

Universal Tropical R map of $\mathfrak{sl}_2$ and Prehomogeneous Geometric Crystals

Dedicated to Professor Masaki Kashiwara on the occasion of his 60th birthday

By

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§ 1. Introduction

The notion of geometric crystal is a sort of geometric lifting of crystal base theory. If a geometric crystal χ is "positive", it can be "ultra-discretized" to a crystal $\mathcal{UD}(\chi)$ (for more details, see below). For a geometric crystal $\chi = (X, \{e_i\}, \{\gamma_i\}, \{\varepsilon_i\})$, if there exists an open dense orbit by e_i^c 's, we call χ *prehomogeneous*, where $e_i : \mathbb{C}^\times \times X \rightarrow X$ ($e_i^c(x) = e_i(c, x)$) defines certain rational $\mathbb{C}^\times$ -action on X . For isomorphic prehomogeneous geometric crystals χ_1 and χ_2 ($f : \chi_1 \xrightarrow{\sim} \chi_2$), assume that there exists another isomorphism $f' : \chi_1 \rightarrow \chi_2$ such that $f(p) = f'(p)$ for some point $p \in Z \subset X_1$ where Z is an open dense orbit in X_1 . Then it is easy to see that $f = f'$ as a rational morphism, which resembles Schur's lemma. In this sense, prehomogeneity of geometric crystals corresponds to irreducibility of modules. In general, it is not easy to show the prehomogeneity of a geometric crystal directly. Here we obtain a sufficient condition for prehomogeneity of a "positive" geometric crystal χ , which is, indeed, that the ultra-discretized crystal $\mathcal{UD}(\chi)$ is connected.

Perfect crystals are invented to treat the problem in some physical models from the crystal theoretical point of view ([KMN1],[KMN2]). They possess several remarkable properties; one of the most crucial ones among them is connectedness. Perfect crystals are not only connected themselves but also their tensor product are again connected. Let $\{B_l\}_{l \geq 1}$ be a coherent family of perfect crystals and B_∞ its limit([KKM]). Then the crystal B_∞ holds similar properties to perfect crystals, *e.g.*, connectedness.

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Let $\{\chi_\lambda\}_{\lambda \in \Lambda}$ be a family of geometric crystals indexed by a set Λ . Tropical R map $R_{\lambda\mu}$ is an isomorphism $R_{\lambda\mu} : \chi_\lambda \times \chi_\mu \rightarrow \chi_\mu \times \chi_\lambda$ ($\lambda, \mu \in \Lambda$) satisfying the Yang-Baxter equation. We have not obtained a kind of "universal objects" for geometric crystals just like universal R-matrix for modules of quantum groups except for the $\mathfrak{sl}_2$ - case. The universal tropical R map of $\mathfrak{sl}_2$ is introduced in the last section of this article.

In [KNO],[KNO2], we construct the affine geometric crystals $\mathcal{V}(\mathfrak{g})_L$ ($L \in \mathbb{C}^\times$) for $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, (C_n^{(1)}), D_n^{(1)}, D_{n+1}^{(2)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}$. Each of them is equipped with a positive structure and then there exists the corresponding affine crystal, which is, indeed, isomorphic to $B_\infty(\mathfrak{g}^L)$ the limit of perfect crystal of certain perfect crystals for the Langlands dual affine Lie algebra $\mathfrak{g}^L$. Since $B_\infty \otimes B_\infty$ is a connected crystal, product of geometric crystals $\mathcal{V}_L \times \mathcal{V}_M$ ($L, M \in \mathbb{C}^\times$) is prehomogeneous and then for another tropical R map R'_{LM} such that $R_{LM}(x_0, y_0) = R'_{LM}(x_0, y_0)$, one has $R_{LM} = R'_{LM}$.

This article is basically a review of [KNO],[KNO2] except the last section, where we introduce the universal tropical R-map of $\mathfrak{sl}_2$.

§ 2. Geometric Crystals and Crystals

The notations and definitions here follow [N1],[N2],[N3],[KNO].

§ 2.1. Geometric Crystals

Fix a symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$ with a finite index set I . Let $(\mathfrak{t}, \{\alpha_i\}_{i \in I}, \{\alpha_i^\vee\}_{i \in I})$ be the associated root data satisfying $\alpha_j(\alpha_i^\vee) = a_{ij}$. Let $\mathfrak{g} = \mathfrak{g}(A) = \langle \mathfrak{t}, e_i, f_i (i \in I) \rangle$ be the Kac-Moody Lie algebra associated with A . Let $P \subset \mathfrak{t}^*$ be a weight lattice such that $\mathbb{C} \otimes P = \mathfrak{t}^*$ and $P \subset \{\lambda \mid \lambda(Q^\vee) \subset \mathbb{Z}\}$, whose element is called a weight.

Define the simple reflections $s_i \in \text{Aut}(\mathfrak{t})$ ($i \in I$) by $s_i(h) := h - \alpha_i(h)\alpha_i^\vee$, which generate the Weyl group W . Let G be the Kac-Moody group associated with $(\mathfrak{g}, P)$ ([PK]). Let $U_\alpha := \exp \mathfrak{g}_\alpha$ ($\alpha \in \Delta^{\text{re}}$) be the one-parameter subgroup of G . The group G is generated by U_α ($\alpha \in \Delta^{\text{re}}$). For any $i \in I$, there exists a unique group homomorphism $\phi_i : SL_2(\mathbb{C}) \rightarrow G$ such that

$$\phi_i \left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right) = \exp(te_i), \quad \phi_i \left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \right) = \exp(tf_i) \quad (t \in \mathbb{C}).$$

Set $\alpha_i^\vee(c) := \phi_i \left(\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \right)$, $x_i(t) := \exp(te_i)$, $y_i(t) := \exp(tf_i)$, $G_i := \phi_i(SL_2(\mathbb{C}))$, $T_i := \alpha_i^\vee(\mathbb{C}^\times)$ and $N_i := N_{G_i}(T_i)$. Let T be the subgroup of G with P as its weight lattice which is called a *maximal torus* in G , and let $B^\pm (\supset T)$ be the Borel subgroup of G . The following definition is equivalent to the ones in [N1],[KNO].

Definition 2.1. Let X be an ind-variety over $\mathbb{C}$, γ_i and ε_i ($i \in I$) rational functions on X , and $e_i : \mathbb{C}^\times \times X \rightarrow X$ a rational $\mathbb{C}^\times$ -action. A quadruple $(X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ is a G (or $\mathfrak{g}$)-geometric crystal if

- (i) $\{1\} \times X \cap \text{dom}(e_i)$ is open dense in $\{1\} \times X$ for any $i \in I$. Here $\text{dom}(e_i)$ is the domain of definition of $e_i : \mathbb{C}^\times \times X \rightarrow X$.
- (ii) The rational function γ_i ($i \in I$) satisfies $\gamma_j(e_i^c(x)) = c^{a_{ij}} \gamma_j(x)$ for any $i, j \in I$.
- (iii) e_i and e_j satisfy the following relations:

$$\begin{aligned} e_i^{c_1} e_j^{c_2} &= e_j^{c_2} e_i^{c_1} && \text{if } a_{ij} = a_{ji} = 0, \\ e_i^{c_1} e_j^{c_1 c_2} e_i^{c_2} &= e_j^{c_2} e_i^{c_1 c_2} e_j^{c_1} && \text{if } a_{ij} = a_{ji} = -1, \\ e_i^{c_1} e_j^{c_1^2 c_2} e_i^{c_1 c_2} e_j^{c_2} &= e_j^{c_2} e_i^{c_1 c_2} e_j^{c_1^2 c_2} e_i^{c_1} && \text{if } a_{ij} = -2, a_{ji} = -1, \\ e_i^{c_1} e_j^{c_1^3 c_2} e_i^{c_1^2 c_2} e_j^{c_1^2 c_2} e_i^{c_1 c_2} e_j^{c_2} &= e_j^{c_2} e_i^{c_1 c_2} e_j^{c_1^3 c_2} e_i^{c_1^2 c_2} e_j^{c_1^2 c_2} e_i^{c_1} && \text{if } a_{ij} = -3, a_{ji} = -1, \end{aligned}$$

- (iv) The rational function ε_i ($i \in I$) satisfies $\varepsilon_i(e_i^c(x)) = c^{-1} \varepsilon_i(x)$.

The relations in (iii) is called *Verma relations*. If $\chi = (X, \{e_i\}, \{\gamma_i\}, \{\varepsilon_i\})$ satisfies the conditions (i), (ii) and (iv), we call χ a *pre-geometric crystal*.

§ 2.2. Crystals

We recall the notion of crystals.

Definition 2.2. A *crystal* B is a set endowed with the following maps:

$$\begin{aligned} \text{wt} : B &\longrightarrow P, \\ \varepsilon_i : B &\longrightarrow \mathbb{Z} \sqcup \{-\infty\}, \quad \varphi_i : B \longrightarrow \mathbb{Z} \sqcup \{-\infty\} \quad \text{for } i \in I, \\ \tilde{e}_i : B \sqcup \{0\} &\longrightarrow B \sqcup \{0\}, \quad \tilde{f}_i : B \sqcup \{0\} \longrightarrow B \sqcup \{0\} \quad \text{for } i \in I, \\ \tilde{e}_i(0) &= \tilde{f}_i(0) = 0. \end{aligned}$$

Those maps satisfy the following axioms: for all $b, b_1, b_2 \in B$, we have

$$\begin{aligned} \varphi_i(b) &= \varepsilon_i(b) + \langle \alpha_i^\vee, \text{wt}(b) \rangle, \\ \text{wt}(\tilde{e}_i b) &= \text{wt}(b) + \alpha_i \text{ if } \tilde{e}_i b \in B, \\ \text{wt}(\tilde{f}_i b) &= \text{wt}(b) - \alpha_i \text{ if } \tilde{f}_i b \in B, \\ \tilde{e}_i b_2 = b_1 &\iff \tilde{f}_i b_1 = b_2, \\ \varepsilon_i(b) = -\infty &\implies \tilde{e}_i b = \tilde{f}_i b = 0. \end{aligned}$$

Example 2.3.

- (i) If (L, B) is a crystal base, then B is a crystal.
- (ii) For the crystal base $(L(\infty), B(\infty))$ of the subalgebra $U_q^-(\mathfrak{g})$ of the quantum algebra $U_q(\mathfrak{g})$, $B(\infty)$ is a crystal.
- (iii) For $\lambda \in P$, set $T_\lambda := \{t_\lambda\}$. We define a crystal structure on T_λ by

$$\tilde{e}_i(t_\lambda) = \tilde{f}_i(t_\lambda) = 0, \quad \varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty, \quad \text{wt}(t_\lambda) = \lambda.$$

Definition 2.4.

- (i) To a crystal B , a colored oriented graph is associated by

$$b_1 \xrightarrow{i} b_2 \iff \tilde{f}_i b_1 = b_2.$$

We call this graph the *crystal graph* of B .

- (ii) A crystal B is said to be *connected*, if its crystal graph is connected as a graph.
- (iii) A crystal B is free if for any $b \in B$, $i \in I$ and $n > 0$, $\tilde{e}_i^n(b) \neq 0$ and $\tilde{f}_i^n(b) \neq 0$.

§ 2.3. Positive structure, Ultra-discretization and Tropicalization

Let us recall the notions of positive structure and ultra-discretization/tropicalization. The setting below is the same as in [KNO2]. Set $R := \mathbb{C}(c)$ and define

$$\begin{aligned} v: R \setminus \{0\} &\longrightarrow \mathbb{Z} \\ f(c) &\mapsto \deg(f(c)). \end{aligned}$$

Here $\deg$ is the degree of poles at $c = \infty$. Note that for $f_1, f_2 \in R \setminus \{0\}$, we have

$$(2.1) \quad v(f_1 f_2) = v(f_1) + v(f_2), \quad v\left(\frac{f_1}{f_2}\right) = v(f_1) - v(f_2).$$

We say that a non-zero rational function $f(c) \in \mathbb{C}(c)$ is *positive* if f can be expressed as a ratio of polynomials with positive coefficients. Note that $f \in \mathbb{C}(c)$ is positive if and only if any pole of f is not a positive number and $f(x) > 0$ for any $x > 0$.

If $f_1, f_2 \in R$ are positive, then we have

$$(2.2) \quad v(f_1 + f_2) = \max(v(f_1), v(f_2)).$$

Let $T \simeq (\mathbb{C}^\times)^l$ be an algebraic torus over $\mathbb{C}$ and $X^*(T) := \text{Hom}(T, \mathbb{C}^\times)$ (resp. $X_*(T) := \text{Hom}(\mathbb{C}^\times, T)$) be the lattice of characters (resp. co-characters) of T . We denote by T_+ the set of points x in T such that $\chi(x) > 0$ for any character χ . Then $((\mathbb{C}^\times)^n)_+ = (\mathbb{R}_{>0})^n$.

A non-zero rational function on an algebraic torus T is called *positive* if it is written as g/h where g and h are a positive linear combination of characters of T .

Definition 2.5. Let $f: T \rightarrow T'$ be a rational mapping between two algebraic tori T and T' . We say that f is *positive*, if $\chi \circ f$ is positive for any character $\chi: T' \rightarrow \mathbb{C}^\times$.

Denote by $\text{Mor}^+(T, T')$ the set of positive rational mappings from T to T' . Note that any $f \in \text{Mor}^+(T, T')$ induces a real analytic map $f_+: T_+ \rightarrow T'_+$.

Lemma 2.6 ([BK]). *For any $f \in \text{Mor}^+(T_1, T_2)$ and $g \in \text{Mor}^+(T_2, T_3)$, the composition $g \circ f$ is well-defined and belongs to $\text{Mor}^+(T_1, T_3)$.*

By Lemma 2.6, we can define a category $\mathcal{T}_+$ whose objects are algebraic tori over $\mathbb{C}$ and arrows are positive rational mappings. The category $\mathcal{T}_+$ admits products. For two algebraic tori T and T' , their product in $\mathcal{T}_+$ coincides with the usual product of T and T' .

Note that $T \mapsto T_+$ gives a functor from $\mathcal{T}_+$ to the category of real analytic manifolds.

Let $f: T \rightarrow T'$ be a positive rational mapping of algebraic tori T and T' . We define a map $\widehat{f}: X_*(T) \rightarrow X_*(T')$ by

$$\langle \chi, \widehat{f}(\xi) \rangle = v(\chi \circ f \circ \xi),$$

where $\chi \in X^*(T')$ and $\xi \in X_*(T)$. Note that $\chi \circ f \circ \xi$ is a rational map from $\mathbb{C}^\times$ to itself.

Lemma 2.7 ([BK]). *For any algebraic tori T_1, T_2, T_3 , and positive rational mappings $f \in \text{Mor}^+(T_1, T_2)$, $g \in \text{Mor}^+(T_2, T_3)$, we have $\widehat{g \circ f} = \widehat{g} \circ \widehat{f}$.*

By this lemma, we obtain a functor

$$\begin{aligned} \mathcal{UD}: \quad \mathcal{T}_+ &\longrightarrow \text{Set} \\ T &\mapsto X_*(T) \\ (f: T \rightarrow T') &\mapsto (\widehat{f}: X_*(T) \rightarrow X_*(T')). \end{aligned}$$

Let us come back to the situation in §2.1. Hence G is a Kac-Moody group and T is its Cartan subgroup.

Definition 2.8 ([BK]). Let $\chi = (X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ be a G (or $\mathfrak{g}$)-geometric crystal, T' an algebraic torus and $\theta: T' \rightarrow X$ a birational mapping. The mapping θ is called a *positive structure* on χ if it satisfies

- (i) for any $i \in I$ the rational functions $\gamma_i \circ \theta: T' \rightarrow \mathbb{C}$ and $\varepsilon_i \circ \theta: T' \rightarrow \mathbb{C}$ are positive,
- (ii) for any $i \in I$, the rational mapping $e_{i,\theta}: \mathbb{C}^\times \times T' \rightarrow T'$ defined by $e_{i,\theta}(c, t) := \theta^{-1} \circ e_i^c \circ \theta(t)$ is positive.

Let $\theta: T' \rightarrow X$ be a positive structure on a geometric crystal $\chi = (X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$. Applying the functor $\mathcal{UD}$ to the positive rational mappings $e_{i,\theta}: \mathbb{C}^\times \times T' \rightarrow T'$ and $\gamma_i \circ \theta, \varepsilon_i \circ \theta: T' \rightarrow \mathbb{C}^\times$, we obtain

$$\begin{aligned}\tilde{e}_i &:= \mathcal{UD}(e_{i,\theta}): \mathbb{Z} \times X_*(T') \rightarrow X_*(T') \\ \text{wt}_i &:= \mathcal{UD}(\gamma_i \circ \theta), \quad \varepsilon_i := \mathcal{UD}(\varepsilon_i \circ \theta): X_*(T') \rightarrow \mathbb{Z}.\end{aligned}$$

Hence the quadruple $(X_*(T'), \{\tilde{e}_i\}_{i \in I}, \{\text{wt}_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ is a free pre-crystal structure (see [BK, 2.2]) and we denote it by $\mathcal{UD}_{\theta,T'}(\chi)$. As for the definition of crystal, see 3.4 or [KKM],[K1], [K2]. We have thus the following theorem:

Theorem 2.9 ([BK][N1]). *For any geometric crystal $\chi = (X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ and a positive structure $\theta: T' \rightarrow X$, the associated pre-crystal $\mathcal{UD}_{\theta,T'}(\chi)$ is a crystal (see [BK, 2.2]).*

Now, let $\mathcal{GC}^+(\mathfrak{g})$ be the category whose object is a triplet (χ, T', θ) where $\chi = (X, \{e_i\}, \{\gamma_i\}, \{\varepsilon_i\})$ is a $\mathfrak{g}$ -geometric crystal and $\theta: T' \rightarrow X$ is a positive structure on χ , and morphism $f: (\chi_1, T'_1, \theta_1) \rightarrow (\chi_2, T'_2, \theta_2)$ is given by a morphism $\varphi: X_1 \rightarrow X_2$ of geometric crystals such that

$$f := \theta_2^{-1} \circ \varphi \circ \theta_1: T'_1 \rightarrow T'_2,$$

is a positive rational mapping. Let $\mathcal{CR}(\mathfrak{g})$ be the category of $\mathfrak{g}$ -crystals. Then by the theorem above, we have

Corollary 2.10. *$\mathcal{UD}$ defines a functor*

$$\begin{aligned}\mathcal{UD}: \mathcal{GC}^+(\mathfrak{g}) &\longrightarrow \mathcal{CR}(\mathfrak{g}^L) \\ (\chi, T', \theta) &\mapsto \mathcal{UD}_{\theta,T'}(\chi) = (X_*(T'), \{\tilde{e}_i\}_{i \in I}, \{\text{wt}_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I}) \\ (f: (\chi_1, T'_1, \theta_1) \rightarrow (\chi_2, T'_2, \theta_2)) &\mapsto (\widehat{f}: X_*(T'_1) \rightarrow X_*(T'_2)),\end{aligned}$$

where $\mathfrak{g}^L$ is the Langlands dual for $\mathfrak{g}$.

We call the functor $\mathcal{UD}$ “ultra-discretization” as in [N1],[N2]. While for a crystal B , if there exists a geometric crystal χ and a positive structure $\theta: T' \rightarrow X$ on χ such that $\mathcal{UD}(\chi, T', \theta) \simeq B$ as crystals, an object (χ, T', θ) in $\mathcal{GC}^+(\mathfrak{g})$ is called a *tropicalization* of B .

§ 3. Perfect Crystals

§ 3.1. Affine weights

Let $\mathfrak{g}$ be an affine Lie algebra. The sets $\mathfrak{t}$, $\{\alpha_i\}_{i \in I}$ and $\{\alpha_i^\vee\}_{i \in I}$ be as in 2.1. We take $\mathfrak{t}$ so that $\dim \mathfrak{t} = \sharp I + 1$. Let $\delta \in Q_+$ be a unique element satisfying

$$\{\lambda \in Q \mid \langle \alpha_i^\vee, \lambda \rangle = 0 \text{ for any } i \in I\} = \mathbb{Z}\delta,$$

and let $\mathbf{c} \in \sum_i \mathbb{Z}_{\geq 0} \alpha_i^\vee \subset \mathfrak{g}$ be a unique central element satisfying

$$\{h \in Q^\vee \mid \langle h, \alpha_i \rangle = 0 \text{ for any } i \in I\} = \mathbb{Z}\mathbf{c}.$$

We write ([Kac, 6.1])

$$(3.1) \quad \mathbf{c} = \sum_i a_i^\vee \alpha_i^\vee, \quad \delta = \sum_i a_i \alpha_i.$$

Let $(\cdot, \cdot)$ be the non-degenerate W -invariant symmetric bilinear form on $\mathfrak{t}^*$ normalized by $(\delta, \lambda) = \langle \mathbf{c}, \lambda \rangle$ for $\lambda \in \mathfrak{t}^*$. Let us set $\mathfrak{t}_{\text{cl}}^* := \mathfrak{t}^* / \mathbb{C}\delta$ and let $\text{cl}: \mathfrak{t}^* \longrightarrow \mathfrak{t}_{\text{cl}}^*$ be the canonical projection. Then we have $\mathfrak{t}_{\text{cl}}^* \cong \bigoplus_i (\mathbb{C}\alpha_i^\vee)^*$. Set $\mathfrak{t}_0^* := \{\lambda \in \mathfrak{t}^* \mid \langle \mathbf{c}, \lambda \rangle = 0\}$, $(\mathfrak{t}_{\text{cl}}^*)_0 := \text{cl}(\mathfrak{t}_0^*)$. Then we have a positive-definite symmetric form on $(\mathfrak{t}_{\text{cl}}^*)_0$ induced by the one on $\mathfrak{t}^*$. Let $\Lambda_i \in \mathfrak{t}_{\text{cl}}^*$ ($i \in I$) be a weight such that $\langle \alpha_i^\vee, \Lambda_j \rangle = \delta_{i,j}$, which is called a *fundamental weight*. We choose P so that $P_{\text{cl}} := \text{cl}(P)$ coincides with $\bigoplus_{i \in I} \mathbb{Z}\Lambda_i$ and we call P_{cl} the *classical weight lattice*.

§ 3.2. Affinization

Let $U_q(\mathfrak{g}) = \langle e_i, f_i, q^h \mid i \in I, h \in P \rangle$ be the quantum affine algebra associated with P and $U'_q(\mathfrak{g}) = \langle e_i, f_i, q^h \mid i \in I, h \in (P_{\text{cl}})^* \rangle$ its subalgebra associated with $(P_{\text{cl}})^*$. Set $\text{Mod}^f(\mathfrak{g}, P_{\text{cl}})$ the category of a finite dimensional $U'_q(\mathfrak{g})$ -module M with a weight decomposition $M = \bigoplus_{\lambda \in P_{\text{cl}}} M_\lambda$.

Let M be an object in $\text{Mod}^f(\mathfrak{g}, P_{\text{cl}})$. For $l \in \mathbb{C}^\times$, define the $U'_q(\mathfrak{g})$ -module M_l as follows: There exists a $\mathbb{C}$ -linear bijective homomorphism $\Phi_l: M \longrightarrow M$, such that

$$\begin{aligned} q^h \Phi_l(u) &= \Phi_l(q^h u) \quad \text{for } h \in P_{\text{cl}}^*, \\ e_i \Phi_l(u) &= l^{\delta_{i,0}} \Phi_l(e_i u), \\ f_i \Phi_l(u) &= l^{-\delta_{i,0}} \Phi_l(f_i u). \end{aligned}$$

The module $M_l := \Phi_l(M)$ is said to be an *affinization* of M ([KMN1],[K1]).

§ 3.3. Fundamental representation $W(\varpi_1)_l$

Here we consider the following affine Lie algebras $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, D_{n+1}^{(2)}, A_{2n}^{(2)}$. Let $\{\Lambda_i \mid i \in I\}$ be the set of fundamental weights as in §3.1. Let $\varpi_1 := \Lambda_1 - a_1^\vee \Lambda_0$ be the (level 0) fundamental weight, where $i = 1$ is the node of the Dynkin diagram as in [KNO][KNO2].

Let $W(\varpi_1)$ be the fundamental representation of $U'_q(\mathfrak{g})$ (see [K1, Sect 5.]). By [K1, Theorem 5.17], $W(\varpi_1)$ is a finite-dimensional irreducible integrable $U'_q(\mathfrak{g})$ -module, an object in $\text{Mod}^f(\mathfrak{g}, P_{\text{lc}})$ and has a global basis with a simple crystal. Thus, we can consider its affinization $W(\varpi_1)_l$ ($l \in \mathbb{C}^\times$) specialized at $q = 1$. Then we obtain a

finite-dimensional $\mathfrak{g}$ -module denoted also by $W(\varpi_1)_l$ in which we shall construct affine geometric crystals in the next section.

Note that $W(\varpi_1)$ is an irreducible $\mathfrak{g}$ -module for any $\mathfrak{g}$ as above. But, for some i and $\mathfrak{g}$, $W(\varpi_i)$ is not necessarily irreducible.

§ 3.4. Limit of perfect crystals

We review the limits of perfect crystals following [KKM]. (See also [KMN1],[KMN2].)

Let $\mathfrak{g}$ be an affine Lie algebra, P_{cl} the classical weight lattice as above and for $l \in \mathbb{Z}_{>0}$ set $(P_{\text{cl}})_l^+ := \{\lambda \in P_{\text{cl}} \mid \langle \mathbf{c}, \lambda \rangle = l, \langle \alpha_i^\vee, \lambda \rangle \geq 0\}$.

Definition 3.1. We say that a crystal B is *perfect* of level l if

- (i) $B \otimes B$ is connected as a crystal graph.
- (ii) There exists $\lambda_0 \in P_{\text{cl}}$ such that

$$\text{wt}(B) \subset \lambda_0 + \sum_{i \neq 0} \mathbb{Z}_{\leq 0} \text{cl}(\alpha_i), \quad \#B_{\lambda_0} = 1$$

- (iii) There exists a finite-dimensional $U'_q(\mathfrak{g})$ -module V with a crystal pseudo-base B_{ps} such that $B \cong B_{ps}/\pm 1$
- (iv) The maps $\varepsilon, \varphi: B^{\min} := \{b \in B \mid \langle \mathbf{c}, \varepsilon(b) \rangle = l\} \longrightarrow (P_{\text{cl}}^+)_l$ are bijective, where $\varepsilon(b) := \sum_i \varepsilon_i(b) \Lambda_i$ and $\varphi(b) := \sum_i \varphi_i(b) \Lambda_i$.

Theorem 3.2 ([KMN1],[KMN2]). *For each affine Lie algebra $\mathfrak{g}$ as above and for any positive integer l , there exists a perfect crystal $B_l(\mathfrak{g})$ of level l .*

For an affine Lie algebra $\mathfrak{g}$, let $\{B_l(\mathfrak{g})\}_{l \geq 1}$ be a family of perfect crystals of level l and set $J := \{(l, b) \mid l > 0, b \in B_l^{\min}\}$.

Definition 3.3. A crystal $B_\infty = B_\infty(\mathfrak{g})$ with an element b_∞ is called the *limit* of $\{B_l\}_{l \geq 1}$ if

- (i) $\text{wt}(b_\infty) = \varepsilon(b_\infty) = \varphi(b_\infty) = 0$.
- (ii) For any $(l, b) \in J$, there exists an embedding of crystals:

$$\begin{aligned} f_{(l,b)}: T_{\varepsilon(b)} \otimes B_l \otimes T_{-\varphi(b)} &\hookrightarrow B_\infty \\ t_{\varepsilon(b)} \otimes b \otimes t_{-\varphi(b)} &\mapsto b_\infty \end{aligned}$$

- (iii) $B_\infty = \bigcup_{(l,b) \in J} \text{Im} f_{(l,b)}$.

As for the crystal T_λ , see Example 2.3 (iii). If the limit of a family $\{B_l\}$ exists, we say that $\{B_l\}$ is a *coherent family* of perfect crystals.

Proposition 3.4 ([KKM]). *For each affine Lie algebra $\mathfrak{g}$ as above there exists a coherent family of perfect crystals $\{B_l(\mathfrak{g})\}_{l>0}$ and its limit $B_\infty(\mathfrak{g})$.*

§ 4. Affine Geometric Crystals

Following the method in [KNO2], we shall see how to construct the affine geometric crystal $\mathcal{V}(\mathfrak{g})_l$ ($l \in \mathbb{C}^\times$) in $W(\varpi_1)_l$.

§ 4.1. Translation $t(\tilde{\varpi}_1)$

For $\xi_0 \in (\mathfrak{t}_{\text{cl}}^*)_0$, let $t(\xi_0)$ be as in [K1, Sect.4]:

$$t(\xi_0)(\lambda) := \lambda + (\delta, \lambda)\xi - (\xi, \lambda)\delta - \frac{(\xi, \xi)}{2}(\delta, \lambda)\delta$$

for $\xi \in \mathfrak{t}^*$ such that $\text{cl}(\xi) = \xi_0$. Then $t(\xi_0)$ does not depend on the choice of ξ , and it is well-defined.

Let $c_i^\vee$ be as follows:

$$(4.1) \quad c_i^\vee := \max(1, \frac{2}{(\alpha_i, \alpha_i)}).$$

Then $t(m\varpi_i)$ belongs to the extended Weyl group $\widetilde{W}$ if and only if $m \in c_i^\vee \mathbb{Z}$. Setting $\tilde{\varpi}_i := c_i^\vee \varpi_i$ ($i \in I$) ([K2]), $t(\tilde{\varpi}_1)$ is expressed as follows:

$$t(\tilde{\varpi}_1) = \begin{cases} \iota(s_n s_{n-1} \cdots s_2 s_1) & A_n^{(1)} \text{ case,} \\ \iota(s_1 \cdots s_n)(s_{n-1} \cdots s_2 s_1) & B_n^{(1)}, A_{2n-1}^{(2)} \text{ cases,} \\ (s_0 s_1 \cdots s_n)(s_{n-1} \cdots s_2 s_1) & C_n^{(1)}, D_{n+1}^{(2)} \text{ cases,} \\ \iota(s_1 \cdots s_n)(s_{n-2} \cdots s_2 s_1) & D_n^{(1)} \text{ case,} \\ (s_0 s_1 \cdots s_n)(s_{n-1} \cdots s_2 s_1) & A_{2n}^{(2)} \text{ case,} \end{cases}$$

where ι is certain Dynkin diagram automorphism. Now, we know that each $t(\tilde{\varpi}_1)$ is in the form w_1 or $\iota \cdot w_1$ for some $w_1 \in W$, e.g., $w_1 = s_n \cdots s_1$ for $A_n^{(1)}$, $w_1 = (s_1 \cdots s_n)(s_{n-1} \cdots s_1)$ for $B_n^{(1)}$, etc.,

In the case $A_{2n}^{(2)}$, $\eta := \text{wt}(v_{\bar{n}}) = \Lambda_{n-1} - \Lambda_n$ (resp. $\Lambda_{n-1} - 2\Lambda_n$) is a unique weight of $W(\varpi_1)_l$ which satisfies $\langle \alpha_i^\vee, \eta \rangle \geq 0$ for $i \neq n$. For this η we have

$$(4.2) \quad t(\eta) = (s_n s_{n-1} \cdots s_1)(s_0 s_1 \cdots s_{n-1}) =: w_2.$$

§ 4.2. Affine geometric crystals in $W(\varpi_1)_l$

Let σ be the Dynkin diagram automorphism as in [KNO2] and $w_1 = s_{i_1} \cdots s_{i_k}$ be as in the previous subsection. Let $H \in \mathfrak{t}$ be an element as in [KNO2].

$$\alpha_i(H) = \begin{cases} 1 & \text{if } i = 1 \text{ and } \mathfrak{g} \neq D_{n+1}^{(2)}, A_{2n}^{(2)} \\ 2 & \text{if } i = 1 \text{ and } \mathfrak{g} = D_{n+1}^{(2)}, A_{2n}^{(2)} \\ -1 & \text{if } i = 0 \text{ and } \mathfrak{g} \neq D_{n+1}^{(2)}, A_{2n}^{(2)} \\ -2 & \text{if } i = 0 \text{ and } \mathfrak{g} = D_{n+1}^{(2)}, A_{2n}^{(2)} \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$(4.3) \quad \mathcal{V}(\mathfrak{g})_l := \{v(x_1, \dots, x_k) := Y_{i_1}(x_1) \cdots Y_{i_k}(x_k) l^H v_1 \mid x_1, \dots, x_k \in \mathbb{C}^\times\} \subset W(\varpi_1)_l$$

where the vector v_1 is a highest weight vector in $W(\varpi_1)_l$ as a $\mathfrak{g}_0$ -module and then $\mathcal{V}(\mathfrak{g})$ has a G_0 -geometric crystal structure, where $\mathfrak{g}_0 \subset \mathfrak{g}$ (resp. $G_0 \subset G$) is a simple Lie algebra (resp. simple algebraic group) corresponding to the index set $I_0 := I \setminus \{0\}$. Moreover $(\mathbb{C}^\times)^k \rightarrow \mathcal{V}(\mathfrak{g})_l$ given by $(x_1, \dots, x_k) \mapsto v(x_1, \dots, x_k)$ is a birational morphism. We shall define a G -geometric crystal structure on $\mathcal{V}(\mathfrak{g})_l$ by using the Dynkin diagram automorphism σ except for $A_{2n}^{(2)}$. This σ induces an automorphism of $W(\varpi_1)_l$, which is denoted by $\sigma_l: W(\varpi_1)_l \rightarrow W(\varpi_1)_l$.

Theorem 4.1 ([KNO2]).

- (i) *Case $\mathfrak{g} \neq A_{2n}^{(2)}$. For $x = (x_1, \dots, x_k) \in (\mathbb{C}^\times)^k$, there exist a unique $y = (y_1, \dots, y_k) \in (\mathbb{C}^\times)^k$ and a positive rational function $a(x)$ such that*

$$(4.4) \quad v(y) = a(x) \sigma_l(v(x)), \quad \varepsilon_{\sigma(i)}(v(y)) = \varepsilon_i(v(x)) \quad \text{if } i, \sigma(i) \neq 0.$$

- (ii) *Case $\mathfrak{g} = A_{2n}^{(2)}$. Associated with w_1 and w_2 as in the previous subsection, we define*

$$\mathcal{V}(\mathfrak{g})_l := \{v_1(x) = Y_0(x_0) Y_1(x_1) \cdots Y_n(x_n) Y_{n-1}(\bar{x}_{n-1}) \cdots Y_1(\bar{x}_1) l^H v_1 \mid x_i, \bar{x}_i \in \mathbb{C}^\times\},$$

$$\mathcal{V}_2(\mathfrak{g})_l := \{v_2(y) = Y_n(y_n) \cdots Y_1(y_1) Y_0(y_0) Y_1(\bar{y}_1) \cdots Y_{n-1}(\bar{y}_{n-1}) l^{H'} v_{\bar{n}} \mid y_i, \bar{y}_i \in \mathbb{C}^\times\},$$

where $\alpha_0(H') = 2$, $\alpha_n(H') = -4$ and $\alpha_i(H') = 0$ otherwise. (Note that $\text{wt}(v_1)(H) = \text{wt}(v_{\bar{n}})(H')$.) For any $x \in (\mathbb{C}^\times)^{2n}$ there exist a unique $y \in (\mathbb{C}^\times)^{2n}$ and a rational function $a(x)$ such that $v_2(y) = a(x) v_1(x)$.

Now, using this theorem, we define the rational mapping

$$(4.5) \quad \begin{aligned} \bar{\sigma}: \mathcal{V}(\mathfrak{g})_l &\longrightarrow \mathcal{V}(\mathfrak{g})_l, & \bar{\sigma}: \mathcal{V}(\mathfrak{g})_l &\longrightarrow \mathcal{V}_2(\mathfrak{g})_l, \\ v(x) &\mapsto v(y) \quad (\mathfrak{g} \neq A_{2n}^{(2)}), & v_1(x) &\mapsto v_2(y) \quad (\mathfrak{g} = A_{2n}^{(2)}), \end{aligned}$$

Associated with an element $w \in W$, let B_w^- be the geometric crystal as in [N1], [KNO], which is isomorphic to the geometric crystal on the Schubert cell X_w (see [N1]).

Theorem 4.2 ([KNO2]). *The rational mapping $\bar{\sigma}$ is birational. If we define a $\mathfrak{g}_0$ -geometric crystal structure on $\mathcal{V}(\mathfrak{g})_l$ by the one on $B_{w_1}^- \cdot l^H$ and a rational $\mathbb{C}^\times$ -action $e_0 : \mathbb{C}^\times \times \mathcal{V}(\mathfrak{g})_l \rightarrow \mathcal{V}(\mathfrak{g})_l$ and rational functions wt_0 and ε_0 on $\mathcal{V}(\mathfrak{g})_l$ by*

$$(4.6) \quad \begin{cases} e_0^c := \bar{\sigma}^{-1} \circ e_{\sigma(0)}^c \circ \bar{\sigma}, & \varepsilon_0 := \varepsilon_{\sigma(0)} \circ \bar{\sigma}, & \gamma_0 := \gamma_{\sigma(0)} \circ \bar{\sigma}, & \text{for } \mathfrak{g} \neq A_{2n}^{(2)}, \\ e_0^c := \bar{\sigma}^{-1} \circ e_0^c \circ \bar{\sigma}, & \varepsilon_0 := \varepsilon_0 \circ \bar{\sigma}, & \gamma_0 := \gamma_0 \circ \bar{\sigma}, & \text{for } \mathfrak{g} = A_{2n}^{(2)}. \end{cases}$$

then $(\mathcal{V}(\mathfrak{g})_l, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ turns out to be a positive affine $\mathfrak{g}$ -geometric crystal.

Remark. In the case $\mathfrak{g} = A_{2n}^{(2)}$, $\mathcal{V}_2(\mathfrak{g})$ has a $\mathfrak{g}_{I \setminus \{n\}}$ -geometric crystal structure. Thus, $e_0, \gamma_0, \varepsilon_0$ are well-defined on $\mathcal{V}_2(\mathfrak{g})$.

§ 4.3. Product structure on affine geometric crystals

In general, if χ_1 and χ_2 are geometric crystals induced from unipotent crystals, the product $\chi_1 \times \chi_2$ possesses a geometric crystal structure ([BK]). More precisely, let $\chi_1 = (X, \{e_i\}, \{\gamma_i\}, \{\varepsilon_i\})$ and $\chi_2 = (Y, \{e_i\}, \{\gamma_i\}, \{\varepsilon_i\})$ be geometric crystals induced from unipotent crystals. For $x \in X$ and $y \in Y$ set

$$(4.7) \quad \gamma_i(x, y) := \gamma_i(x)\gamma_i(y),$$

$$(4.8) \quad \varepsilon_i(x, y) := \varepsilon_i(x) + \frac{\varepsilon_i(x)\varepsilon_i(y)}{\varphi_i(x)} \quad (\varphi_i(x) = \gamma(x)\varepsilon_i(x)),$$

$$(4.9) \quad e_i^c(x, y) := (e_i^{c_1}(x), e_i^{c_2}(y)) \quad \text{where } c_1 := \frac{c\varphi_i(x) + \varepsilon_i(y)}{\varphi_i(x) + \varepsilon_i(y)}, \quad c_2 := \frac{c}{c_1}.$$

Then, (4.7)–(4.9) endow the product $X \times Y$ with a structure of a geometric crystal.

As for the affine geometric crystal $\mathcal{V}_l$ in the previous subsection, its data $e_i, \gamma_i, \varepsilon_i$ ($i = 1, \dots, n$) are obtained from the ones of the geometric crystal $B_{\mathbf{i}}^- \cdot l^H$ which is induced from the unipotent crystal on some $X_w \times l^H$ where $\mathbf{i}$ is a reduced word for w and X_w is the Schubert cell associated with $w \in W$ and note that $\{l^H\}$ holds trivial unipotent crystal structure. We can check the $i = 0$ -case directly and then obtain:

Theorem 4.3 ([KNO2]). *For any $k \in \mathbb{Z}_{\geq 0}$ and $L_1 \cdots, L_k \in \mathbb{C}^\times$, the product $\mathcal{V}_{L_1} \times \cdots \times \mathcal{V}_{L_k}$ has an affine geometric crystal structure.*

§ 4.4. Ultra-discretization of $\mathcal{V}(\mathfrak{g})_l$

Let us see the ultra-discretization of $\mathcal{V}(\mathfrak{g})_l$.

By Theorem 4.2 on $\mathcal{V}(\mathfrak{g})_l$, for $l > 0$ it holds a natural positive structure $\theta_l : (\mathbb{C}^\times)^m \rightarrow \mathcal{V}(\mathfrak{g})_l$ ($x \mapsto v(x)$) where $m = \dim \mathcal{V}(\mathfrak{g})_l$. Then we have the following theorem:

Theorem 4.4 ([KNO2]). *For $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, D_n^{(1)}, D_{n+1}^{(2)}, A_{2n-1}^{(2)}$ and $A_{2n}^{(2)}$, suppose that $l > 0$. Then the ultra-discretization $\mathcal{UD}_{\theta_l}(\mathcal{V}(\mathfrak{g})_l)$ associated with the positive structure θ_l is isomorphic to the crystal $B_\infty(\mathfrak{g}^L)$.*

§ 5. Tropical R Maps

§ 5.1. Definition of tropical R maps

Definition 5.1. Let $\{(X_\lambda, \{e_i^\lambda\}, \{\gamma_i^\lambda\}, \{\varepsilon_i^\lambda\})\}_{\lambda \in \Lambda}$ be a family of geometric crystals equipped with the product structures, where Λ is a certain index set and its element is called a *spectral parameter*. A birational isomorphism $\mathcal{R}_{\lambda\mu} : X_\lambda \times X_\mu \longrightarrow X_\mu \times X_\lambda$ ($\lambda, \mu \in \Lambda$) is said to be a *tropical R map* if it satisfies the following conditions:

$$(5.1) \quad (e_i^{X_\mu \times X_\lambda})^c \circ \mathcal{R}_{\lambda\mu} = \mathcal{R}_{\lambda\mu} \circ (e_i^{X_\lambda \times X_\mu})^c,$$

$$(5.2) \quad \varepsilon_i^{X_\lambda \times X_\mu} = \varepsilon_i^{X_\mu \times X_\lambda} \circ \mathcal{R}_{\lambda\mu},$$

$$(5.3) \quad \gamma_{X_\lambda \times X_\mu} = \gamma_{X_\mu \times X_\lambda} \circ \mathcal{R}_{\lambda\mu},$$

$$(5.4) \quad \mathcal{R}^{(12)} \mathcal{R}^{(23)} \mathcal{R}^{(12)} = \mathcal{R}^{(23)} \mathcal{R}^{(12)} \mathcal{R}^{(23)} \quad \text{on } X_\lambda \times X_\mu \times X_\nu,$$

for any $i \in I$ and any $\lambda, \mu, \nu \in \Lambda$. Here $\mathcal{R}^{(ij)}$ means that it acts on i -th and j -th components of the product.

In [KNO2], we obtain tropical R maps $\{\overline{\mathcal{R}}_{LM}\}_{L, M > 0}$ of the family of affine geometric crystals $\{\mathcal{V}_L\}_{L > 0}$ for the above affine Lie algebras.

§ 5.2. Prehomogeneous Geometric Crystal

Definition 5.2. Let $\chi = (X, \{e_i^c\}, \{\gamma_i\}, \{\varepsilon_i\})$ be a geometric crystal. We say that χ is *prehomogeneous* if there exists a Zariski open dense subset $\Omega \subset X$ which is an orbit by the actions of the e_i^c 's.

The following lemma is obvious.

Lemma 5.3 ([KNO2]). *Let $\chi_j = (X_j, \{e_i^c\}, \{\gamma_i\}, \{\varepsilon_i\})$ ($j = 1, 2$) be prehomogeneous geometric crystals. Let $\Omega_1 \subset X_1$ be an open dense orbit in X_1 . For isomorphisms of geometric crystals $\phi, \phi' : \chi_1 \rightarrow \chi_2$, suppose that there exists $p_1 \in \Omega_1$ such that $\phi(p_1) = \phi'(p_1) \in X_2$. Then, we have $\phi = \phi'$ as rational morphisms.*

The following is the key of this article:

Theorem 5.4 ([KNO2]). *Let $\chi = (X, \{e_i^c\}, \{\gamma_i\}, \{\varepsilon_i\})$ be a finite-dimensional positive geometric crystal with the positive structure $\theta : T \rightarrow X$ and $B := \mathcal{UD}_\theta(\chi)$ the crystal obtained as the ultra-discretization of χ . χ is prehomogeneous if B is a connected crystal.*

§ 5.3. Uniqueness of the birational R-maps

Theorem 5.5 ([KNO2]). *Let $\overline{\mathcal{R}}_{LM}$ be the tropical R map as above. Set $z_0 := \overline{\mathcal{R}}(\mathbf{1}, \mathbf{1})$ where $\mathbf{1} := v(1, \dots, 1) \in \mathcal{V}_L$. Let $\mathcal{R}'$ be a tropical R map such that $\mathcal{R}'(\mathbf{1}, \mathbf{1}) = z_0$. Then we have $\overline{\mathcal{R}} = \mathcal{R}'$ as birational morphisms.*

Proof. Let $\mathcal{V}_l, \mathcal{V}_m$ ($l, m \in \mathbb{C}^\times$) be the affine geometric crystals constructed in Sect.4 and $\mathcal{R}$ the tropical R map as in this section and $\mathcal{R}' : \mathcal{V}_l \times \mathcal{V}_m \rightarrow \mathcal{V}_m \times \mathcal{V}_l$ other tropical R map such that $\mathcal{R}(\mathbf{1}, \mathbf{1}) = \mathcal{R}'(\mathbf{1}, \mathbf{1})$. By Theorem 4.4, we have that $\mathcal{UD}_{(\Theta_l, \Theta_m)}(\mathcal{V}_l(\mathfrak{g}) \times \mathcal{V}_m(\mathfrak{g}))$ is isomorphic to the crystal $B_\infty(\mathfrak{g}^L) \otimes B_\infty(\mathfrak{g}^L)$.

It follows from that for any positive integer m and any affine Lie algebra $\mathfrak{g}$ appearing in this article tensor product of crystals $\bigotimes_{i=1}^m \mathcal{UD}_{\Theta_{l_i}}(\mathcal{V}(\mathfrak{g})_{l_i}) \cong B_\infty(\mathfrak{g}^L)^{\otimes m}$ is connected. By Theorem 5.4 we obtain that the geometric crystal $\mathcal{V}_l \times \mathcal{V}_m$ is prehomogeneous. Therefore, by Lemma 5.3 we obtain $\mathcal{R} = \mathcal{R}'$, which completes the proof of Theorem 5.5. $\square$

§ 6. Universal tropical R map of $\mathfrak{sl}_2$

Only for $\mathfrak{sl}_2$ -case, we found a kind of "universal" tropical R map. For arbitrary $\mathfrak{sl}_2$ -geometric crystals $\mathbb{X} = (X, e_X, \gamma_X, \varepsilon_X)$ and $\mathbb{Y} = (Y, e_Y, \gamma_Y, \varepsilon_Y)$, the morphism $\mathcal{R} : X \times Y \rightarrow Y \times X$ is defined by

$$(6.1) \quad \mathcal{R}(x, y) := (e_Y^{a(x, y)} y, e_X^{b(x, y)} x), \quad \text{where} \quad a(x, y) := \frac{\varepsilon_Y(y) + \frac{\varepsilon_X(x)}{\gamma_Y(y)}}{\varepsilon_X(x) + \frac{\varepsilon_Y(y)}{\gamma_X(x)}}, \quad b(x, y) = \frac{1}{a(x, y)}.$$

Theorem 6.1. *The morphism $\mathcal{R}$ satisfies the following:*

$$(6.2) \quad \gamma_{Y \times X}(\mathcal{R}(x, y)) = \gamma_{X \times Y}(x, y),$$

$$(6.3) \quad \varepsilon_{Y \times X}(\mathcal{R}(x, y)) = \varepsilon_{X \times Y}(x, y),$$

$$(6.4) \quad \mathcal{R} \circ e^c = e^c \circ \mathcal{R},$$

$$(6.5) \quad \mathcal{R}^{(12)} \mathcal{R}^{(23)} \mathcal{R}^{(12)} = \mathcal{R}^{(23)} \mathcal{R}^{(12)} \mathcal{R}^{(23)} \quad (\text{Yang-Baxter eq.})$$

$$(6.6) \quad \mathcal{R} \circ \mathcal{R} = \text{id}_{X \times Y}$$

This implies that there exists the canonical morphism with the same properties as a universal R-matrix for modules of quantum groups. Furthermore, the morphism $\mathcal{R}$ is an isomorphism of geometric crystals.

Proof. Let us show (6.2). We have

$$\gamma_{Y \times X}(\mathcal{R}(x, y)) = \gamma_Y(e^{a(x, y)} y) \gamma_X(e^{b(x, y)} x) = a(x, y)^2 b(x, y)^2 \gamma_X(x) \gamma_Y(y) = \gamma_{X \times Y}(x, y).$$

Next, let us see (6.3).

$$\begin{aligned} \varepsilon_{Y \times X}(\mathcal{R}(x, y)) &= \varepsilon_Y(e^{a(x, y)} y) + \frac{\varepsilon_X(e^{b(x, y)} x)}{\gamma_Y(e^{a(x, y)} y)} = a(x, y)^{-1} \varepsilon_Y(y) + \frac{b(x, y)^{-1} \varepsilon_X(x)}{a(x, y)^2 \gamma_Y(y)} \\ &= a(x, y)^{-1} \left(\varepsilon_Y(y) + \frac{\varepsilon_X(x)}{\gamma_Y(y)} \right) = \varepsilon_X(x) + \frac{\varepsilon_Y(y)}{\gamma_X(x)} = \varepsilon_{X \times Y}(x, y). \end{aligned}$$

Let us show (6.4). For $c \in \mathbb{C}^\times$ and $(x, y) \in X \times Y$, let $c_1 := c_1(c, x, y)$, $c_2 := c_2(c, x, y)$ be as in (4.9). We have

$$(6.7) \quad \mathcal{R}(e^c(x, y)) = (e^{a(e^{c_1}x, e^{c_2}y)c_2}y, e^{b(e^{c_1}x, e^{c_2}y)c_1}x),$$

$$(6.8) \quad e^c \mathcal{R}(x, y) = (e^{c'_1 a(x, y)}y, e^{c'_2 b(x, y)}x),$$

where $c'_1 := c_1(c, e^{a(x, y)}y, e^{b(x, y)}x)$ and $c'_2 := c_2(c, e^{a(x, y)}y, e^{b(x, y)}x)$. By simple calculations, one has

$$(6.9) \quad a(e^{c_1}x, e^{c_2}y) = \frac{c_1}{c_2} \frac{c\varepsilon_Y(y) + \frac{\varepsilon_X(x)}{\gamma_Y(y)}}{c\varepsilon_X(x) + \frac{\varepsilon_Y(y)}{\gamma_X(x)}}, \quad c'_1 = \frac{c\varepsilon_Y(y) + \frac{\varepsilon_X(x)}{\gamma_Y(y)}}{\varepsilon_Y(y) + \frac{\varepsilon_X(x)}{\gamma_Y(y)}},$$

which means $a(e^{c_1}x, e^{c_2}y)c_2 = c'_1 a(x, y)$ and then $b(e^{c_1}x, e^{c_2}y)c_1 = c'_2 a(x, y)$. We obtained (6.4).

Let us show (6.5). Denote

$$\begin{aligned} \mathcal{R}^{(12)}\mathcal{R}^{(23)}\mathcal{R}^{(12)}(x, y, z) &= (e^A z, e^B y, e^C x), \\ \mathcal{R}^{(23)}\mathcal{R}^{(12)}\mathcal{R}^{(23)}(x, y, z) &= (e^{A'} z, e^{B'} y, e^{C'} x). \end{aligned}$$

It is easy to see that $ABC = 1 = A'B'C'$ and then we may show that $A = A'$ and $C = C'$. By simple calculations, we have

$$\begin{aligned} A &= a(e^{a(x, y)}y, e^{a(e^{b(x, y)}x, z)}z)a(e^{b(x, y)}x, z) = \frac{\varepsilon_Z(z) + \frac{\varepsilon_X(x)}{b(x, y)\gamma_Z(z)}}{\frac{\varepsilon_X(x)}{b(x, y)} + \frac{\varepsilon_Z(z)}{b(x, y)^2\gamma_X(x)}} \\ &= \frac{\varepsilon_Z(z) \left(\varepsilon_Z(z) + \frac{\varepsilon_Y(y)}{\gamma_Z(z)} \right) + \frac{\varepsilon_X(x)}{\gamma_Z(z)} \left(\varepsilon_Y(y) + \frac{\varepsilon_Z(z)}{\gamma_Y(y)} \right)}{\varepsilon_X(x) \left(\varepsilon_Z(z) + \frac{\varepsilon_Y(y)}{\gamma_Z(z)} \right) + \frac{\varepsilon_Z(z)}{\gamma_X(x)} \left(\varepsilon_Y(y) + \frac{\varepsilon_Z(z)}{\gamma_Y(y)} \right)}, \\ A' &= a(x, e^{a(y, z)}z)a(y, z) = \frac{\varepsilon_Z(z) + \frac{\varepsilon_X(x)}{a(y, z)\gamma_Z(z)}}{\varepsilon_X(x) + \frac{\varepsilon_Z(z)}{a(y, z)\gamma_X(x)}} \\ &= \frac{\varepsilon_Z(z) \left(\varepsilon_Z(z) + \frac{\varepsilon_Y(y)}{\gamma_Z(z)} \right) + \frac{\varepsilon_X(x)}{\gamma_Z(z)} \left(\varepsilon_Y(y) + \frac{\varepsilon_Z(z)}{\gamma_Y(y)} \right)}{\varepsilon_X(x) \left(\varepsilon_Z(z) + \frac{\varepsilon_Y(y)}{\gamma_Z(z)} \right) + \frac{\varepsilon_Z(z)}{\gamma_X(x)} \left(\varepsilon_Y(y) + \frac{\varepsilon_Z(z)}{\gamma_Y(y)} \right)}, \end{aligned}$$

which means $A = A'$.

Next, we have

$$\begin{aligned} C &= b(e^{b(x, y)}x, z)b(x, y), \\ C' &= b(e^{b(x, e^{a(y, z)}z)}x, e^{b(y, z)}y)b(x, e^{a(y, z)}z) \end{aligned}$$

It follows from the explicit forms of $a(x, y)$ and $b(x, y)$ that $a(x, y) = b(y, x)$. Thus we

obtain

$$\begin{aligned} C|_{x \leftrightarrow z} &= b(e^{b(z,y)}z, x)b(z, y) = a(x, e^{a(y,z)}z)a(y, z) = A' \\ C'|_{x \leftrightarrow z} &= b(e^{b(z, e^{a(y,x)}x)}z, e^{b(y,x)}y)b(z, e^{a(y,x)}x) \\ &= a(e^{a(x,y)}y, e^{a(e^{b(x,y)}x, z)}z)a(e^{b(x,y)}x, z) = A, \end{aligned}$$

which implies $C = C'$. Thus, we also obtain $B = B'$.

Finally, let us show (6.6). Set $(y', x') = \mathcal{R}(x, y)$. Then we have

$$\mathcal{R} \circ \mathcal{R}(x, y) = (e_X^{a(y', x')} e_X^{b(x, y)} x, e_Y^{b(y', x')} e_Y^{a(x, y)} y).$$

By the formula (6.9) and $a(x, y)b(x, y) = 1$ we obtain

$$a(y', x') = a(e_y^{a(x, y)} y, e_X^{b(x, y)} x) = \frac{a(x, y)}{b(x, y)} a(y, x) = \frac{a(x, y)}{b(x, y)} b(x, y) = a(x, y),$$

and also $b(y', x') = b(x, y)$. Thus, we have $\mathcal{R} \circ \mathcal{R}(x, y) = (x, y)$ and completed the proof.

□

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