

SOME SUFFICIENT CONDITIONS FOR STRONGLY STARLIKENESS

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Abstract. We consider strongly starlikeness of order α of functions $f(z) = z + a_{n+1}z^{n+1} + \dots$ which are analytic in the unit disc and satisfy the condition of the form

$$\left| f'(z) \left(\frac{z}{f(z)} \right)^{1+\mu} - 1 \right| < \lambda, \quad 0 < \mu < 1, \quad 0 < \lambda < 1.$$

1. INTRODUCTION AND PRELIMINARIES

Let H denote the class of functions analytic in the unit disc $U = \{z: |z| < 1\}$ and let $A \subset H$ be the class of normalized analytic functions f in U such that $f(0) = f'(0) - 1 = 0$. For $n \geq 1$ we put

$$A_n = \left\{ f: f(z) = z + a_{n+1}z^{n+1} + \dots \text{ is analytic in } U \right\}$$

and $A_1 = A$.

A function $f \in A$ is said to be *strongly starlike of order α* , $0 < \alpha \leq 1$, if and only if

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z} \right)^\alpha,$$

where $\prec$ denotes the usual *subordination*. We denote this class by $S(\alpha)$. If $\alpha=1$, then $S(1) \equiv S^*$ is the well-known class of *starlike functions* in U (see, for example, [1]).

In this paper we find a condition so that $f \in A_n$ satisfying

$$(1) \quad f'(z) \left(\frac{z}{f(z)} \right)^{1+\mu} \prec 1 + \lambda z, \quad 0 < \mu < 1, \quad 0 < \lambda < 1,$$

is in $S(\alpha)$. Also, we consider an integral transformation.

We note that the author in [4] determined the values for λ in (1) which implies starlikeness in U . Recently, Ponnusamy and Singh [5] found the condition which implies the strongly starlikeness of order α , but for $\mu < 0$ in (1). By using the similar method as in [5] we consider strongly starlikeness in the case (1).

First, we cite the following

LEMMA A. Let $Q \in H$ satisfy the subordination condition

$$Q(z) \prec 1 + \lambda_1 z, \quad Q(0) = 1,$$

where $0 < \lambda_1 \leq 1$. For $0 < \alpha \leq 1$, let $p \in H$, $p(0) = 1$ and p satisfy the condition

$$Q(z)p^\alpha(z) < 1 + \lambda z, \quad 0 < \lambda \leq 1.$$

Then for

$$(2) \quad \sin^{-1} \lambda + \sin^{-1} \lambda_1 \leq \frac{\alpha\pi}{2}$$

we have $\operatorname{Re}\{p(z)\} > 0$ in U .

This is the special case of the more general lemma given in [5].

2. RESULTS AND CONSEQUENCES

For our results we also need the following two lemmas.

LEMMA 1. Let $p \in H$, $p(z) = 1 + p_n z^n + \dots$, $n \geq 1$, satisfy the condition

$$p(z) - \frac{1}{\mu} z p'(z) < 1 + \lambda z, \quad 0 < \mu < 1, \quad 0 < \lambda \leq 1.$$

Then

$$p(z) < 1 + \lambda_1 z,$$

where

$$\lambda_1 = \frac{\lambda\mu}{n-\mu}.$$

The proof of this lemma for $n=1$ is given by the author in [4]. For any $n \in \mathbb{N}$ we also can apply Jack's lemma [3].

LEMMA 2. If $0 < \mu < 1$, $0 < \lambda \leq 1$ and $Q \in H$ satisfying

$$Q(z) < 1 + \frac{\lambda\mu}{n-\mu} z, \quad Q(0) = 1, \quad n \in \mathbb{N},$$

and if $p \in H$, $p(0) = 1$ and satisfies

$$Q(z)p^\alpha(z) < 1 + \lambda z,$$

where

$$(3) \quad 0 < \lambda \leq \frac{(n-\mu)\sin(\pi\alpha/2)}{|\mu + (n-\mu)e^{i\pi\alpha/2}|},$$

then $\operatorname{Re}\{p(z)\} > 0$ in U .

Proof. If in Lemma A we put $\lambda_1 = \frac{\lambda\mu}{n-\mu}$, then the condition (2) is equivalent to

$$\sin^{-1} \lambda + \sin^{-1} \frac{\lambda\mu}{n-\mu} \leq \frac{\alpha\pi}{2}.$$

This inequality is equivalent to

$$\sin^{-1} \left(\lambda \sqrt{1 - \frac{\lambda^2 \mu^2}{(n-\mu)^2}} + \frac{\lambda \mu}{n-\mu} \sqrt{1-\lambda^2} \right) \leq \frac{\alpha \pi}{2},$$

or to the inequality

$$\lambda \left[\sqrt{(n-\mu)^2 - \lambda^2 \mu^2} + \mu \sqrt{1-\lambda^2} \right] \leq (n-\mu) \sin(\alpha \pi / 2).$$

From there, after some transformations, we get the following equivalent inequality

$$\begin{aligned} & \left([\mu^2 + (n-\mu)^2]^2 - 4\mu^2(n-\mu)^2 \cos^2(\alpha \pi / 2) \right) \lambda^4 - 2(n-\mu)^2 [\mu^2 + (n-\mu)^2] \sin^2(\alpha \pi / 2) \lambda^2 + \\ & + (1-\mu)^4 \sin^4(\alpha \pi / 2) \geq 0 \end{aligned}$$

which is equivalent to the condition (3).

By Lemma A we have that $\operatorname{Re}\{p(z)\} > 0$ in U .

THEOREM 1. Let $f \in A_n$, $0 < \mu < 1$ and f satisfy the subordination

$$f'(z) \left(\frac{z}{f(z)} \right)^{1+\mu} < 1 + \lambda z,$$

where

$$0 < \lambda \leq \frac{n-\mu}{\sqrt{\mu^2 + (n-\mu)^2}}.$$

Then $f \in S^*$.

Proof. If we put $Q(z) = \left(\frac{z}{f(z)} \right)^\mu (= 1 + q_n z^n + \dots)$, then after some calculation, we get

$$Q(z) - \frac{1}{\mu} z Q'(z) = f'(z) \left(\frac{z}{f(z)} \right)^{1+\mu} < 1 + \lambda z.$$

From Lemma 1 we have

$$Q(z) < 1 + \lambda_1 z, \quad \lambda_1 = \frac{\lambda \mu}{n-\mu}.$$

The rest part of the proof is the same as in the case $n=1$ (Theorem 1 in [4]) and we omit the details.

THEOREM 2. Let $0 < \mu < 1$ and $0 < \alpha \leq 1$. If $f \in A_n$ satisfies

$$(4) \quad \left| f'(z) \left(\frac{z}{f(z)} \right)^{1+\mu} - 1 \right| < \frac{(n-\mu) \sin(\pi \alpha / 2)}{|\mu + (n-\mu) e^{i \pi \alpha / 2}|}, \quad z \in U,$$

then $f \in S(\alpha)$.

Proof. If we put $\lambda = \frac{(n-\mu) \sin(\pi \alpha / 2)}{|\mu + (n-\mu) e^{i \pi \alpha / 2}|}$, then, since $0 < \alpha \leq 1$, we have

$0 < \lambda \leq \frac{n-\mu}{\sqrt{\mu^2 + (n-\mu)^2}}$, and by Theorem 1, $f \in S^*$. Later, the function

$$Q(z) = \left(\frac{z}{f(z)} \right)^\mu = 1 + q_n z^n + \dots \text{ is analytic in } U \text{ and satisfies } Q(z) < 1 + \lambda_1 z, \lambda_1 = \frac{\lambda\mu}{n-\mu}.$$

Now, if we define

$$p(z) = \left(\frac{zf'(z)}{f(z)} \right)^{\frac{1}{\alpha}},$$

then p is analytic in U , $p(0)=1$ and the condition (4) is equivalent to

$$Q(z)p^\alpha(z) < 1 + \lambda z.$$

Finally, from Lemma 2 we obtain

$$\left(\frac{zf'(z)}{f(z)} \right)^{\frac{1}{\alpha}} < \frac{1+z}{1-z} \quad \left(\Leftrightarrow \frac{zf'(z)}{f(z)} < \left(\frac{1+z}{1-z} \right)^\alpha \right),$$

i.e. $f \in S(\alpha)$. $\square$

We note that for $\alpha=1$ we have the statement of Theorem 1.

For $n=1, \mu=1/2, \alpha=2/3$ we get the following

COROLLARY 1. Let $f \in A$ and let

$$\left| f'(z) \left(\frac{z}{f(z)} \right)^{\frac{3}{2}} - 1 \right| < \frac{1}{2}, \quad z \in U.$$

Then

$$\left| \arg \left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{\pi}{3}, \quad z \in U,$$

i.e. $f \in S(2/3)$.

THEOREM 3. Let $0 < \mu < 1$, $\operatorname{Re}\{c\} > -\mu$, and $0 < \alpha \leq 1$. If $f \in A_n$ satisfies

$$(5) \quad \left| f'(z) \left(\frac{z}{f(z)} \right)^{1+\mu} - 1 \right| < \left| \frac{n+c-\mu}{c-\mu} \right| \frac{(n-\mu)\sin(\pi\alpha/2)}{|\mu + (n-\mu)e^{i\pi\alpha/2}|}, \quad z \in U,$$

then the function

$$(6) \quad F(z) = z \left[\frac{c-\mu}{z^{c-\mu}} \int_0^z \left(\frac{t}{f(t)} \right)^\mu t^{c-\mu-1} dt \right]^{-\frac{1}{\mu}}$$

belongs to $S(\alpha)$.

Proof. If we put that λ is equal to the right hand side of the inequality (5) and

$$Q(z) = F'(z) \left(\frac{z}{F(z)} \right)^{1+\mu} \quad (= 1 + q_n z^n + \dots)$$

then from (5) and (6) we obtain

$$Q(z) + \frac{1}{c-\mu} zQ'(z) = f'(z) \left(\frac{z}{f(z)} \right)^{1+\mu} < 1 + \lambda z.$$

Hence, by using the result of Hallenbeck and Ruscheweyh [2, Th.1] we have that

$$Q(z) < 1 + \lambda_1 z, \quad \lambda_1 = \frac{|(c-\mu)\lambda|}{|n+c-\mu|} = \frac{(n-\mu)\sin(\pi\alpha/2)}{|\mu + (n-\mu)e^{i\pi\alpha/2}|},$$

and the desired result easily follows from Theorem 2.

REMARK 1. For $\alpha=1$ and $n=1$ we have the corresponding result given earlier in [4].

For $c=\mu+1$, we have

COROLLARY 2. Let $0 < \mu < 1$ and $0 < \alpha \leq 1$. If $f \in A_n$ satisfies the condition

$$\left| f'(z) \left(\frac{z}{f(z)} \right)^{1+\mu} - 1 \right| < \frac{n(n-\mu)\sin(\pi\alpha/2)}{|\mu + (n-\mu)e^{i\pi\alpha/2}|}, \quad z \in U,$$

then the function

$$F(z) = z \left[\frac{1}{z} \int_0^z \left(\frac{t}{f(t)} \right)^\mu dt \right]^{-\frac{1}{\mu}}$$

belongs to $S(\alpha)$.

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REFERENCES

- [1] P. L. Duren, Univalent functions, Springer-Verlag, 1983.
- [2] R. Hallenbeck and St. Ruscheweyh, Subordination by convex functions, *Proc. Amer. Math. Soc.* 52(1975), 191-195.
- [3] I. S. Jack, Functions starlike and convex of order α , *J. London Math. Soc.* (2) 3(1971), 469-474.
- [4] M. Obradović, A class of univalent functions, *Hokkaido Math.J.* 27(2) (1998), 329-335.
- [5] S. Ponnusamy and V. Singh, Criteria for strongly starlike functions, *Complex Variables: Theory and Appln.*, to appear.

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