suppose from now on, and leave the nodal reduction case to next subcase.

Suppose the real discriminant points $\chi_{e+1}, \dots, \chi_{e+k+l+3}$ below X_E also all sit in $K((t))$ after finite base change if necessary. Then in a similar manner as before, with respect to the variable $s'_E := s_E^{-1}$, we set

$$(58) \quad P_E(s'_E) := \prod_{i=1}^{3+k+l} (s'_E - \chi_{e+i}^{-1}),$$

consider its Newton polygon $\text{Newt}(P_E)$, then corresponding toric blow up model $\mathcal{B}_E \rightarrow \mathbb{P}_{s_E}^1 \times \text{Spec}(K[[t]])$ with cosupport at $t = 0, s_E = \infty$. Then, generalizing the stable reduction over $K(s_E)[[t]]$, we extend ambient space $\mathbb{P}_{\mathbb{P}_{s_E}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(3))$ of X_E to that of $\mathcal{B}$ so that its restriction to $\mathbb{P}_{s_E}^1$ is $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(3))$ which includes the t -direction stable reduction of the generic fiber of X_E , and trivial $\mathbb{P}^1$ -bundle over the rest of components of $\mathcal{B}|_{t=0}$. Then it is easy to see the closure inside the ambient model over $K[[t]]$ gives reduction to the surface of type $\mathbb{E}\mathbb{A} \cdots \mathbb{A}$ in [ABE20].

Subcase. ($\mathbb{E}_{k+l}$ to $\mathbb{D}_{k-1}\mathbb{A}_{l_1-1} \cdots \mathbb{A}_{l_m-1}$) Similarly to the previous subcase, we next treat the case when t -direction stable reduction of the generic fiber of X_E becomes nodal (i.e., $j = \infty$). This assumption means

$$(59) \quad \Delta_{12} = (g_4^\nu)^3 - 27(g_6^\nu)^2 = 0$$

hence we can write $g_4^\nu = 3P_2^2, g_6^\nu = P_2^3$. Since we normalized our g_i^ν to give the t -direction stable reduction of the generic fiber, $P_2|_{t=0} \neq 0$ as a polynomial.

If the roots of $P_2|_{t=0}$ remain finite and distinct, then we only need to do toric modifications of the base model $\mathbb{P}_{s_E}^1 \times \text{Spec}K[[t]]$ at cosupport $\infty \times 0$ (closed point). As it is completely similar to the just previous subcase, using Newton polygon of the polynomial of s'_E with roots χ_i^{-1} converging to 0, we omit details.

If at least one the roots of $P_2|_{t=0}$ diverge, then we do toric blow up at $\infty \times 0 \in \mathbb{P}_{s_E}^1 \times \text{Spec}K[[t]]$ so that $\mathcal{B}|_{t=0}$ becomes union of $\mathbb{P}_s^1$ with one or two exceptional divisors to each of which the diverging real discriminant converge. Also, if the roots of P_2 converge to same points q in $\mathbb{P}_{s_E}^1$, we do weighted blow up of the base model surface at the point q so that the roots

converge to different points in the same component which we (still) denote as $\mathbb{P}_s^1$. After that, we contract all irreducible components (curves) of $t = 0$ which do not contain any real discriminant. Then again similarly, we take ambient space whose restriction to $\mathbb{P}_s^1 \times \{t = 0\}$ (resp., other components) is $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(3))$ (resp., trivial $\mathbb{P}^2$ -bundle).

After all these procedures, we obtain the model of reduction type $\mathbb{D}\mathbb{A} \cdots \mathbb{A}$.

Case 4 (From Type II to Type III). Now we deal with the case when the corresponding morphism from $\text{Spec}K[[t]] \rightarrow \overline{M}_W^{\text{SBB}}$, where the target space refers to the Satake-Baily-Borel compactification, maps generic point inside 1-cusp (M_W^{seg} and M_W^{nn} in the [OO21, §7] notation), and maps the closed point to 0-cusp $M_W^{\text{nn,seg}}$. We assume this below and call it $(*_{II,III})$.

Subcase. ($\widetilde{\mathbb{E}_8\mathbb{E}_8}$ to $\mathbb{E}_{9-l}\mathbb{A}_{l_1-1} \cdots \mathbb{A}_{l_m-1}$) First, we treat the case when the generic point of $\text{Spec}K[[t]]$ maps to M_W^{seg} . (Other case when the generic point of $\text{Spec}K[[t]]$ maps to M_W^{nn} , is treated in the **Subcase** after next.) We write the component of $\mathbb{E}_9$ -surface ([ABE20]) i.e., rational elliptic surface with double locus a single smooth fiber, as $X_E \rightarrow B_E \simeq \mathbb{P}^1$ as local notation. We suppose the double locus fibers over ∞ .

In case the reduction $t = 0$ gives divergence of some real discriminants in the base B_E to ∞ , then we again do the toric blow ups of the model completely similarly as in previous steps via Newton polygon technique, so that the real discriminant points only converge finite in the strict transform of B_E and smooth points in $\mathcal{B}|_{t=0}$ in general. Then again in the similar manner, we obtain model of polarization whose restriction to B_E is $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(3)$ while trivial $\mathcal{O}^{\oplus 3}$ otherwise, projectify it and take closure of X_E inside.

If such model is generically smooth over the strict transform of B_E (otherwise, proceed to next subcase). Then by the assumption $(*_{II,III})$ it follows that the fiber over ∞ becomes nodal at $t = 0$ (otherwise, it remains to be in Type II locus i.e., 1-cusps of $\overline{M}_W^{\text{SBB}}$). Hence the reduction for $t = 0$ is the desired fibred surface of type $\mathbb{E}\mathbb{A} \cdots \mathbb{E}$.

Subcase. ($\widetilde{\mathbb{E}_8\mathbb{E}_8}$ to $\mathbb{D}_{8-l}\mathbb{A}_{l_1-1} \cdots \mathbb{A}_{l_m-1}$) If the obtained model of $(X_E, R) \rightarrow B_E$ in the last step is *not* generically smooth over the strict transform of B_E , then the corresponding elements of $H^0(\mathcal{O}_{\mathbb{P}^1}(4))$ (resp., $H^0(\mathcal{O}_{\mathbb{P}^1}(6))$) which we still prefer to write g_4^ν, g_6^ν are of the form $(3P_2^2, P_2^3)$ with some $P_2 \in H^0(\mathcal{O}_{\mathbb{P}^1}(2))$. If P_2 vanishes at ∞ , i.e., degree at most 1 as a polynomial, then it means that one of the root of P_2 which is also a real discriminant point, diverges (or converges) to ∞ . We do toric blow up of the model of B_E at this stage by the Newton polygon of the polynomial whose

roots are diverging real discriminants, as in the previous steps. The process avoids the divergence of real discriminants ∞ while procuding further rational components in the reduction of base $\mathcal{B}|_{t=0}$. If P_2 is *not* squarefree, we do the same process as Case1 Step4. Then we contract all irreducible components of $t = 0$ which do not contain any real discriminants.

Then finally, similarly, we create the model of $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(3)$ at $t \neq 0$ as before, its projectivization, and take the closure of X_E inside, which is our desired model. In this manner, we obtain further degeneration to surface of type $\mathbb{D}\mathbb{A} \cdots \mathbb{A}$.

Subcase. ($\widetilde{\mathbb{D}_{16}}$ to $\mathbb{D}_a \mathbb{A}_{l_1-1} \cdots \mathbb{A}_{l_m-1} \mathbb{D}_b$ with $a + b + l = 16$) Now we treat the case when the generic point of $\text{Spec} K[[t]]$ maps to M_W^{nn} while the closed point maps to $M_W^{\text{seg}} \cap M_W^{\text{nn}}$, i.e., degenerations of $\widetilde{\mathbb{D}_{16}}$ -type surfaces to type III surfaces.

We lift the $K((t))$ -rational point at $\mathcal{M}_W^{\text{nn}}$ to $(g_8 = 3G_4^2, g_{12} = G_4^3)$ with $G_4 \in H^0(\mathcal{O}(4))$ with coefficient $K((t))$. By multiplying appropriate integer power of t , we can first assume that G_4 has all coefficients in $K[[t]]$. We also set the solutions of G_4 as $\sigma_1, \sigma_2, \tau_1, \tau_2$, which we can and do assume to be in $K((t))$ after finite base change of $K[[t]]$ if necessary. We suppose $\sigma_i|_{t=0} = 0, \tau_i|_{t=0} = \infty$.

Similarly to Case1 Step 1, we set

$$(60) \quad f(0) := \min\{\text{val}_t(\sigma_1), \text{val}_t(\sigma_2)\},$$

$$(61) \quad f(\infty) := \min\{\text{val}_t(\tau_1^{-1}), \text{val}_t(\tau_2^{-1})\}.$$

and consider blow up $\mathcal{B}_1 \rightarrow \mathcal{B}_{\text{triv}}$ by

$$(62) \quad \langle s, t^{f(0)} \rangle \cdot \langle s^{-1}, t^{f(\infty)} \rangle.$$

Then, the special fibre of $\mathcal{B}_1$ over $t = 0$ is

$$(63) \quad \mathbb{P}^1_{\frac{s}{t^{f(0)}}} \cup \mathbb{P}^1_s \cup \mathbb{P}^1_{\frac{1}{s \cdot t^{f(\infty)}}}$$

where the two ends are exceptional curves.

Then, as in the Case1 Step 1, the first component contains the limit of $\sigma_i|_{t \neq 0}$ and the last component contains the limiit of $\tau_i|_{t \neq 0}$ both different from the nodal points.

Then, similarly to before, we degenerate $\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(3)$ on the original base $\mathbb{P}^1$ to the whole model so that its reduction restricts to

- (i) $\mathbb{P}^1_{\frac{s}{t^{f(0)}}}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1})$ over $\mathbb{P}^1_{\frac{s}{t^{f(0)}}}$
- (ii) trivial $\mathbb{P}^2$ -bundle over $\mathbb{P}^1_s$ (i.e., $\mathbb{P}^2 \times \mathbb{P}^1_s$)
- (iii) $\mathbb{P}^1_{\frac{s}{t^{f(\infty)}}}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1})$ over $\mathbb{P}^1_{\frac{s'}{t^{f(\infty)}}}$.

Then our first step is to take closure of original X inside the projectivization of the above $\mathbb{P}^2$ -bundle on the rational chain.

After this, we do the same procedures as Step 2, Step 3 and then Step 4 of Case 1. Then we obtain the desired reduction to $\mathbb{D}\mathbb{A} \cdots \mathbb{A}\mathbb{D}$ type surface.

By here, we complete the case by case reproof of stable reduction type Theorem 3.2. $\square$

Therefore, the completion of proof of Theorem 3.1 also follows the above (re)proof of Theorem 3.2 (recall the beginning of our proof). $\square$

The identification of the normalization of $\overline{M}_W^{\text{ABE}}$ with the toroidal compactification in [ABE20, §7] follows from the fact that the relative location of the real discriminants in the broken base chain of $\mathbb{P}^1$ s are encoded as $(\mathbb{G}_m \otimes \Lambda_i)$. This may also follows again from further analysis in addition to above, but since this point overlaps more closely with the arguments in [ABE20] we do not pursue this here. See [ABE20, the proof of Proposition 7.45].

Instead, we do some more explicit description.

Corollary 3.5 (of our reproof of Theorem 3.2). *The boundary strata of $\overline{M}_W^{\text{ABE}}$ which parametrizes degenerated surfaces of the following stable types*

$$(64) \quad \begin{cases} \mathbb{E}\mathbb{A}\mathbb{E} \\ \mathbb{E}\mathbb{D} \\ \mathbb{E}\mathbb{A} \cdots \mathbb{A}\mathbb{D}_k (\text{with } k \geq 9), \end{cases}$$

are not in the closure of two boundary prime divisors of Type II.

Proof. The first two strata are both 17-dimension by the easy computation, while the Type II boundary components are also both 17-dimension, hence the proof follows. The last stratum, the proof follows from our stable reduction arguments (or from the observation below). $\square$

We observe that, in our situation at least, if a surface component which corresponds to the lattice of Λ type degenerates to those of type $\Lambda_1, \dots, \Lambda_m$, $\Lambda_1 \oplus \dots \oplus \Lambda_m$ is a sublattice of Λ . This is partially explained in [ABE20] and also related to Proposition 5.2 to be explained.

Remark 3.6. Recall from [DHT17, §4.1] combined with [CD07, §3.3], the interesting observation that one aspect of the classical Shioda-Inose structure construction to $II_{1,17}$ -lattice polarized (higher Picard rank) K3 surface can be explained by an interesting Jacobian fibration which corresponds to

the strata M_W^{nn} . The correspondence is explained via a part of Dolgachev-Nikulin mirror symmetry [Dol96, especially 7.11] i.e., the fiber of such Jacobian fibration plus the elliptic fiber of element of M_W provides Type II degeneration from M_W to M_W^{nn} . This remark is not essentially new.

Boundary strata of small codimensions. We classify boundary divisors and boundary strata of codimension 2 of the compactification $\overline{M}_W^{\text{ABE}}$. As prime divisors, there are at total 54 of those as follows:

- (i) $\mathbb{E}_{k_1} \mathbb{A}_{k_2} \mathbb{E}_{k_3}$ where $k_1 + k_2 + k_3 = 17, 0 \leq k_1 \leq 8, 0 \leq k_2 \leq 17, 0 \leq k_3 \leq 8$. At total, we have 45 boundary prime divisors of this type. The moduli is the product of Weyl group quotient of at total 17-dimensional algebraic tori (divided by left-right involution if $k_1 = k_3$).
- (ii) $\mathbb{E}_k \mathbb{D}_{17-k}$ where $0 \leq k \leq 8$. 9 of these boundary prime divisors.

The classification of 16-dimensional boundary strata are as follows:

- (i) $\mathbb{E}_{k_1} \mathbb{A}_{k_2} \mathbb{A}_{k_3} \mathbb{E}_{16-k_1-k_2-k_3}$ type with each $k_i \geq 0$.
- (ii) $\mathbb{E}_{k_1} \mathbb{A}_{k_2} \mathbb{D}_{16-k_1-k_2}$ with non-negative index. By [ABE20, §7I], the normalization $\overline{\mathcal{M}}_W^{\text{toroidal}} \rightarrow \overline{M}_W^{\text{ABE}}$ are nontrivial at the 9 irreducible components of those with $k_1 + k_2 = 16, 0 \leq k_1 \leq 8$.
- (iii) $\mathbb{D}\mathbb{D}$ type. Again, by *loc.cit*, the normalizations are non isomorphic at the one component for $\mathbb{D}_0 \mathbb{D}_{16}$.

Hence, the normalization of $\overline{M}_W^{\text{ABE}}$ are non-isomorphic at $9 + 1 = 10$ irreducible components of 16-dimension (which is biggest dimension), and the preimage becomes $18 + 2 = 20$ components.

Part 2. Application to type II degeneration of K3 surfaces

4. LIMIT MEASURE ALONG TYPE II DEGENERATION

4.1. Limit points. While the previous part I focuses on the *elliptic K3 surfaces*, their degenerations and moduli compactification, in this part II, we apply it to study more general K3 surfaces degeneration of type II over $\mathbb{C}$. The main point is, as in [OO21], the elliptic K3 structure appears around boundary as special Lagrangian fibration after suitable hyperKähler rotation, as expected in the context of the Mirror symmetry and shown in [OO21, §4]. If we follow the setup of [OO21, §6], we first observe the following.

Lemma 4.1. *If we naturally send $\mathcal{F}_{2d} \ni (X, L)$ into $\mathcal{M}_{K3}$ by adding $c_1(L)$ as additional period, type II cusps map to the strata $\mathcal{M}_{K3}(d)$ (see [OO21, §6]) of the Satake compactification of adjoint type $\overline{\mathcal{M}}_{K3}^{\text{Sat,adj}}$.*

We refine the statements in Proposition 5.1 which shows the limit existence in a yet another Satake compactification $\overline{\mathcal{M}_{K3}}^{\tau}$ among those non-adjoint types, which especially dominates the above compactification of adjoint type and dilates the 0-dimensional locus $\mathcal{M}_{K3}(d)$ to 17-dimension.

Proof. As it is well-known, for type II degeneration, with some fixed marking, $\langle [\operatorname{Re}\Omega_X], [\operatorname{Im}\Omega_X] \rangle$ converge to isotropic plane while obviously $[\omega_X]$ remains the same class. Comparing with §6.2 of *loc.cit*, we obtain the proof. $\square$

Note that the locus $\mathcal{M}_{K3}(d)$ is nothing but the only 0-cusp of the Satake-Baily-Borel compactification of $\mathcal{M}_{K3}(a)$, which is identified with the moduli of Weierstrass elliptic K3 surfaces modulo the involution (see [OO21, §7]). This is the key point to convert general problem on type II degeneration into type III degeneration of elliptic K3 surfaces. In other words, roughly we divide the diverging isotropic plane into a line plus a line. Note that this conversion is not easy to explicitly describe in general but we refer to related discussion in §7 which partially aims to see the conversion more explicitly.

4.2. Limit measure determination via Satake compactification. We now explicitly determine *measured* Gromov-Hausdorff limits ([Fuk87a]) of tropical K3 surfaces in the sense of [OO21, §4] so that we can justify the desired PL function invariant V . That is, we study the collapse of 2-dimensional spheres S^2 with the McLean metrics to unit intervals, through the algebro-geometric compactifications [ABE20] and its study in the previous section 3 of the asymptotic behaviour of singular fibers. This is an application of above stable reduction theorem after [ABE20], providing one way of understanding of measured Gromov-Hausdorff limits classification (cf., [Osh] for another way).

We recall that Satake compactification of adjoint representation type coincides with certain generalization of Morgan-Shalen type compactification [OO21, Theorem 2.1]. This is the viewpoint we take in this section.

For our purpose, we introduce the geometric realization map in a non-archimedean manner, which we write as $\tilde{\Phi}(a)$, as follows. This is essentially found by [ABE20, §4] and Y.Oshima [Osh] independently in somewhat different forms. The synchronization of the two works was rather surprising (at least to me) since their original aims were totally different, and also the tools are different: the latter was in more Hodge-theoretic context using a yet another Satake compactification as we define and briefly show below (see [Osh] for details). No clear reason of the miraculous coincidence has been found yet, while our works mean to take a first step.

Via a yet another Satake compactification. As a preparation of precise statements, while more details are in [Osh], we consider the irreducible representation τ of $SO_0(3, 19)$ whose highest root is only orthogonal to the left-most one in the Dynkin diagram of [OO21, §6.1]. Then, as [Osh] provides more details, the corresponding Satake compactification [Sat60a, Sat60b] $\overline{\mathcal{M}}_{K3}^{Sat, \tau}$ has (real) 17-dimensional strata $\mathcal{M}_{K3}(d)^\tau$ which is

$$O(\Lambda_{\text{seg}}) \backslash C^+(\Lambda_{\text{seg}}) / \mathbb{R}_{>0},$$

divided by the involution induced by complex conjugation. Here, $\Lambda_{\text{seg}} := p^\perp / p \simeq U \oplus E_8(-1)^{\oplus 2}$ with isotropic plane $p \subset \Lambda_{K3} \simeq U^{\oplus 3} \oplus E_8(-1)^{\oplus 3}$, and

$$(65) \quad C^+(\Lambda_{\text{seg}}) := \{x \in \Lambda_{\text{seg}} \otimes \mathbb{R} \mid x^2 > 0\},$$

hence isomorphic to the 17-dimensional real open unit ball. Its fundamental domain is provided by the classical Vinberg's method ([ABE20, Osh]). Here, we follow [ABE20, 4C] and denote as $P \simeq \mathcal{M}_{K3}(d)^\tau$ which is a subdivided Coxeter chamber (modulo the natural involution). P is of the form: $P := \{x \in C^+(\Lambda_{\text{seg}}) \mid (x, \alpha_i) > 0\}$ for in Λ_{seg} .

By [OO21, 2.1], we can replace the Satake compactification by certain Morgan-Shalen compactification, which is more useful in this paper.

Definition 4.2 (Geometric realization & measure density function). We consider the quotient of

$$(66) \quad \overline{\mathcal{M}}_W^{\text{MSBJ}} \simeq \overline{\mathcal{M}}_W^{\text{Sat, adj}}$$

where the right hand side denotes the Satake compactification with respect to the adjoint representation of $SO_0(3, 19)$ (the isomorphism is proven at [OO21, 2.1] as a general theory) by $O(\Lambda_{\text{seg}}) / O^+(\Lambda_{\text{seg}})$, acting as the complex conjugate involution. Then we obtain compactifications of $\mathcal{M}_{K3}(a)$ in [OO21] respectively, which we denote as

$$(67) \quad \overline{\mathcal{M}}_{K3}(a)^{\text{MSBJ}} \simeq \overline{\mathcal{M}}_{K3}(a)^{\text{Sat, adj}}.$$

Their common boundaries are hence stratified as follows:

$$(68) \quad \mathcal{M}_{K3}(a) \sqcup \mathcal{M}_{K3}(d)^\tau \sqcup \{2 \text{ points } p_{\text{seg}} \text{ and } p_{nn}\}.$$

Note this space (68), away from the two points p_{seg} and p_{nn} , can be also regarded as a subset of $\partial \overline{\mathcal{M}}_{K3}^{Sat, \tau}$. From the left hand side interpretation of (67), p_{seg} (resp., p_{nn}) corresponds to the prime divisor of toroidal compactifications over the 1-cusp M_W^{nn} (resp., M_W^{seg}) as [CM05].

Now, we define *geometric realization map* $\tilde{\Phi}$ from the above space (68) away from p_{nn} to

$$\{(X, d, \nu) \mid (X, d) \text{ is a compact metric space with diameter one} \\ \text{and } \nu \text{ is a Radon measure}\} / \sim,$$

as follows. Here, $\sim$ denotes the natural equivalence relation defined as positive constant multiplication of ν , as follows.

- (i) (S^2 case) For $x \in \mathcal{M}_{K3}(a)$, we define $\tilde{\Phi}(x)$ as the tropical K3 surface $\Phi(x)$ as [OO21, §6] with its Monge-Ampere measure (equivalent to the volume form), as the (a priori) additional data.
- (ii) (Unit interval case) In the following cases, we take $\Phi(x)$ as a unit interval with certain measure.
 - (a) (cf., [ABE20, §7A] and [Osh]) Recall that any $x \in \mathcal{M}_{K3}(d)^\tau$ is represented by some $l \in P \simeq O(\Lambda_{\text{seg}}) \setminus C^+(\Lambda_{\text{seg}}) / \mathbb{R}_{>0}$, which is neither p_{seg} nor p_{nn} . Hence we write x as $x(l)$. To such l , [ABE20, §7A] associates a piecewise linear function $V(l)$ from $[0, 1]$ for some construction of polygon using it. We set $\tilde{\Phi}(x = x(l))$ as $[0, 1]$ with the density function $V(l)$. While we do not reproduce the details of the functions $V(l)$ due to [ABE20, §7A] and leave it to *loc.cit* and [Osh], we describe its basic structures and informations to some extent at the beginning of §4.3.
 - (b) (A special point p_{seg} cf., [Osh]) We set $\tilde{\Phi}(p_{\text{seg}}) := ([0, 1], d, \nu)$ with standart metric d and $\nu \equiv 0$.

As [Osh] discusses in more details, we expect $\tilde{\Phi}$ is continuous in the sense of the measured Gromov-Hausdorff convergence as a refinement of [OO21, §6.2, Conjecture 6.2]. Indeed, the above (i) reflects the collapsing of the Ricci-flat K3 surfaces to tropical K3 surfaces (metrized S^2 s) around $\mathcal{M}_{K3}(a)$ as proven in [OO21, Theorem 6.9], the above (iia) and (iib) reflects the expectation of the limit measures for collapsing of the Ricci-flat K3 surfaces to segments. As one of our main claims in this paper, we give its partial confirmation, depending on our Part 1.

Theorem 4.3 (cf. also [Osh] for another proof). *The geometric realization map $\tilde{\Phi}$ is continuous with respect to the measured Gromov-Hausdorff topology in the sense of [Fuk87a].*

As we mentioned, [Osh] gives a very different proof of this Theorem 4.3, with particular differences notably at the Steps 3, 4. In particular, his proof uses Satake topology and asymptotic analysis of periods along particular 2-cycles, while our proof uses rather the formulation via the Morgan-Shalen type compactification and Part 1 of this paper.

Proof. First, we fix a notation and make a setup: we take a sequence of (g_8, g_{12}) with subindex i whose corresponding points in M_W converge to a point in M_W^{seg} , the union of a 1-cusp and the 0-cusp, in the Satake-Baily-Borel compactification $\overline{M_W}$. Recall that we show it is isomorphic to the GIT quotient compactification of M_W with respect to the Weierstrass model description in [OO21, Theorem 7.9]. Taking $(c_1 s^4, c_2 s^6)$ as a GIT polystable representative of the limit point in M_W^{seg} , by the Luna slice étale theorem at stacky level (cf., [Luna73, Dre04]) for instance, we can and do assume our sequence of (g_8, g_{12}) converges to it. For later use, for each i , we consider the roots of g_8 (resp., g_{12} , Δ_{24}) and denote as $\{\xi_j\}_{j=1,\dots,8}$ (resp., $\{\eta_j\}_{j=1,\dots,12}$, $\{\chi_j\}_{j=1,\dots,24}$) in ascending order of the absolute values. The natural analogues of $e(0)$ (38) and $e(\infty)$ (39) in our stable reduction arguments i.e., sequence version are

$$(69) \quad \epsilon := \max\{|\xi_j|, |\eta_{j'}| \mid 2 \leq j \leq 4, 1 \leq j' \leq 6\},$$

$$(70) \quad \epsilon' := \max\{|\xi_j|^{-1}, |\eta_{j'}|^{-1} \mid 5 \leq j \leq 7, 7 \leq j' \leq 12\}.$$

Step 1. Firstly, this Step 1 focuses on the case when the sequence of $[(g_8, g_{12})]$ converges to a point in the 1-cusp, $M_W^{seg} \setminus M_W^{nn}$.

In this case, [OO21, §7.3.2] shows the corresponding sequence of McLean metrics converges to infinitely long open surface which is asymptotically cylindrical at two ends 0 and ∞ , as minimal non-collapsing pointed Gromov-Hausdorff limit.

In this case, for large enough i i.e., with the McLean metric close enough to the above asymptotically cylindrical surface, [OO21, §7.3.7, notably Lemma 7.26] implies the following: after rescale with fixed diameters, in particular with bounded above distance of $s = 0$ and $s = \infty$, the corresponding renormalized $\rho(r)$ in *loc.cit* uniformly converges to 0 (after making r bounded by rescale) so that even the full measure of the (rescaled) McLean metric also tends to 0 for $i \rightarrow \infty$.

Hence we obtain desired convergence to the interval with 0 measure, as metric measure space, in the sense of e.g. [Fuk87a].

Step 2. This Step 2 provides the first step analysis of the “maximally degenerate” case when $c_1 = 3c_2 = 3$, i.e., when the sequence of (g_8, g_{12}) converges to $(3s^4, s^6)$. The arguments below is borrowed from [Osh], which we follow and leave for the proof. (Our later steps are different from [Osh], with more algebro-geometric or non-archimedean perspectives.) We thank Y.Oshima for the permission to write also here. For each i , we define a

cut-off function on $\mathbb{R}_{>0}$ as

$$\varphi(r) := \begin{cases} -1 & r < \epsilon, \\ \frac{\log r}{|\log \epsilon|} & \epsilon \leq r \leq \epsilon'^{-1}, \\ 1 & r > \epsilon'^{-1} \end{cases}$$

Here, for each j , suppose $\lim_{i \rightarrow \infty} \varphi(|\chi_j|) =: x_j$ (the appearance of two indices i, j are not typo as j is fixed here while χ_j depends on i) which is negative for $j \leq k$ and non-negative otherwise. In addition, we may assume that $-\frac{\log |D_i|}{|\log \epsilon_i|} \rightarrow d \in [0, +\infty]$, where D_i denotes the top coefficient of Δ_{24} . Then [Osh] determines the limit measure on the interval by using the approximate description of the McLean metric [OO21, §7.3.3, notably Lemma 7.16]. The limit measure can be described as (up to positive constants multiplication) V on $[-1, 1]$ by

$$V(w) = 12w + d - \sum_{j=1}^k \max\{w, x_j\} - \sum_{j=k+1}^{24} \max\{0, w - x_j\}.$$

and as in [HSZ19], metric d and measure ν on the interval $[-1, 1]$ as

$$\begin{aligned} d &= V(w)^{\frac{1}{2}} dw, \quad \nu = V(w) dw, \quad \text{if } V \not\equiv 0, +\infty, \\ d &= dw, \quad \nu = dw \quad \text{if } V \equiv 0 \text{ (or } V \equiv +\infty). \end{aligned}$$

Lemma 4.4 ([Osh], compare with [HSZ19]). *For the given and fixed sequence of (g_8, g_{12}) , consider the sequence of the underlying base $\mathbb{P}^1$ with McLean metric of the Weierstrass elliptic K3 surfaces which we rescale to make their diameters 2.³ Then the sequence converges to the above $([-1, 1], d, \nu)$ as the metric measure space, up to the multiple of ν .*

For convenience, we sketch the proof of [Osh] which depends on [OO21, §7.3.1, 7.3.3], but we leave the details to [Osh]. The key claim used in the proof is [OO21, Lemma 7.16], which is a good approximation of the McLean metric $\mu(g_8, g_{12})ds \otimes \overline{ds}$ on the base $\mathbb{P}^1$ of Weierstrass elliptic K3 surfaces. We denote the metrize base $\mathbb{P}^1$ as $B_{g_8, g_{12}}$ but it refers to $\tilde{\Phi}(x)$ in the case of (i). The key term in the approximation of μ [OO21, Lemma 7.16] is $\log |\Delta_{24}(:= g_8^3 - 27g_{12}^2)|$ which diverges at the discriminant locus of $B_{g_8, g_{12}}$. Hence, we can approximate the McLean metric $\mu(g_8, g_{12})ds \otimes \overline{ds}$ by some $\mathbb{C}(1)(:= \{z \in \mathbb{C} \mid |z| = 1\})$ -invariant simpler metric $\mu'(g_8, g_{12})ds \otimes \overline{ds}$ where $\mu'(g_8, g_{12})$ is defined in terms of $\log |\chi_j|s$ and $\log |s|$, away from $s = 0, \chi_j (j = 1, \dots, 24), \infty$. Then the limit measure of $(B_{g_8, g_{12}}, \mu(g_8, g_{12})ds \otimes \overline{ds})$ with respect to the sequence $(g_8, g_{12}) \rightarrow (3s^4, s^6)$

³This convention is for the compatibility with [Osh].

is calculatable by replacing μ by simpler μ' so that we obtain Lemma 4.4. We leave the detailed calculations and estimates to [Osh].

Step 3. We consider the normalized compact moduli $\overline{M}_W^{\text{ABE},\nu}$ and its stacky refinement $\overline{\mathcal{M}}_W^{\text{ABE},\nu}$ (a proper Deligne-Mumford algebraic stack) which comes from the construction of $\overline{\mathcal{M}}_W^{\text{ABE}}$ in §3.3 i.e., by the log KSBA moduli interpretation after [ABE20].

Take an étale chart $\overline{\mathcal{U}}$ of the stack $\overline{\mathcal{M}}_W^{\text{ABE},\nu}$ which contains the preimage of the 0-cusp of $\overline{M}_W$. We denote the preimage of the open part M_W as $\mathcal{U} \subset \overline{\mathcal{U}}$. Denote the corresponding coarse moduli as $U \subset \overline{U}$. Now we apply the Morgan-Shalen compactification as [Odk18, Appendix] to $\mathcal{U} \subset \overline{\mathcal{U}}$ and denote it simply as $U \subset \overline{U}^{\text{MSBJ}}$.

As preparation, now we define the following modified Newton polygon of the discriminant Δ_{24} for a sequence of (g_8, g_{12}) with respect to $i = 1, 2, \dots$ converging to $(3s^4, s^6)$. For $\Delta_{24}(s) = \sum_{j=1}^{24} d_j s^j$, we set

$$\text{Newt}(\Delta_{24}) := \{(j, -\log |d_j|) \mid 0 \leq j \leq 24\} + \mathbb{R}_{\geq 0}(0, 1)$$

as an analogue of (45) and modify it by using ϵ, ϵ' of (69), a sequence analogue of $e(0), e(\infty)$ (during the proof of Claim 3.3), as follows: first we regard the above $\text{Newt}(\Delta_{24})$ as a graph of PL convex function $\varphi_\Delta^{\mathbb{C}}: [0, 24] \rightarrow \mathbb{R} \cup \{\infty\}$ and modification is defined below. We set similarly as before

$$(71) \quad i_\epsilon := \max\{i \mid \varphi_\Delta(i) - \varphi_\Delta(i+1) \geq \epsilon\},$$

$$(72) \quad i_{\epsilon'} := \min\{i \mid \varphi_\Delta(i+1) - \varphi_\Delta(i) \geq \epsilon'\}.$$

Again as before, we modify $\varphi_\Delta^{\mathbb{C}}$ to $\overline{\varphi}_\Delta^{\mathbb{C}}: [0, 24] \rightarrow \mathbb{R} \cup \{\infty\}$ as

$$(73) \quad \overline{\varphi}_\Delta^{\mathbb{C}}(i) := \begin{cases} \varphi_\Delta^{\mathbb{C}}(i_\epsilon) - \epsilon(i_\epsilon - i) & (\text{if } 0 \leq i \leq i_\epsilon) \\ \varphi_\Delta^{\mathbb{C}}(i) & (\text{if } i_\epsilon \leq i \leq i_{\epsilon'}) \\ \varphi_\Delta^{\mathbb{C}}(i_{\epsilon'}) + \epsilon'(i - i_{\epsilon'}) & (\text{if } i_{\epsilon'} \leq i \leq 24). \end{cases}$$

We are actually only concerned about it modulo positive constant multiplication, but anyhow denote the graph of $\overline{\varphi}_\Delta^{\mathbb{C}}$ as $\text{Newt}'(\Delta_{24})$. Note that, from the definition using the (archimedean) logarithm, the non-differentiable points in the domain is not necessarily integers. For instance, along any holomorphic punctured family of (g_8, g_{12}) converging to $(3s^4, s^6)$, the obtained limit of the above $\text{Newt}'(\Delta_{24})$ modulo rescale (fixing the height) becomes our $P_{\Delta,0}$ in (49), the epigraph of $\overline{\varphi}_\Delta$ in (73). We can and do assume our sequence sits in a neighborhood U'' of $((3, \vec{0}); (1, \vec{0}))$ in $\mathbb{A}^{22} = \mathbb{A}^9 \times \mathbb{A}^{13}$ describing the coefficients of g_8 s and g_{12} s for each i . We consider the rational map from U'' to some (arbitrarily fixed) toroidal compactification $\overline{M}_W^{\text{AMRT},\{\Sigma\}}$ and

replace U'' by its blow up to make it a morphism. We denote the preimage of the boundary as $D'' \subset U''$, and set $U''' := U'' \setminus D''$.

Now, we apply the functoriality of MSBJ construction [Odk18, Appendix §A.2, A.15] (more precisely, the analytic extension in [Od20b]), we obtain a continuous map $\overline{U}'''^{\text{MSBJ}}(U'') \rightarrow \overline{M}_W^{\text{MSBJ}}(\overline{M}_W^{\text{AMRT}, \{\Sigma\}})$.

From the previous Step 2, the limit of $\text{Newt}'(\Delta_{24})$ for $i \rightarrow \infty$ decides the measured Gromov-Hausdorff limit of McLean metrics sequence (4.4), which is metrically the interval. Thus, from the case-by-case proof of Claim 3.3 during that of Theorem 3.2, above discussion readily implies that:

Claim 4.5. *The measured Gromov-Hausdorff limit of McLean metrics sequence (4.4) is determined by the limit point inside the Morgan-Shalen type compactification $\overline{U}'''^{\text{MSBJ}}(U'')$ (if exists).*

Step 4. If we consider the set of points of the boundary $\partial \overline{U}'''^{\text{MSBJ}}(U'')$, whose (given integral) affine coordinates valued in $\mathbb{Q}$, it is obviously dense. On the other hand, recall from the previous Step 3 that there is a natural continuous map $\overline{U}'''^{\text{MSBJ}}(U'') \rightarrow \overline{M}_W^{\text{MSBJ}}(\overline{M}_W^{\text{AMRT}, \{\Sigma\}})$. Hence, it is enough to show the following claim:

Claim 4.6. *For any point $p \in \partial \overline{U}'''^{\text{MSBJ}}(U'')$ with rational affine coordinates, if we describe its image in $\overline{M}_W^{\text{MSBJ}}(\overline{M}_W^{\text{AMRT}, \{\Sigma\}})$ as $\bar{l} := \mathbb{R}l$ with $(0 \neq)l = l(p) \in C^+(\Lambda_{\text{seg}}) \cap \Lambda_{\text{seg}} \otimes \mathbb{Q}$ (we also denote $\bar{l} = \overline{l(p)}$) the limit measure density function V ([HSZ19], our previous Step 2) for some sequence in M_W converging to p , coincides with $\tilde{\Phi}(\bar{l(p)})$.*

To prove the Claim 4.6, recall that [ABE20, Theorem 1.2] shows that the normalization of the log KSBA compactification of the Weierstrass elliptic K3 surfaces with their “(weighted) rational curves cycle” type boundaries is the toroidal compactification ([AMRT]) with respect to the rational curves cone. As its first step, they construct, for given $(0 \neq)l \in C^+(\Lambda_{\text{seg}}) \cap \Lambda_{\text{seg}} \otimes \mathbb{Q}$, a certain Kulikov model $X_{LR}(l)$ (and its flop $X'_{LR}(l)$ after a base change). For l , we take such models as the one in Claim 4.6. And one can assume the image of t in Δ^* converges to p for $t \rightarrow 0$. Indeed, we can take $X_{LR}(l)$ to be the pull back of the Kulikov (semistable) model family, constructed in [ABE20], to a generic analytic curve transversally intersecting the open strata of the prime divisor of U'' corresponding to p (if such divisor does not exist, we simply replace U'' by blow up satisfying it). Then *loc.cit* showed that its monodromy invariant (cf., e.g., [FriSca86]) is nothing but l modulo $O(\Lambda_{\text{seg}})$ in Corollary 7.33 *loc.cit*. It is done using the crucial diffeomorphism from degenerating elliptic K3 surface to a corresponding Symington type Lagrangian fibration by bare hand [EF19] and then calculating the intersection numbers on the Lagrangian fibration side. Recall

from [OO21, Theorem 2.8, Corollary 4.25] that the limit inside the Morgan-Shalen-Boucksom-Jonsson compactification of toroidal compactifications $\overline{M}_W^{\text{MSBJ}}(\overline{M}_W^{\text{AMRT}, \{\Sigma\}})$, which is independent of the choice of the admissible decompositions Σ , is equivalent to the information of the monodromy on $U^\perp$ of signature $(2, 18)$.

For each $X_{LR}(l)$ as above, one can directly see the limit measure density function by our previous Steps combined with the case-by-case explicit proof of Claim 3.3, and coincides with $\tilde{\Phi}(\bar{l})$ which is determined by the monodromy. Hence, it is determined by the limit inside $\overline{M}_W^{\text{MSBJ}}(\overline{M}_W^{\text{AMRT}, \{\Sigma\}})$ by [OO21, Theorem 2.8, Corollary 4.25] and the claim 4.6 for general sequence, the desired coincidence (Theorem 4.3) finally follows. $\square$

4.3. Explicit description and examples. Recall that, in particular, $\tilde{\Phi}(l)$ in case (iia) of Definition 4.2 is as follows, as [ABE20, §7A], [Osh], which describes all the details from which we borrow. The fundamental polygon P is divided into $9 = 3^{1+1}$ maximal chambers, say $\{P'_a\}_a$, and the points of $[0, 1]$ where $(\tilde{\Phi}(l))(0)$ is non-differentiable can be written as

$$(74) \quad 0 = \frac{q_{-2}}{q_{22}} = \frac{q_{-1}}{q_{22}} = \frac{q_0}{q_{22}} \leq \frac{q_1}{q_{22}} \leq \dots \leq \frac{q_{19}}{q_{22}} \leq \frac{q_{20} = q_{21} = q_{22}}{q_{22}} = 1.$$

The definitions also imply

$$(75) \quad \frac{q_1}{q_{22}} = \max \left\{ (l, -\frac{1}{3}\beta_L), 0 \right\},$$

with $\beta_L \in \Lambda_{\text{seg}}$ (see [ABE20, §4C]) and every q_j are linear at each P'_a with respect to the description (65).

The values and slopes of the function satisfy

$$(76) \quad (\tilde{\Phi}(l))(0) = \max\{(l, \beta_L), 0\},$$

$$(77) \quad \frac{d\tilde{\Phi}(l)(x)}{dx} = 9 - i \text{ for any } x \in (\frac{q_i}{q_{22}}, \frac{q_{i+1}}{q_{22}}).$$

In particular, $\tilde{\Phi}(l)$ is convex. Indeed:

- if $(l, \beta_L) \leq 0$, for generic l under such assumption, $\tilde{\Phi}(l)(0) = 0$ and the slope of $\tilde{\Phi}(l)$ starts with 9 and decrease by 1 at each wall crossing through q_j .
- if $(l, \beta_L) \geq 0$, then for generic l under such assumption, the slope of $\tilde{\Phi}(l)$ starts with 8 and decrease by 1 at each wall crossing through q_j .

In the case $\tilde{\Phi}(l)(0) = \tilde{\Phi}(l)(1) = 0$ (e.g., §6), then note that the barycenter of q_i is the middle point $\frac{1}{2}$. The behaviour of the function $(\tilde{\Phi}(l))$ around the opposite end 1 (denoted by R in [ABE20]) is completely similar.

Remark 4.7 (Relation with [CM05, §5]). For one parameter Type III degenerations from M_W to the locus inside the closure of M_W^{nn} , we expect that the corresponding limit point in $M_W(d)^\tau$ can be explained by the collision of 18 blow up centers p_i s for the stable type II degeneration of those elliptic K3 surfaces introduced in [CM05, §5]. For the combinatorial type of such type III degenerations, recall Corollary 3.5.

Example 4.8 (Via Davenport-Stothers triple). Here we see simple examples of degenerating Weierstrass elliptic K3 surfaces and apply above to obtain the limit measures of the family of McLean metrized spheres.

In the following two examples, let us denote

$$(78) \quad g_4(s) := 3(s^4 + 2s),$$

$$(79) \quad g_6(s) := s^6 + 3s^2 + \frac{3}{2},$$

so that

$$(80) \quad g_4^3 - 27g_6^2 = -27(s^3 + \frac{9}{4}).$$

Up to affine transformation, this is known to be the only pair of degree 4, degree 6 polynomials with the degree of $g_4^3 - 27g_6^2$ is 3. It is an easy example of “Davenport-Stothers triple” (cf., e.g., [Dav65],[Sto81],[Zan95],[Shi05]).

Our first example is as follows:

$$(81) \quad g_8(s) := g_4\left(\frac{s}{t}\right)g_4\left(\frac{1}{ts}\right)s^4,$$

$$(82) \quad g_{12}(s) := g_6\left(\frac{s}{t}\right)g_6\left(\frac{1}{ts}\right)s^6$$

for $t \rightarrow 0$. Then we see that the density function V of the limit measure of the tropical K3 surfaces is as follows (modulo rescale):

$$(83) \quad V(a) = \begin{cases} a & 0 \leq a \leq \frac{1}{2} \\ 1 - a & \frac{1}{2} \leq a \leq 1, \end{cases}$$

which is directly checkable after our arguments in §3.3 and [ABE20, §7A].

Example 4.9 (Via Davenport-Stothers triple again). We use the same g_4, g_6 as above Ex 4.8 while construct different g_8, g_{12} s. Note

$$\left(s + \frac{1}{s}\right)^{-1} = \frac{s}{(s^2 + 1)},$$

$(s + \frac{1}{s})^{-1}$ is near 0 if and only if s is near 0 or ∞ . Thus $(s + \frac{1}{s})^{-1}$ is near ∞ if and only if s is near $\sqrt{-1}$. In this example, we define g_8, g_{12} as follows:

$$g_8(s) := g_4\left(\frac{s}{t(s^2 + 1)}\right) \cdot (s^2 + 1)^4,$$

$$g_{12}(s) := g_6\left(\frac{s}{t(s^2 + 1)}\right) \cdot (s^2 + 1)^6.$$

Then

$$\Delta_{24}(s) = g_8^3 - 27g_{12}^2 = 0 \in \mathcal{O}_{\mathbb{P}^1}(24)|_s$$

if and only if

$$\frac{s}{t(s^2 + 1)} = \chi_i (i = 1, 2, 3)$$

or

$$\frac{s}{(s^2 + 1)} = \infty$$

with multiplicity 18 if and only if

$$(84) \quad s + \frac{1}{s} = (t\chi_i)^{-1} (i = 1, 2, 3)$$

or

$$(85) \quad s + \frac{1}{s} = 0$$

where, the latter with multiplicities 18. The former (84) happens if and only if

$$s = \frac{1 \pm \sqrt{1 - 4t^2\chi_i^2}}{2}$$

and the latter happens when either $s = \sqrt{-1}$ with the multiplicity 9 or $s = -\sqrt{-1}$ with the multiplicity 9 again. Therefore, if we $t \rightarrow 0$, we get $[0, 1]$ with the corresponding V (modulo rescale) as same again:

$$V(a) = \begin{cases} a & 0 \leq a \leq \frac{1}{2} \\ 1 - a & \frac{1}{2} \leq a \leq 1. \end{cases}$$

In next §6, we observe that above two cases are close to the direction of collapsing of [HSVZ18].

Example 4.10 (Simplest D type). On the other hand, as another simple our instance of our discussion in the proof of Theorem 3.2, we obtain a different type of V with $V(0) = V(1) \neq 0$.

Set

$$g_8(s) = 3((s - ta_1)(s - ta_2)(ts - a_3)(ts - a_4))^2,$$

$$g_{12}(s) = ((s - ta_1)(s - ta_2)(ts - a_3)(ts - a_4))^3,$$

for $a_1 \neq a_2, a_3 \neq a_4$, all lie in K . Then, the Newton polygon of Δ_{24} has only two slopes, so that the proof (Case 1, 2) of Theorem 3.2 shows the corresponding V for $t \rightarrow 0$ is a constant function.

Indeed, this is the simplest prototypical example of D type degeneration of elliptic K3 surfaces.

From the definition 4.2 of our $\tilde{\Phi}$, and compare with [ABE20, §7A] or [Osh], Theorem 3.2 ensures that V can have much more varieties in general.

5. LIMITS ALONG TYPE II DEGENERATION AND ASSOCIATED LATTICES

As claimed in our introduction, we are now ready to give general considerations on limits along $\mathcal{F}_{2d}$ to make sense of the V function for type II degenerations. Suppose we are in the repeated setup as $(\mathcal{X}, \mathcal{L}) \rightarrow \Delta$ in $\mathcal{F}_{2d}$ is a type II polarized degeneration of K3 surfaces, dominated by a Kulikov model $\tilde{\mathcal{X}}$ and the pull back $\tilde{\mathcal{L}}$ of $\mathcal{L}$ to $\tilde{\mathcal{X}}$, and a stable type II degeneration $\mathcal{X}_0 = V_0 \cup V_1$. Then, refining Lemma 4.1, the following holds.

Proposition 5.1. *For the given $\pi: (\mathcal{X}, \mathcal{L}) \rightarrow \Delta$ as above, the naturally associated continuous map φ^o from $\Delta \setminus 0$ to $\mathcal{M}_{K3}$ continuously extends to a map φ from Δ with $\varphi(0) = c(\mathcal{X}, \mathcal{L})$ in $\mathcal{M}_{K3}(d)^\tau \subset \overline{\mathcal{M}_{K3}}^{\text{Sat}, \tau}$. In other words, the limit point inside $\overline{\mathcal{M}_{K3}}^{\text{Sat}, \tau}$ for $t \rightarrow 0$ is well-defined. In particular, there is the well-defined function $V = V_\pi = V(\mathcal{X}, \mathcal{L}) := \tilde{\Phi}(c(\mathcal{X}, \mathcal{L}))$ on the segment for this $(\mathcal{X}, \mathcal{L})$ as we noted in the beginning of the paper.*

Proof. The proof is easy as Lemma 4.1, as through a marking α of $H^2(\mathcal{X}_1, \mathbb{Z})$, $\varphi^o(t)$ clearly converges to the image of the Kähler class $\alpha(c_1(\mathcal{L}|_{\mathcal{X}_1}))$ for $t \rightarrow 0$. $\square$

We remark that in the collaboration with Oshima, the above limit is expected to describe the limit measure and more generally $\tilde{\Phi}$ to be continuous on whole $\mathcal{M}_{K3} \sqcup \mathcal{M}_{K3}(d)^\tau (\subset \overline{\mathcal{M}_{K3}}^{\text{Sat}, \tau})$ with respect to the *measured* Gromov-Hausdorff topology so that the above V_π determines the limit measure of the hyperKähler metrics on general fibers. [Osh] provides related discussions.

Furthermore, we take a marking $H^2(\mathcal{X}_1, \mathbb{Z}) \simeq \Lambda_{K3}$ so that the corresponding isotropic plane is

$$\mathbb{Z}e'' \oplus \mathbb{Z}e'$$

and we denote the image of $c_1(\mathcal{L}|_{\mathcal{X}_1})$ as v_{2d} of norm $2d$. Recall the canonical isomorphism

$$\langle e'', e' \rangle^\perp / \langle e'', e' \rangle \simeq \Lambda_{\text{seg}} = II_{1,17} \simeq U \oplus E_8^{\oplus 2}.$$

We write $\langle e'', e' \rangle =: p$. Then, $v_{2d}^\perp \subset p^\perp / p$ is studied classically in e.g. [Fri84], which we denote as $\Lambda_{\text{per}}(c) = \Lambda_{\text{per}}(c(\mathcal{X}, \mathcal{L}))$.

As a hyperKähler rotated side, we take a type III degeneration $\mathcal{X}^\vee \rightarrow \Delta$ of Weierstrass elliptic K3 surfaces which we suppose to be Kulikov degeneration, i.e., $(\mathcal{X}^\vee, \mathcal{X}_0^\vee)$ is log smooth and is minimal. We put a marking on the smooth fibers so that the elliptic fiber class is e'' and the zero-section class is f'' . Recall that from [ABE20, §7], an irreducible decomposition of $\mathcal{X}_0^\vee$ which we write as $\cup_i V_i^\vee$ satisfies each V_i (or its pair) are either of the following forms:

- $XI \cdots IX$,
- $Y_2 Y_a I \cdots IX$,
- $Y_2 Y_a I \cdots IY_2 Y_a$.

It is easy to confirm that after appropriate flops, we can and do assume that the non-toric component (i.e., those with positive charges) all remains at the stable model of [ABE20]. Then, such remaining rational surfaces V_i with normal crossing boundary $\cup_j D_{i,j}$ and are encoded as slightly generalized root lattice of type either $\mathbb{DA} \cdots \mathbb{AD}$, $\mathbb{DA} \cdots \mathbb{AE}$, $\mathbb{EA} \cdots \mathbb{AE}$ with possibly indices 0s. This is encoded in *loc.cit* as $P_{LR}(l)$ (resp., piecewise linear function V). We denote such lattice as $\Lambda_{\text{ABE}}(\mathcal{X}^\vee)$. Note that its rank is generally 0 and at most 17. On the other hand, as a hyperKähler rotation of $(\mathcal{X}_t^\vee, \omega_t^\vee)$ with $[\omega_t^\vee] = m_t e'' + f''$ with $m_t \rightarrow \infty$, we set $\{(\mathcal{X}_t, \omega_t)\}_t$ of type II for $t \rightarrow 0$ (as in [OO21, §4]). We anyhow denote the limit inside the Satake compactification $\overline{\mathcal{M}}_{K3}^{\text{Sat}, \tau}$ formally as $c(\mathcal{X}, \mathcal{L})$. Then, the following holds.

Proposition 5.2. *In the above setup, the two associated negative definite lattices has canonical inclusion which respects the bilinear forms:*

$$\Lambda_{\text{ABE}}(\mathcal{X}^\vee) \subset \Lambda_{\text{per}}(c(\mathcal{X}, \mathcal{L})).$$

Proof. Recall that the $\Lambda_{\text{ABE}}(\mathcal{X}^\vee)$ ([ABE20, §7G, §7H]) is the direct sum of the slightly generalized ADE lattices $(\sum_j \mathbb{Z}[D_{i,j}])^\perp \subset H^2(V_i, \mathbb{Z})$. We use Clemens contraction map $\mathcal{X}_1^\vee \rightarrow \mathcal{X}_0^\vee$, and the marking of $\mathcal{X}_1^\vee$ so that we can regard $H^2(\mathcal{X}_0^\vee, \mathbb{Z})$ canonically⁴ as a sublattice of Λ_{K3} .

Any $(\sum_j \mathbb{Z}[D_{i,j}])^\perp \subset H^2(V_i, \mathbb{Z})$ lies in $(1, 1)$ -part. On the other hand, from the construction of the hyperKähler rotation $\mathcal{X}^\vee$, one of its period (real part of the cohomology of the holomorphic volume form) converges to v_{2d} as $(2, 0)$ -part. Hence they are orthogonal. This completes the proof. $\square$

Example 5.3. If $2d = 4$, i.e., degenerations of quartics, there are certainly examples where the above two lattices $\Lambda_{\text{ABE}}(\mathcal{X}^\vee)$ and $\Lambda_{\text{per}}(\mathcal{X}, \mathcal{L})$ do not coincide. For instance, if $v_{2d} = 2e'' + f''$, then

$$\Lambda_{\text{ABE}}(\mathcal{X}^\vee) \simeq E_8(-1)^{\oplus 2}$$

⁴modulo the monodromy, but the classes in our actual concern are all monodromy invariant and further if one fixes a continuous path connecting 0 and 1 in Δ , then it becomes canonical.

while

$$\Lambda_{\text{per}}(\mathcal{X}, \mathcal{L}) \simeq \langle -4 \rangle \oplus E_8(-1)^{\oplus 2}.$$

Also, there is another example with $2d = 4$ such that

$$\Lambda_{\text{ABE}}(\mathcal{X}^\vee) \simeq D_8(-1)^{\oplus 2}$$

while

$$\Lambda_{\text{per}}(\mathcal{X}, \mathcal{L}) \simeq \langle -4 \rangle \oplus D_8(-1)^{\oplus 2}.$$

Remark 5.4. Similar even negative definite lattices appear also in a slightly different context of Dolgachev-Nikulin mirror symmetry for lattice polarized K3 surfaces [Dol96]. Recall that the Dolgachev-Nikulin mirror ([Dol96, 7.11], [DHT17, 4.1]) of $\mathcal{F}_{2d}$ says, to each type II degeneration in $\mathcal{F}_{2d}$, there is an associated isotropic element $e(\mathcal{X}, \mathcal{L})$ in Λ_{2d} modulo $\tilde{O}(\Lambda_{2d})$.

From the arguments in [OO21, 4.14, 4.18, 6.10], in an open neighborhood of 0-cusp, $e(\mathcal{X}, \mathcal{L})$ induces elliptic fibrations. Then, we expect that the direct sum of ADE lattices which represent the Kodaira type of reducible degenerations of fibers, coincides with $\Lambda_{\text{ABE}}(\mathcal{X}^\vee)$. Indeed, in every $2d \leq 4$ case, they coincide by the calculation of [Dol96, §7].

We conclude the section by making an easy but important remark.

Proposition 5.5 (Denseness of algebraic limits). *Note that for each $d \geq 1$, we can consider $\overline{\mathcal{F}_{2d}} \rightarrow \overline{\mathcal{M}_{K3}}^{\text{Sat}, \tau}$ (see Lemma 4.1, also §5). If we consider the union of such limits:*

$$\bigcup_{d \in \mathbb{Z}_{>0}} (\partial \overline{\mathcal{F}_{2d}} \cap \mathcal{M}_{K3}(d)^\tau),$$

then this countable set is dense inside the whole 17-dimensional strata $\mathcal{M}_{K3}(d)^\tau$.

Proof. This easily follows since $\mathcal{M}_{K3}(d)^\tau$ is the quotient of

$$\{\lambda \in \Lambda_{\text{seg}} \otimes \mathbb{R} \mid \lambda^2 > 0\},$$

while Λ_{seg} is an even *integral* lattice. □

This implies the following straightforwardly.

Corollary 5.6 (Possible PL invariants for type II degenerations). *Possible PL invariants for type II degenerations of polarized K3 surfaces run over a dense subset of which appears in [ABE20, §7A] and [Osh].*

This result in particular gives *negative* answer to the first question of [HSZ19, §2.6] on the behaviour of V .

6. [HSVZ18] GLUED METRIC AND TYPE II LIMITS OF ALGEBRAIC K3 SURFACES

The recent work of Hein-Sun-Viaclovsky-Zhang [HSVZ18] gives construction of compact K3 surfaces at the level of hyperKähler structures, by glueings of Tian-Yau metrics and Taub-NUT type metrics, which maps and collapses to an interval.

In this section, we reveal how [HSVZ18] fits into our picture, therefore giving more structures. As a result, *loc.cit* roughly corresponds to two following aspects simultaneously:

Aspect 1. the special stable type $\mathbb{E}\mathbb{A}\mathbb{E}$ in [ABE20] (cf., also our §3, §4.2),

Aspect 2. also the pushforward of two Lagrangian fibrations on the limiting K3 surfaces.

6.1. Review of [HSVZ18] construction. First, we recall their construction here (while we leave full details to *loc.cit*). They construct compact hyperKähler manifolds (hence homeomorphic to the K3 surfaces) by glueing, which maps to an interval, from the following set of data:

- two arbitrary DelPezzo surfaces X_1 with the degrees $d_1 := (-K_{X_1})^2$ and $d_2 := (-K_{X_2})^2$,
- choice of their (isomorphic) smooth anticanonical divisors $D_i \subset X_i (i = 1, 2)$ with an isomorphism $D_1 \simeq D_2$,
- Tian-Yau metrics ([TY90]) on $X_i \setminus D_i$ (note $\chi(X_i \setminus D_i) = 12 - d_i$) which is cohomologically zero in $H^2(X_i \setminus D_i, \mathbb{R})$,
- a transition region $\mathcal{N}$ whose general fibers over the interval are $(T^2 \times \mathbb{R})$ away from $(d_1 + d_2)$ -points in the base,
- a hyperKähler metric on $\mathcal{N}$ constructed by the Gibbons-Hawking ansatz,
- (parameter specifying the attaching parameter for the S^1 -rotation),
- the “collapsing parameter” $\beta \in (0, 1]$.

As for the Tian-Yau metric of above situation, they analyzed its asymptotic at the boundary D_i to identify with ALH (or ALG* suggested by [CC21]) with exactly quadratic curvature decay and the non-integer volume growth $\sim r^{\frac{4}{3}}$ (cf., also [Hein12, Theorem 1.5(iii), I_b -case]), where r denotes the distance from some arbitrary base point.

From the above data, *loc.cit* glues the Tian-Yau hyperKähler metrics on $(X_i \setminus D_i)$ and some Gibbons-Hawking metrics with several (multi-)Taub-NUT asymptotics on $\mathcal{N}$, which collapses to the interval $[0, 1]$ when $\beta \rightarrow 0$ (also see earlier expectation by R.Kobayashi [Kob90b, p223]), which we here write S_β with its hyperKähler metric g_β . Furthermore, they provide a continuous map $F_\beta: S_\beta \rightarrow [0, 1]$ which satisfies:

- (i) the fibers over ends $F_\beta^{-1}(0)$ and $F_\beta^{-1}(1)$ are closure of open locus in the Tian-Yau spaces $X_i \setminus D_i$,
- (ii) for $\beta \rightarrow 0$, (S_β, g_β) converges in the Gromov-Hausdorff sense to the unit interval with natural affine structure (induced from the behaviour of harmonic functions on S_β),⁵
- (iii) the limit measure on the interval is written as $\sqrt{V(x)}dx$ with a convex PL function on $[0, 1]$ with $V(0) = V(1) = 0$, where dx stands for the affine structure above.

Remark 6.1. With respect to this affine structure dx , assuming the Gromov-Hausdorff limit of rescaled metrics with fixed diameters is identified with the dual graph, the natural affine structure with respect to the latter perspective is $V(x)dx$ (see [BJ17]).

Recall that [TY90] first constructed the hermitian metric h on the normal bundle for $D_i \subset X_i$ whose curvature form is Ricci-flat, then solved the complex Monge-Ampère equation with the reference metric of Calabi-ansatz type via h .

As [Fuk87b, Fuk89, CFG92] show, the fibers are infranilmanifolds, indeed simply Heisenberg nilmanifolds (cf., also [HSZ19, §2.2]). In particular, they also confirmed their hyperKähler manifolds are parametrized by 57-dimensional data (plus rescaling data), i.e., at least containing some open subset of $\mathcal{M}_{K3}$. Note that $V(0) = V(1) = 0$ condition of the above (iii) infers, as [Osh] shows logically, it should only gives a neighborhood of $E(A)E$ type subcone of the fundamental polygon $P(\simeq \mathcal{M}_{K3}(d)^\tau)$ in the whole $\mathcal{M}_{K3}^{\text{Sat}, \tau}$, hence the direction which involves D type is missing.

Then after [HSVZ18], more recent work of Honda-Sun-Zhang [HSZ19] proved similar PL structure for all possible *limit measure* on the Gromov-Hausdorff limit when it is 1-dimensional (interval). In §2.6 of *loc.cit*, they raise some questions regarding the function V to which we answer:

- First question in *loc.cit* asks if $V(p) = 0$ at the boundary point p in the case when V is not constant. This is far from true, from the presence of D type region combined with Theorem 4.3.
- The second question in *loc.cit*, in the situation of [HSVZ18], asks if V is singular at the $d_1 + d_2$ points in the interval. The answer is yes from our conclusion.
- Their third question is about the ratio of slopes. As our analysis so far, the slopes can be normalized to $0, \pm 1, \dots, \pm 9$ and the ratios are rational as expected.

⁵We observe in general this affine structure is not same as the one induced from non-archimedean structure as used in [BJ17].

6.2. Our interpretation of [HSVZ18]. Now, we discuss the aspects 1, 2 of the beginning of this §6.

For Aspect1 - Landau-Ginzburg model. Recall that [CJL21, Theorem 6.4] relates the above Tian-Yau metrics and those of $\frac{4}{3}$ -order volume growth gravitational instanton on rational elliptic surfaces ([Hein12]) by hyperKähler rotations (cf., [CJL21, 6.9], [HSVZ18, 2.5]). We expect our viewpoint may help to clarify relation with the Landau-Ginzburg models [EHX97], as we partially give observation here.

As first instance, we observe the following. For type II degeneration with one component isomorphic to $\mathbb{P}^2$, consider its underlying $\mathbb{R}^2$ below our degenerate elliptic K3 surface of $X_3/\mathbb{E}_0$ -type ([ABE20, 7.4], §3, §3.2). Then it is the limit of the affine structures of Gross-Siebert program type at [CPS, Example 2.4] (see also [LLL20, §3.1]), which has 3 I_1 -type singularities of affine structure. Indeed, if three of them collide via moving worms [KS06], it becomes the abovementioned $X_3/\mathbb{E}_0$ -type singularity of affine structures. [LLL20] also identified it with the affine structure coming from special Lagrangian fibration of a complement of cubic curve in $\mathbb{P}^2$ constructed in [CJL21]. See the details at [CPS, CJL21, LLL20].

Also, [ABE20] with the arguments in this paper provide further evidence to a variant of Doran-Harder-Thompson expectation [Dol96, DHT17] for K3 surfaces, where “mirror” is replaced or specialized to be hyperKähler rotation, with slight refinement by putting $\mathbb{A}$ -type surfaces between. In particular, this picture applies for general type II degenerations, with possibly many irreducible components, hence not necessarily Tjurin degeneration in the sense of [DHT17].

Indeed, recall that in [ABE20, §7] moduli compactification and our reconstruction in (3.1), the main role was played by the singular fibers behaviour. Such fact together with our interpretation of M_W as limits of hyperKähler rotated K3 surfaces may naturally invoke the homological mirror symmetry type phenomenon after [Sei01], that the Lefschetz vanishing cycles around the degenerations of the elliptic curves reflect the B-model pictures of the degeneration of K3 surfaces. We hope to have further understanding of it in our context in more systematic way in future.

For Aspect 2 - relation with two Lagrangian fibrations. We take a sequence of $(g_8, g_{12}) \in H^0(\mathbb{P}_s^1, \mathcal{O}(8)) \times H^0(\mathbb{P}_s^1, \mathcal{O}(12))$ converging to $(3s^4, s^6)$ and the associated Weierstrass K3 surface

$$\begin{aligned} \pi'' : X &:= [y^2z = 4x^3 - g_8(s)xz^2 + g_{12}(s)z^3] \\ &\subset \mathbb{P}_{\mathbb{P}_s^1}(\mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(6) \oplus \mathcal{O}_{\mathbb{P}^1}) \\ &\rightarrow B \simeq \mathbb{P}_s^1 \end{aligned}$$

converging to $\lambda \in \mathcal{M}_{K3}(d)^\tau$ in the Satake compactification $\overline{\mathcal{M}_{K3}}^{Sat, \tau}$. For $i \gg 0$, we have two Lagrangian fibrations:

- (i) As we showed in [OO21, §4], for fixed $m \gg 0$, we obtain a hyperKähler rotation $X_m^\vee$ of X which is canonically diffeomorphic to X (so that we can keep the corresponding marking to original φ for X) whose holomorphic form $\Omega_m^\vee$ has cohomology class as

$$(86) \quad [\Omega_m^\vee] = |\log \epsilon|^{-1} \text{Re} \Omega + \sqrt{\frac{-1}{2m}} c(f'' + me'').$$

Here, ϵ is as (69) and c_i is uniquely determined positive constant which automatically converges to 1 for $i \rightarrow \infty$. By the same argument as [OO21, §4], we obtain a fibration structure $\pi': X_m^\vee \rightarrow \mathbb{P}_s^1 = B_m^\vee$ defined by the pencil $|e'|$ with the fiber class e' . Note that this is a special Lagrangian fibration with respect to the original complex structure, as in [OO21, §4].

- (ii) Original $\pi'': X \rightarrow B \simeq \mathbb{P}_s^1$,⁶ the Weierstrass elliptic fibration structure. The fiber class is e'' and is determined as $|e''|$. This is Lagrangian fibration with respect to the holomorphic volume form Ω .

As [HSVZ18] confirms, its glued K3 surfaces form a subset of $\mathcal{M}_{K3}$ which includes an open subset U_{HSVZ} whose closure is in the EAE region of $\mathcal{M}_{K3}(d)^{Sat, \tau}$. We can and do assume that U_{HSVZ} is close to the boundary enough so that its any point has the special Lagrangian fibration π' in (i).

Conjecture 6.2. *For any glued fibration of K3 surface to the segment as in [HSVZ18] so that $p = (F_\beta: X \rightarrow [0, 1]) \in U_{HSVZ}$, F_β factors through both π' and π'' . There is a 1-homology class, which we denote $e' \cap e''$, such that*

- $e' \cap e''$ is primitive in both $H_1(e', \mathbb{Z})$ and $H_1(e'', \mathbb{Z})$.
- $e' \cap e''$ is monodromy invariant with respect to both π' and π'' .

The above conjecture would clarify an interpretation of the nilmanifold (Heisenberg manifold) fiber of [HSVZ18] as S^1 -bundle over an elliptic curve.

Remark 6.3. It would be interesting to see if the conjectural map $B_m^\vee$ coincides with a moment map for a $\mathbb{C}^*$ -action on it with the McLean metric and the limit measure is comparable to its Duistermaat-Heckman measure.

Remark 6.4. The domain wall crossing [HSVZ18, Theorem 1.5] (also treated in Type II superstring theory before according to [HSVZ18, Remark 1.6]) is now reflected as the formation of the singularity of affine structure of I_w type.

⁶Recall that in our first sections, the symbol π was used as a one parameter degeneration of K3 surfaces.

7. ROOT LATTICE TYPE AND TYPE II DEGENERATIONS

Suppose we have a type II polarized degeneration of K3 surfaces $\pi: (\mathcal{X}, \mathcal{L}) \rightarrow \Delta$. As an example case, suppose the end component of $\mathcal{X}_0$ is $\mathbb{F}_1$. Consider the ample cone of the $\mathbb{F}_1$, which gives the simplest classical instance of 2-ray game (cf., [Take89] for higher dimensional work) of Fano variety: Denote the natural projection $\varphi: \mathbb{F}_1 \rightarrow \mathbb{P}^2$, $\psi: \mathbb{F}_1 \rightarrow \mathbb{P}^1$, and H the hyperplane in $\mathbb{P}^2$ passing through the center of φp , E the exceptional curve, and set the strict transform of H as H' so that $\varphi^*H = H' + E$, as local notation. Then as is well-known and easy, the ample cone is

$$\text{Amp}(\mathbb{F}_1) = \mathbb{R}_{\geq 0}[\varphi^*H] + \mathbb{R}_{\geq 0}[\pi^*\mathcal{O}_{\mathbb{P}^1}(1)]$$

so that each extremal ray corresponds to φ and π .

Our point here is that if we consider MMP of $\mathbb{F}_1$ with scaling in $|L|$, then depending on the terminal objects (either $\mathbb{P}^1$ or $\mathbb{P}^2$), we have a subdivision of the cone:

$$\begin{aligned} \text{Amp}(\mathbb{F}_1) &= (\mathbb{R}_{\geq 0}[\varphi^*H] + \mathbb{R}_{\geq 0}[-K_{\mathbb{F}_1}]) \\ &\quad + (\mathbb{R}_{\geq 0}[-K_{\mathbb{F}_1}] + \mathbb{R}_{\geq 0}[\psi^*\mathcal{O}_{\mathbb{P}^1}(1)]). \end{aligned}$$

We denote the first cone as $\mathcal{C}_1$ and the second as $\mathcal{C}_2$. Then we observe the following: if type II Kulikov degeneration with nef (but generically ample [She83]) polarization $\mathcal{L}$ has end component $V \simeq \mathbb{F}_1$, then

- $[\mathcal{L}|_V] \in \mathcal{C}_1$ if and only if the corresponding PL function has $\mathbb{E}$ type singularity at the end and
- $[\mathcal{L}|_V] \in \mathcal{C}_2$ if and only if the corresponding PL function has $\mathbb{D}$ type singularity at the end.

Now we conjecture the following.

Conjecture 7.1 ($\mathbb{D}$ vs $\mathbb{E}$ conjecture). ⁷ *We consider type II polarized degeneration of K3 surfaces $(\mathcal{X}, \mathcal{L}) \rightarrow \Delta$ in $\mathcal{F}_{2d}$. Take a simultaneous resolution after base change to make it Kulikov model $\tilde{\mathcal{X}}$. We denote the pull back of $\mathcal{L}$ to $\tilde{\mathcal{X}}$ as $\tilde{\mathcal{L}}$ and $\mathcal{X}_0 = V_0 \cup V_1$, the stable type II degeneration ([Fri84, Kon85]).*

Suppose that if we run the MMP with scaling ⁸ in $\tilde{\mathcal{L}}|_{V_i}$ to V_i , it ends with ruled surface structure (resp., birational contraction). Then our hyperKähler rotation of $(\mathcal{X}_t, \mathcal{L}_t)$ limits to $\mathbb{D}$ type end of interval (resp., $\mathbb{E}$ type end of interval).

⁷(Added notes in revision) We note here again that in version 4 of [ABE20], the authors change the notation of type $\mathbb{D}$ to type C. See §3.2.4.

⁸As in Example 7.2, $\tilde{\mathcal{L}}|_{V_i}$ can be only semiample and big, as the pullback of ample line bundle on some crepant contraction. Nevertheless, the MMP with scaling still makes sense. If not preferred, one can pass to the crepant contraction induced by $\tilde{\mathcal{L}}|_{V_i}$ and discuss on it.

Example 7.2. Indeed, it at least matches to the 4 cases of degree 2 examples: see [Fri84, 5.2] (cf., also [Sha80, AET19]).

Remark 7.3 (Strong open K-polystable degenerations on M_W^{nn}). For $X := \mathbb{F}_2$, D an elliptic bi-section for the ruling, then $X^\circ := X \setminus D$, for certain range of ample L , $(X^\circ, L^\circ := L|_{X^\circ})$ is strongly open K-polystable [Od20a], as in the arguments of *loc.cit.* Indeed, [AP06] applied to the crepant contraction to the quadric cone $X \rightarrow \mathbb{P}(1, 1, 2)$ implies that. This appears as M_W^{nn} in [OO21, §7]. We expect that these $\mathbb{D}$ type degenerating family bubble off different ALH gravitational instantons along minimal non-collapsing rescaling in the sense of [Od20a, §6].

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